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A Non-Probabilistic Proof of the Assouad Embedding Theorem with Bounds on the Dimension

Abstract

We give a non-probabilistic proof of a theorem of Naor and Neiman that asserts that if (E, d) is a doubling metric space, there is an integer $N > 0$, depending only on the metric doubling constant, such that for each exponent $\alpha \in (1/2, 1)$, one can find a bilipschitz mapping $F = (E, d^\alpha) \to \mathbb{R}^N$.

Keywords

Assouad Embedding • doubling metric spaces • snowflake distance

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Guy David[1]*, Marie Snipes[2]†

1 Univ Paris-Sud,
Laboratoire de mathématiques UMR-8628,
Orsay F-91405, France
and Institut Universitaire de France

2 Department of Mathematics,
Hayes Hall, Kenyon College,
Gambier, Ohio 43022, USA

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1. Introduction

The purpose of this paper is to give a simpler proof of a theorem proved by Naor and Neiman [3] that asserts that if (E, d) is a doubling metric space, there is an integer $N > 0$, depending only on the metric doubling constant, such that for each exponent $\alpha \in (1/2, 1)$, one can find a bilipschitz mapping $F : (E, d^\alpha) \to \mathbb{R}^N$.
Here $\mathbb{R}^N$ is equipped with its Euclidean metric, the snowflake distance d^α is simply defined by $d^\alpha(x, y) = d(x, y)^\alpha$ for $x, y \in E$, and metrically doubling means that there is an integer $C_0 \geq 1$ such that for every $r > 0$, every (closed) ball of radius $2r$ in E can be covered with no more than C_0 balls of radius r. We call C_0 a metric doubling constant for (E, d).

Remark 1.

Notice that in a doubling metric space with doubling constant C_0, every ball of radius $2^m r$ can be covered with C_0^m balls of radius r. More generally, for $\lambda > 0$, every ball of radius λr can be covered by $C_0 \lambda^{N_0}$ balls of radius r, where $N_0 = \log_2 C_0$. To see this replace λ by the next power of 2.

A consequence of Naor and Neiman's main result in [3] is the following.

Theorem 2.

For each $C_0 \geq 1$, there is an integer N and, for $1/2 < \alpha < 1$, a constant $C = C(C_0, \alpha)$ such that if (E, d) is a metric space that admits the metric doubling constant C_0, we can find an injection $F : E \to \mathbb{R}^N$ such that

$$C^{-1} d(x, y)^\alpha \leq |F(x) - F(y)| \leq C d(x, y)^\alpha$$

for $x, y \in E$.

* E-mail: Guy.David@math.u-psud.fr
† E-mail: snipesm@kenyon.edu

If we did not ask N to be independent of α, this would just be the usual Assouad embedding theorem from [1]. The theorem of Naor and Neiman is more general than Theorem 2, but its proof is also more complicated. In particular, the existence of the bilipschitz embedding F is demonstrated in [3] by constructing a random embedding into $\mathbb{R}^N$ and showing (via a version of the Lovász Local Lemma) that with positive probability the map is bilipschitz.

In the present paper we do not try to get optimal values of N and C; see [3] for better control of these constants, and for more information on the context. There is nothing special about the constant $1/2$ in our statement; we just do not want to consider the case when α is close to 0, for which the dimension independence fails. Indeed, even when $E = \mathbb{R}$, with the usual distance, one needs many dimensions to construct an α-snowflake (or equivalently, a mapping F as above) with α small.

The proof will be a rather simple modification of the standard proof of Assouad's theorem (see, for example [2]), but it took a surprising amount of energy to the authors, plus the knowledge of the fact that the result is true, to make it work. Rather than using a probabilistic proof, we use an adaptive argument, and work at small relative scales to use the fact that there is a lot of space in $\mathbb{R}^N$; the difficult part (for the authors) was to realize that using a very sparse collection of scales would not kill the argument, and instead helps control the residual terms. The constant $C(C_0, \alpha)$ gets very large (when α gets close to 1), but this is expected.

2. Acknowledgements

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3. Proof of Theorem 2

We need some notation before we start the proof. Here the doubling metric space (E, d) is fixed, and $B(x, r)$ will denote the closed ball with center x and radius r.

We note that, in fact, if one considers values of α bounded away from both 0 and 1, one can use the standard Assouad proof (as in the proof of Lemma 12.7 in [2]) to show that the resulting snowflake spaces can all be embedded in a fixed dimension Euclidean space. Hence we need only consider α very close to 1.

We shall use a small parameter $\tau > 0$, with $\tau \leq 1 - \alpha$, and work at the scales

$$r_k = \tau^{2k}, k \in \mathbb{Z}. \tag{1}$$

We first prove Theorem 2 in the case that E has finite diameter; this allows us to choose an initial scale $k = k_0$, where k_0 is such that $r_{k_0} \geq \mathrm{diam}(E)$. But our construction, and the main constants, will not depend on this choice. We shall dispense with this extra assumption and treat the general case near (21).

For each $k \geq k_0$, select a maximal collection $\{x_j\}$, $j \in J_k$, of points of E, with $d(x_i, x_j) \geq r_k$ for $i \neq j$. Thus, by maximality

$$E \subset \bigcup_{j \in J_k} B(x_j, r_k). \tag{2}$$

For convenience, we write $B_j = B(x_j, r_k)$ and $\lambda B_j = B(x_j, \lambda r_k)$ for $\lambda > 0$.

Letting $N(x)$ denote the number of indices $j \in J_k$ such that $d(x_j, x) \leq 10 r_k$, we now check that

$$N(x) \leq C_0^5 \quad \text{for } x \in E. \tag{3}$$

Indeed, we can cover $B(x, 10r_k)$ with fewer than C_0^5 balls D_l of radius $r_k/3$. Each D_l contains at most one x_j (because $d(x_i, x_j) \geq r_k$ for $i \neq j$). Because all the x_j that lie in $B(x, 10r_k)$ are contained in some D_l, (3) follows.

From (3) and the assumption (for the present case) that E is bounded, we conclude J_k is finite. Note, however, that even if E were unbounded, we could use the preceding discussion to construct the collection $\{x_j\}$ without using the axiom of choice.

Set $\Xi = \{1, 2, \dots C_0^5\}$ (a set of colors). Now enumerate J_k, and for $j \in J_k$ let $\xi(j)$ be the first color not taken by an earlier close neighbor; specifically, choose $\xi(j)$ to be the first color not used by an earlier $i \in J_k$ such that $d(x_i, x_j) \le 10 r_k$. By construction we have

$$\xi(i) \neq \xi(j) \quad \text{for } i, j \in J_k \text{ with } i \neq j \text{ and } d(x_i, x_j) \le 10 r_k.$$

Finally, for each $\xi \in \Xi$, define the set $J_k(\xi) := \{j \in J_k\,;\, \xi(j) = \xi\}$. Thus

$$d(x_i, x_j) > 10 r_k \quad \text{for } i, j \in J_k(\xi) \text{ such that } i \neq j. \tag{4}$$

For each $j \in J_k$, set $\varphi_j(x) = \max\{0, 1 - r_k^{-1} \operatorname{dist}(x, B_j)\}$. The formula is not important; we just want to make sure that

$$\begin{aligned} 0 \le \varphi_j(x) \le 1 & \quad \text{everywhere,} \\ \varphi_j(x) = 1 & \quad \text{for } x \in B_j, \\ \varphi_j(x) = 0 & \quad \text{for } x \in E \setminus 2B_j, \end{aligned} \tag{5}$$

and

$$\varphi_j \text{ is Lipschitz, with } ||\varphi_j||_{lip} \le r_k^{-1}. \tag{6}$$

We continue with the non-surprising part of the construction. For each $\xi \in \Xi$, we will construct two mappings: $F^\xi : E \to \mathbb{R}^M$ and a slightly modified version $\widetilde{F}^\xi : E \to \mathbb{R}^M$, where M is a very large integer depending only on the metric doubling constant. Our final mapping $F : E \to \mathbb{R}^N$, will be the tensor product of these $2C_0^5$ mappings. Thus, the dimension N is $2C_0^5 M$, which can probably be improved.
We decide that F^ξ will be of the form:

$$F^\xi(x) = \sum_{k \ge k_0} r_k^\alpha f_k^\xi(x),$$

where

$$f_k^\xi(x) = \sum_{j \in J_k(\xi)} v_j \varphi_j(x), \tag{7}$$

with vectors $v_j \in \mathbb{R}^M$ that will be carefully chosen later. The extra room in $\mathbb{R}^M$ will be used to give lots of different choices of v_j. $\widetilde{F}$ will take on the same form as F, but with a different choice of the vectors $\{v_j\}$. However, for both functions we will choose the v_j inductively, and so that

$$v_j \in B(0, \tau^2) \subset \mathbb{R}^M,$$

with the same very small $\tau > 0$ as in the definition of $r_k = \tau^{2k}$ above; τ will be chosen near the end. With this choice, we immediately see that

$$||f_k^\xi||_\infty \le \tau^2$$

because the φ_j, $j \in J_k(\xi)$, have disjoint supports by (4) and (5); hence the series in (7) converges. Moreover, if we set

$$F_k^\xi(x) = \sum_{k_0 \le \ell \le k} r_\ell^\alpha f_\ell^\xi(x), \tag{8}$$

we get that

$$||F^\xi - F_k^\xi||_\infty \le \sum_{\ell > k} r_\ell^\alpha \tau^2 = r_{k+1}^\alpha \tau^2 \sum_{\ell \ge 0} \tau^{2\ell\alpha} \le 2\tau^2 r_{k+1}^\alpha \tag{9}$$

(because $r_l = \tau^{2l}$ and $\tau^{2\alpha} < 1/2$ when τ is small). Also, the Lipschitz norm of f_k^ξ is

$$||f_k^\xi||_{lip} \le \tau^2 r_k^{-1}$$

by (6) and because the φ_j are supported in disjoint balls; we sum brutally and get that

$$||F_k^\xi||_{lip} \leq \sum_{\ell \leq k} r_\ell^\alpha ||f_\ell^\xi||_{lip} \leq \tau^2 \sum_{\ell \leq k} r_\ell^{\alpha-1} = \tau^2 r_k^{\alpha-1} \sum_{\ell \leq k} \tau^{2(\ell-k)(\alpha-1)} = \tau^2 r_k^{\alpha-1}(1-\tau^{2(1-\alpha)})^{-1}$$

by (8). We take $\tau \leq 1-\alpha$ (many other choices would do, the main point is to have a control in (10) below by a power of τ, which could even be negative); then

$$\ln(\tau^{2(1-\alpha)}) = 2(1-\alpha)\ln(\tau) = -2(1-\alpha)\ln\left(\frac{1}{\tau}\right) \leq -2\tau \ln\left(\frac{1}{\tau}\right);$$

we exponentiate and get that $\tau^{2(1-\alpha)} \leq e^{-2\tau\ln(\frac{1}{\tau})} \leq 1-\tau\ln(\frac{1}{\tau})$ if τ is small enough, hence $1-\tau^{2(1-\alpha)} \geq \tau\ln(\frac{1}{\tau})$, and finally

$$||F_k^\xi||_{lip} \leq \frac{\tau}{\ln(\frac{1}{\tau})} r_k^{\alpha-1} \leq r_k^{\alpha-1} \tag{10}$$

(again if τ is small enough; for example, $\tau < 1/2$ works).

We now describe how to choose the vectors v_j, $j \in J_k$, so that the differences $|F_k^\xi(x) - F_k^\xi(y)|$ will be as large as possible (toward proving that $(F_k^\xi)^{-1}$ is Lipschitz).
Fix $k \geq k_0$, suppose that the F_{k-1}^ξ were already constructed, and fix $\xi \in \Xi$. Put any order $<$ on the finite set $J_k(\xi)$. We shall construct F_k^ξ with the order $<$. Specifically, we will choose the v_j in Lemma 3 using the order $<$. For $\widetilde{F}_k^\xi$, we will use the reverse order.
Recall that we defined

$$F_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha f_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha \sum_{i \in J_k(\xi)} v_i \varphi_i(y) \tag{11}$$

in (8) and (7); for each $j \in J_k(\xi)$, we shall also consider the partial sum $G_{k,j}^\xi$ defined by

$$G_{k,j}^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha \sum_{i \in J_k(\xi)\,;\, i<j} v_i \varphi_i(y), \tag{12}$$

which we therefore assume to be known when we choose v_j.

Lemma 3.

For each $j \in J_k(\xi)$, we can choose $v_j \in B(0, \tau^2)$ so that

$$|F_k^\xi(x) - G_{k,j}^\xi(y)| \geq \tau^3 r_k^\alpha \quad \text{for } x \in B_j \text{ and } y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j. \tag{13}$$

Observe that for $x \in B_j$, $\varphi_j(x) = 1$ and $\varphi_i(x) = 0$ for the other indices $i \neq j \in J_k(\xi)$ (because φ_i is supported in $2B_i$ by (5), and $2B_i$ never meets B_j by (4)). Thus

$$F_k^\xi(x) = F_{k-1}^\xi(x) + r_k^\alpha f_k^\xi(x) = F_{k-1}^\xi(x) + r_k^\alpha v_j \tag{14}$$

by (8) and (7). By (10), $||F_k^\xi||_{lip} \leq r_k^{\alpha-1}$, but the proof of (10) also yields

$$||G_{k,j}^\xi||_{lip} \leq r_k^{\alpha-1}$$

(we just add fewer terms). We shall use this to replace B_j and $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$ with discrete sets. Set $\eta = \tau^3 r_k$, and pick an η-dense set X in B_j, and an η-dense set Y in $B(x_j, 10\tau^{-2}r_k) \setminus 2B_j$. We shall soon prove that we can choose v_j so that

$$|F_k^\xi(x') - G_{k,j}^\xi(y')| \geq 3\tau^3 r_k^\alpha \quad \text{for } x' \in X \text{ and } y' \in Y, \tag{15}$$

and let us first check that the lemma will follow.

Proof of Lemma 3. Notice that for $x \in B_j$, we can find $x' \in X$ such that

$$|F_k^\xi(x') - F_k^\xi(x)| \le ||F_k^\xi||_{lip}\, \eta \le r_k^{\alpha-1} \cdot \tau^3 r_k = \tau^3 r_k^\alpha,$$

and similarly, for $y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j$ we can find $y' \in Y$ such that

$$|G_{k,j}^\xi(y) - G_{k,j}^\xi(y')| \le ||G_{k,j}^\xi||_{lip}\, \eta \le \tau^3 r_k^\alpha.$$

Then (13) for x and y follows from (15), as needed.
So we want to arrange (15). First we bound $|X|$, the number of elements in X. By Remark 1, we can cover $B_j = B(x_j, r_k)$ by $C_0(2\tau^{-3})^{N_0}$ balls of radius $\eta/2$; we just keep those that meet B_j, pick an element of B_j in each such ball, and get an η-dense net X, with $|X| \le C_0(2\tau^{-3})^{N_0}$. Similarly, we can find Y so that

$$|Y| \le C_0 \left(2\, \frac{10\tau^{-2} r_k}{\eta}\right)^{N_0} = C_0 \left(\frac{20\tau^{-2} r_k}{\tau^3 r_k}\right)^{N_0} = C_0 (20\tau^{-5})^{N_0}.$$

The total number of pairs (x', y') for which we have to check (15) is thus

$$|X||Y| \le C_0^2 (40\tau^{-8})^{N_0}.$$

Now pick a maximal finite set V in $B(0, \tau^2) \subset \mathbb{R}^M$, whose points lie at mutual distances at least $7\tau^3$ from each other. For each pair (x', y') as above, the different choices of $v_j \in V$ yield the same value of $G_{k,j}^\xi(y')$ (because $G_{k,j}^\xi$ does not depend on v_j, by (12)), and values of $F_k^\xi(x')$ that differ by at least $7\tau^3 r_k^\alpha$, by (14). Thus (15) for this pair (x', y') cannot fail for more than one choice of $v_j \in V$, and it is now enough to show that V has more than $|X||Y|$ elements. Taking C_M to be the doubling constant of $\mathbb{R}^M$, it follows from Remark 1 that $|V| \ge C_M (1/7\tau)^{\log_2 C_M}$, which is indeed larger than $|X||Y|$ if $C_M > C_0^8$ and τ is small enough (depending on M). This completes our verification of (15); as noted earlier, Lemma 3 follows. □

For each color ξ, choose the vectors v_j, and hence the mappings F_k^ξ, as in Lemma 3. Also define a second version $\widetilde{F}_k^\xi$ of F_k^ξ using the opposite order on $J_k(\xi)$. By (9), $F^\xi : E \to \mathbb{R}^M$ is the limit $F^\xi = \lim_{k\to\infty} F_k^\xi$. We are ready to check that F (the tensor product of the maps $\{F^\xi, \widetilde{F}^\xi : \xi \in \Xi\}$) is bilipschitz from the snowflaked space (E, d^α).

Lemma 4.
We have that

$$\frac{\tau^5}{8}\, d(x,y)^\alpha \le |F(x) - F(y)| \le 5N\tau^{-2(1-\alpha)} d(x,y)^\alpha \quad \text{for } x, y \in E.$$

Proof. Let $x, y \in E$ be given; we may assume that $x \neq y$. Let k be such that

$$4r_k \le d(x,y) \le 4r_{k-1} = 4\tau^{-2} r_k, \tag{16}$$

where the last part comes from (1). Then $r_k \le 4r_k \le d(x,y) \le \mathrm{diam}(E) \le r_{k_0}$ by our definition of k_0, and so $k \ge k_0$. By (10) and (16),

$$|F_k^\xi(x) - F_k^\xi(y)| \le ||F_k^\xi||_{lip}\, d(x,y) \le r_k^{\alpha-1} d(x,y) \le \left(\frac{d(x,y)}{4\tau^{-2}}\right)^{\alpha-1} d(x,y) = (4\tau^{-2})^{1-\alpha} d(x,y)^\alpha. \tag{17}$$

But (9) also says that for sufficiently small τ,

$$\big||F^\xi(x) - F^\xi(y)| - |F_k^\xi(x) - F_k^\xi(y)|\big| \le 2||F^\xi - F_k^\xi||_\infty \le 4\tau^2 r_{k+1}^\alpha = 4\tau^2 \tau^{2\alpha} r_k^\alpha \le 2\tau^2 d(x,y)^\alpha. \tag{18}$$

Inequalities (17) and (18) (and similar estimates for the $\widetilde{F}_k^\xi$) give the upper bound in the statement of Lemma 4.
For the lower bound, consider the same fixed $x, y \in E$. Notice that by (2) we can find $j \in J_k$ such that $x \in B_j$. Let $\xi \in \Sigma$ be the color such that $j \in J_k(\xi)$. We will consider separately the case where $y \in 2B_i$ for some $i \in J_k(\xi)$ and where $y \notin 2B_i$ for all $i \in J_k(\xi)$.
We'll need to know that

$$y \in B(x_j, 10\tau^{-2} r_k) \setminus 2B_j. \tag{19}$$

That $y \in B(x_j, 10\tau^{-2} r_k)$ follows from (16), because $d(x, x_j) \le r_k$ since $x \in B_j$. Moreover, if $y \in 2B_j$, then $d(x, y) \le d(x, x_j) + d(x_j, y) \le 3r_k$, which would contradict (16). So (19) holds.
If $y \in 2B_i$ for some $i \in J_k(\xi)$, then $i \neq j$, by (19). Let us assume that $i < j$; otherwise, we would use $\widetilde{F}_k^\xi$ instead of F_k^ξ in the following calculations. Recall that all the $\varphi_l(y)$, $l \neq i$, are equal to 0, by (5) and (4). Then (11) and (12) yield

$$F_k^\xi(y) = F_{k-1}^\xi(y) + r_k^\alpha v_i \varphi_i(y) = G_{k,j}^\xi(y) \tag{20}$$

(with $F_{k-1}^\xi(y) = 0$ if $k = k_0$). By (19), we can apply (13), which says that

$$|F_k^\xi(x) - F_k^\xi(y)| = |F_k^\xi(x) - G_{k,j}^\xi(y)| \ge \tau^3 r_k^\alpha.$$

We then combine this with (18) and get that

$$|F^\xi(x) - F^\xi(y)| \ge |F_k^\xi(x) - F_k^\xi(y)| - 4\tau^2\tau^{2\alpha} r_k^\alpha \ge \tau^3 r_k^\alpha - 4\tau^2 r_{k+1}^\alpha = \tau^3 r_k^\alpha(1 - 4\tau^{-1}\tau^{2\alpha}) \ge \frac{\tau^3 r_k^\alpha}{2}$$

by (1) and because we can take $\alpha > 2/3$ and τ small (recall that when $1/2 \le \alpha \le 2/3$, we could simply use the proof of [1] or [2]). Now

$$|F(x) - F(y)| \ge |F^\xi(x) - F^\xi(y)| \ge \frac{\tau^3 r_k^\alpha}{2} \ge \frac{\tau^3}{2}\left(\frac{d(x, y)}{4\tau^{-2}}\right)^\alpha \ge \frac{\tau^5}{8} d(x, y)^\alpha$$

by (16). This proves Lemma 4 when $y \in 2B_i$ for some $i \in J_k(\xi)$. If not, all the $\varphi_i(x)$ vanish by (5), and $F_k^\xi(y) = G_{k,j}^\xi(y) = F_{k-1}^\xi(y)$ for all j, by (11) and (12). That is, (20) still holds (with j again chosen so that $x \in B_j$), and we can continue just as in the previous case. Lemma 4 follows. □

This completes the proof of Theorem 2 for bounded E.
Now suppose E is an unbounded metric space with doubling constant C_0. Fix an origin x_0, and apply the construction above to the sets $E_m = E \cap B(x_0, 2^m)$.
The set E_m is itself doubling, with doubling constant C_0^2. To see this note that if $x \in E_m$ and $r > 0$, we can cover $E_m \cap B(x, 2r)$ with C_0^2 balls of radius $r/2$, which (when they meet E_m) we can replace with balls of radius r whose centers are in E_m.
We get from the proof above a mapping F_m such that

$$C^{-1} d(x, y)^\alpha \le |F_m(x) - F_m(y)| \le C d(x, y)^\alpha \tag{21}$$

for $x, y \in E_m$, where C depends on C_0 and α but not on m. We may assume that $F_m(x_0) = 0$, after possibly adding a constant, which would not destroy (21).
Now define for each $k \in \mathbb{Z}$, a maximal collection $\{x_j\} \subset E$, $j \in J_k$, with $d(x_i, x_j) \ge r_k$ for $i \neq j$. Although E is unbounded, each J_k is still at most countable.
Notice also that for each x_j, the sequence $\{F_m(x_j)\}$ is bounded (by (21) and because $F_m(x_0) = 0$); hence we can extract a subsequence $\{m_j\}$, so that the sequence $F_{m_j}(x_j)$ converges for each x_j. By (21) again, the convergence is uniform on each bounded subset of E, so (21) passes to the limit, and this limit F satisfies the conclusion of Theorem 2. This completes our proof.

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