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FRACTIONAL ORDER CONVERGENCE RATE ESTIMATES OF FINITE DIFFERENCE METHOD ON NONUNIFORM MESHES

DEJAN BOJOVIĆ

University of Kragujevac, Faculty of Science
Radoja Domanovića 12, 34000 Kragujevac, Yugoslavia
E-mail: bojovicd@ptt.yu

BOŠKO S. JOVANOVIĆ

University of Belgrade, Faculty of Mathematics
Studentski trg 16, 11000 Belgrade, Yugoslavia
E-mail: bosko@matf.bg.ac.yu

Abstract — In this paper we show how the theory of interpolation of function spaces can be used to establish convergence rate estimates for finite difference schemes on nonuniform meshes. As a model problem we consider the first boundary value problem for the Poisson equation. We assume that the solution of the problem belongs to the Sobolev space W_2^s, $2 \leqslant s \leqslant 3$. Using the interpolation theory we construct a fractional-order convergence rate estimate which is consistent with the smoothness of data.

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1. Introduction

For a class of finite difference schemes for elliptic boundary value problems, the convergence rate estimates consistent with the smoothness of data

$$\|u - v\|_{W_2^r(\Omega_h)} \leqslant C h^{s-r} \|u\|_{W_2^s(\Omega)}, \quad s \geqslant r \tag{1}$$

are of major interest. Here $u = u(x)$ denotes the solution of the original boundary value problem, v is the solution of corresponding finite difference scheme, h is the step size of uniform mesh, $W_2^s(\Omega)$ and $W_2^s(\Omega_h)$ denote Sobolev spaces of functions of continuous and discrete variables, respectively, and C is a positive generic constant, independent of h and u. For problems with variable coefficients the constant C depends on the coefficients. Standard technique for derivation of such estimates (see [2, 6]) is based on the Bramble–Hilbert lemma [1]. An alternative technique (see [3, 9]) is based on the theory of interpolation of Banach spaces [7]. Estimates of the type (1) are obtained for the broad class of difference schemes on uniform meshes.

In the present paper analogous results are obtained for some difference schemes on nonuniform meshes. As a model problem we consider the first boundary value problem for the Poisson equation in a rectangular domain. The problem is approximated on nonuniform rectangular mesh using difference schemes proposed in [4]. For the solutions from the Sobolev–Slobodetskiĭ space a convergence rate estimate of the form

$$\|u-v\|_{W_2^r(\Omega_h)} \leqslant C h_{\max}^{s-r} \|u\|_{W_2^s(\Omega)}\,, \quad s \geqslant r \tag{2}$$

is obtained for $r=1$ and $2 \leqslant s \leqslant 3$. Here $h_{\max}$ is the maximal step size in Ω_h. In such a way, the convergence rate estimates obtained in [4] for $s=3$ are extended to the case of solutions from Sobolev–Slobodetskiĭ spaces.

2. State of the problem and preliminary results

We consider the Poisson equation with homogenous boundary condition in the domain $\Omega = (0,1)^2$:

$$-\Delta u = f \quad \text{in} \quad \Omega\,, \qquad u=0 \quad \text{on} \quad \Gamma = \partial\Omega\,. \tag{3}$$

We assume that $f \in W_2^{s-2}(\Omega)\,,\ 2 \leqslant s \leqslant 3\,,\ s \neq 5/2$. Then the unique weak solution u belongs to the Sobolev space $W_2^s(\Omega)$.

In the domain $\overline{\Omega} = \Omega \cup \Gamma$ we define the nonuniform mesh:

$$\hat{\bar{\omega}} = \big\{ x = x_{i_1 i_2} = (x_1^{i_1},\, x_2^{i_2})\,;\ x_j^{i_j} = x_j^{i_j-1} + h_j^{i_j}\,,\ i_j = 1,\dots,N_j-1\,,\ x_j^0 = 0\,,\ x_j^{N_j} = 1\,;\ j=1,\,2 \big\},$$

$\sum_{i_j=1}^{N_j} h_j^{i_j} = 1\,,\ j=1,\,2$. Let $\hat{\omega}$ be the set of interior mesh points, and $\hat{\gamma}$ — the set of boundary mesh points. We define $\Omega_0 = (0,1] \times (0,1]\,,\ \Omega_1 = (0,1] \times (0,1)$, $\Omega_2 = (0,\,1) \times (0,\,1]$ and $\hat{\omega}_j = \hat{\bar{\omega}} \cap \Omega_j$, $j = 0,1,2$. We define finite differences in the usual manner [5]:

$$v_{x_j} = \big(v^{(+1_j)} - v\big)/h_{j+}, \quad v_{\bar{x}_j} = \big(v - v^{(-1_j)}\big)/h_j, \quad v_{\hat{x}_j} = \big(v^{(+1_j)} - v\big)/\hbar_j\,,$$

where

$$h_j = h_j^{i_j}\,,\ h_{j\pm} = h_j^{i_j \pm 1},\ \hbar_j = (h_j + h_{j+})/2,$$
$$v = v(x), \quad v^{(\pm 1_1)} = v(x_1^{i_1 \pm 1},\, x_2^{i_2}), \quad v^{(\pm 1_2)} = v(x_1^{i_1},\, x_2^{i_2 \pm 1}).$$

Let S^h, denote the linear space of real-valued functions defined on the mesh $\hat{\bar{\omega}}$, and let S_0^h be the linear space of those functions defined on $\hat{\bar{\omega}}$ which are equal to zero on $\hat{\gamma}$. We define inner products:

$$(u,\,v)_* = \sum_{x \in \hat{\omega}} u\,v\,\hbar_1\,\hbar_2, \quad (u,\,v]_1 = \sum_{x \in \hat{\omega}_1} u\,v\,h_1\,\hbar_2,$$
$$(u,\,v]_2 = \sum_{x \in \hat{\omega}_2} u\,v\,\hbar_1\,h_2, \qquad (u,v] = \sum_{x \in \hat{\omega}_0} u\,v\,h_1\,h_2\,,$$

and coresponding norms $\|v\|_*$, $\|v]|_1$, $\|v]|_2$, and $\|v]|$. We also define discrete Sobolev norms and seminorms:

$$\|v\|_{L_2(\hat{\omega})} = \|v\|_*\,, \quad |v|^2_{W_2^1(\hat{\omega})} = \|v_{\bar{x}_1}]|_1^2 + \|v_{\bar{x}_2}]|_2^2\,, \quad \|v\|^2_{W_2^1(\hat{\omega})} = \|v\|^2_{L_2(\hat{\omega})} + |v|^2_{W_2^1(\hat{\omega})}\,.$$

In this paper we use some well-known assertions (see [4]):

Lemma 1. *For arbitrary functions $v, w \in S^h$, which vanish for $x_j = 0, 1,\ j = 1, 2$, the following identities hold:*

$$(v_{\hat{x}_j}, w)_* = -(v, w_{\bar{x}_j}]_j, \quad (v_{\bar{x}_j\hat{x}_j}, w)_* = -(v_{x_j}, w_{\bar{x}_j}]_j.$$

Lemma 2. *For every function $v \in S_0^h$, the following identity holds:*

$$\|\Delta_h v\|_*^2 = \|v_{\bar{x}_1\hat{x}_1}]|_*^2 + \|v_{\bar{x}_2\hat{x}_2}]|_*^2 + 2\|v_{\bar{x}_1\bar{x}_2}]|^2\,, \quad \Delta_h v = v_{\bar{x}_1\hat{x}_1} + v_{\bar{x}_2\hat{x}_2}\,.$$

Lemma 3. *For every function $v \in S_0^h$, the following inequalities hold:*

$$4\|v\|_*^2 \leqslant \|v_{\bar{x}_j}]|_j^2, \quad j = 1, 2, \quad 8\|v\|_*^2 \leqslant |v|^2_{W_2^1(\hat{\omega})}\,.$$

Lemma 4. *For every function $v \in S_0^h$, the following inequality holds:*

$$\|v\|^2_{W_2^1(\hat{\omega})} \leqslant \frac{9}{8}|v|^2_{W_2^1(\hat{\omega})}\,.$$

Lemma 5. *For every function $v \in S_0^h$, the following inequalities hold:*

$$\|h_{3-j}v_{\bar{x}_1\bar{x}_2}]|^2 \leqslant 4\|v_{\bar{x}_j}]|_j^2, \quad \|h_{3-j}v_{\bar{x}_{3-j}\bar{x}_j\hat{x}_j}]|_{3-j}^2 \leqslant 4\|v_{\bar{x}_j\hat{x}_j}]|_*^2, \quad j = 1, 2.$$

Lemma 6. *If $h_{j+} > h_j$, $j = 1, 2$, then for every function $v \in S_0^h$, the following inequalities hold:*

$$\|(h_{j+} - h_j)v_{x_j\bar{x}_{3-j}}]|_{3-j}^2 \leqslant 4\|v_{\bar{x}_{3-j}}]|_{3-j}^2, \quad \|(h_{j+} - h_j)v_{x_j}]|_*^2 \leqslant 4\|v\|_*^2, \quad j = 1, 2\,.$$

We define Steklov's smoothing operators:

$$S_2^{(1)} f(x_1,\, x_2) = \frac{1}{\hbar_1} \int\limits_{x_1 - h_1}^{x_1 + h_{1+}} K_1(x_1')\, f(x_1',\, x_2)\, dx_1',$$

$$S_2^{(2)} f(x_1,\, x_2) = \frac{1}{\hbar_2} \int\limits_{x_2 - h_2}^{x_2 + h_{2+}} K_2(x_2')\, f(x_1,\, x_2')\, dx_2',$$

where

$$K_j(x_j') = \begin{cases} 1 + \frac{x_j' - x_j}{h_j}, & x_j, \in (x_j - h_j,\, x_j), \\ 1 - \frac{x_j' - x_j}{h_{j+}}, & x_j' \in (x_j,\, x_j + h_{j+}). \end{cases}$$

A simple calculation shows that $S_2^{(j)} \frac{\partial^2 u}{\partial x_j^2} = u_{\bar{x}_j\hat{x}_j}$.

3. Convergence of the finite-difference scheme

We approximate the problem (3) with the following finite-difference scheme [4]:

$$-\Delta_h^\theta v = S_2^{(1)} S_2^{(2)} f \ \text{ in } \ \hat{\omega}, \quad v = 0 \ \text{ on } \ \hat{\gamma}, \tag{4}$$

where

$$\Delta_h^\theta v = v_{\bar{x}_1\hat{x}_1} + v_{\bar{x}_2\hat{x}_2} + \theta\big[(h_1^2 v_{\bar{x}_1})_{\hat{x}_1\bar{x}_2\hat{x}_2} + (h_2^2 v_{\bar{x}_2})_{\hat{x}_2\bar{x}_1\hat{x}_1}\big].$$

The error $z = u - v$ satisfies the conditions

$$-\Delta_h^\theta z = \varphi_{1,\bar{x}_1\hat{x}_1} + \varphi_{2,\bar{x}_2\hat{x}_2} + \psi_{1,\hat{x}_1\hat{x}_2} + \psi_{2,\hat{x}_1\hat{x}_2} \quad \text{in} \;\; \hat{\omega}, \qquad z = 0 \;\; \text{on} \;\; \hat{\gamma}, \tag{5}$$

where

$$\varphi_1 = S_2^{(2)} u - u - \left(\frac{h_2^2}{6} u_{\bar{x}_2}\right)_{\hat{x}_2}, \quad \varphi_2 = S_2^{(1)} u - u - \left(\frac{h_1^2}{6} u_{\bar{x}_1}\right)_{\hat{x}_1},$$
$$\psi_1 = (1/6 - \theta) h_1^2 u_{\bar{x}_1\bar{x}_2}, \qquad \psi_2 = (1/6 - \theta) h_2^2 u_{\bar{x}_1\bar{x}_2}.$$

Note that $\psi_1 = \psi_2 = 0$ for $\theta = 1/6$.

We begin the analysis of the scheme (4) by investigating its stability in the discrete W_2^1 norm. Applying Lemmas 1 and 5, we have

$$(-\Delta_h^\theta z, z)_* = \|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2 - \theta \|h_1 z_{\bar{x}_1\bar{x}_2}]|^2 - \theta \|h_2 z_{\bar{x}_1\bar{x}_2}]|^2 \geqslant C_\theta \big(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\big),$$

where

$$C_\theta = \min\{1, 1 - 4\theta\}, \quad C_\theta > 0 \;\; \text{for} \;\; \theta < 1/4\,.$$

Further,

$$\begin{aligned}&(\varphi_{1,\bar{x}_1\hat{x}_1} + \varphi_{2,\bar{x}_2\hat{x}_2} + \psi_{1,\hat{x}_1\hat{x}_2} + \psi_{2,\hat{x}_1\hat{x}_2}, z)_* \\ &\quad = -(\varphi_{1,\bar{x}_1}, z_{\bar{x}_1}]_1 - (\varphi_{2,\bar{x}_2}, z_{\bar{x}_2}]_2 - (\psi_{1,\hat{x}_2}, z_{\bar{x}_1}]_1 - (\psi_{2,\hat{x}_1}, z_{\bar{x}_2}]_2 \\ &\quad \leqslant \sqrt{2}\big(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2\big)^{1/2}\big(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\big)^{1/2}.\end{aligned}$$

From the previous inequalities we have:

$$|z|_{W_2^1(\hat{\omega})} = \big(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\big)^{1/2} \leqslant \frac{\sqrt{2}}{C_\theta}\big(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2\big)^{1/2} \tag{6}$$

Applying Lemma 4 we obtain the estimate

$$\|z\|_{W_2^1(\hat{\omega})} \leqslant C_{0,\theta}\big(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2\big)^{1/2}, \quad C_{0,\theta} = \frac{3}{2C_\theta}. \tag{7}$$

Note that $C_{0,\theta} = C_{0,\theta}(l_1, l_2)$, if the domain $\Omega = (0, l_1) \times (0, l_2)$.

In such a way, the problem of deriving a convergence rate estimate for the finite-difference scheme (4) is now reduced to estimating the right-hand side terms in (7).

Estimates of right-hand side terms in (7) for solution $u \in W_2^3(\Omega)$ were derived in [4]:

$$\|\varphi_{1,\bar{x}_1}]|_1 \leqslant C h_{2,\max}^2 \|u\|_{W_2^3(\Omega)}, \tag{8}$$
$$\|\varphi_{2,\bar{x}_2}]|_2 \leqslant C h_{1,\max}^2 \|u\|_{W_2^3(\Omega)}, \tag{9}$$
$$\|\psi_{1,\hat{x}_2}]|_1 \leqslant C h_{1,\max}^2 \|u\|_{W_2^3(\Omega)}, \tag{10}$$
$$\|\psi_{2,\hat{x}_1}]|_2 \leqslant C h_{2,\max}^2 \|u\|_{W_2^3(\Omega)}, \tag{11}$$

where $h_{j,\max} = \max_{i_j}\left\{h_j^{i_j}\right\}$, $j = 1, 2$.

Here, first of all, we derive analogous estimates for the case $u \in W_2^2(\Omega)$. We decompose the term $\varphi_{1,\bar{x}_1}$ in the following manner:

$$\varphi_{1,\bar{x}_1} = \Big[(S_2^{(2)} u - u)_{\bar{x}_1}\Big] - \left[\left(\frac{h_2^2}{6} u_{\bar{x}_2}\right)_{\hat{x}_2\bar{x}_1}\right] = \eta_1 - \eta_2.$$

The value of η_1 at the mesh point $(x_1,x_2)\in\hat{\omega}_1$ can be represented in the form

$$\eta_1(x_1,x_2)=\frac{1}{h_1\hbar_2}\int\limits_{x_2-h_2}^{x_2+h_{2+}}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2}^{\xi_2}K_2(\xi_2)\frac{\partial^2u(\xi_1,\rho_2)}{\partial x_1\partial x_2}d\rho_2d\xi_1d\xi_2.$$

Herefrom, applying the Cauchy–Schwartz inequality we obtain

$$|\eta_1(x_1,x_2)|\leqslant C\left(\frac{\hbar_2}{h_1}\right)^{1/2}\left\|\frac{\partial^2u}{\partial x_1\partial x_2}\right\|_{L_2(e_1)},\quad e_1=(x_1-h_1,x_1)\times(x_2-h_2,x_2+h_{2+}).$$

Summation over the mesh $\hat{\omega}_1$ yields

$$\|\eta_1]|_1\leqslant Ch_{2,\max}\|u\|_{W_2^2(\Omega)}.$$

The value of η_2 at the mesh point $(x_1,x_2)\in\hat{\omega}_1$ can be represented in the form

$$\eta_2(x_1,x_2)=\frac{h_{2+}}{6h_1\hbar_2}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2}^{x_2+h_{2+}}\frac{\partial^2u(\xi_1,\xi_2)}{\partial x_1\partial x_2}d\xi_2d\xi_1-\frac{h_2}{6h_1\hbar_2}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2-h_2}^{x_2}\frac{\partial^2u(\xi_1,\xi_2)}{\partial x_1\partial x_2}d\xi_2d\xi_1\,.$$

Applying the Cauchy–Schwartz inequality and summing over the mesh $\hat{\omega}_1$ we obtain

$$\|\eta_2]|_1\leqslant Ch_{2,\max}\|u\|_{W_2^2(\Omega)}\,.$$

From previous estimates we have

$$\|\varphi_{1,\bar{x}_1}]|_1\leqslant Ch_{2,\max}\|u\|_{W_2^2(\Omega)}.\tag{12}$$

Analogous estimate holds true for the term $\varphi_{2,\bar{x}_2}$:

$$\|\varphi_{2,\bar{x}_2}]|_2\leqslant Ch_{1,\max}\|u\|_{W_2^2(\Omega)}.\tag{13}$$

The value of $\psi_{1,\hat{x}_2}$ at the mesh point $(x_1,x_2)\in\hat{\omega}_1$ can be represented in the form

$$\psi_{1,\hat{x}_2}(x_1,x_2)=\left(\frac{1}{6}-\theta\right)\left\{\frac{h_1}{h_{2+}\hbar_2}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2}^{x_2+h_{2+}}\frac{\partial^2u(\xi_1,\xi_2)}{\partial x_1\partial x_2}\,d\xi_2d\xi_1-\right.$$
$$\left.-\frac{h_1}{h_2\hbar_2}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2-h_2}^{x_2}\frac{\partial^2u(\xi_1,\xi_2)}{\partial x_1\partial x_2}\,d\xi_2d\xi_1\right\}.$$

Suppose that the mesh $\hat{\omega}$ is quasiuniform, i.e., there exists a positive constant c_1, such that

$$\max\{h_1^{i_1},\,h_2^{i_2}\}\leqslant c_1\min\{h_1^{i_1},\,h_2^{i_2}\}.\tag{14}$$

Then from the previous integral representation we obtain the estimate

$$\|\psi_{1,\hat{x}_2}]|_1\leqslant Cc_1h_{1,\max}\|u\|_{W_2^2(\Omega)}.\tag{15}$$

Analogous estimate holds true for the term $\psi_{2,\bar{x}_1}$:

$$\|\psi_{2,\hat{x}_1}]|_2 \leqslant Cc_1 h_{2,\max} \|u\|_{W_2^2(\Omega)} \,. \tag{16}$$

Further, we interpolate estimates (8) and (12). Let us define the operator A as follows:

$$A(u) = \varphi_{1,\bar{x}_1} \,.$$

The operator A is, obviously, linear. From (12) it follows that A is a bounded linear operator from $W_2^2(\Omega)$ to $L_2(\hat{\omega}_1)$, and

$$\|A\|_{W_2^2(\Omega)\mapsto L_2(\hat{\omega}_1)} \leqslant C h_{2,\max}. \tag{17}$$

From (8) it follows that A is a bounded linear operator from $W_2^3(\Omega)$ to $L_2(\hat{\omega}_1)$, and

$$\|A\|_{W_2^3(\Omega)\mapsto L_2(\hat{\omega}_1)} \leqslant C h_{2,\max}^2. \tag{18}$$

Applying the K-method of real interpolation [7], from (17) and (18) it follows that A is a bounded linear operator from $\big(W_2^3(\Omega), W_2^2(\Omega)\big)_{q,2} = W_2^{3-q}(\Omega)$ to $\big(L_2(\hat{\omega}_1), L_2(\hat{\omega}_1)\big)_{q,2} = L_2(\hat{\omega}_1)$, and

$$\|A\|_{W_2^{3-q}(\Omega)\mapsto L_2(\hat{\omega}_1)} \leqslant C h_{2,\max}^{2-q}, \quad 0 < q < 1. \tag{19}$$

In such a way we have

$$\|\varphi_{1,\bar{x}_1}]|_1 \leqslant C h_{2,\max}^{2-q} \|u\|_{W_2^{3-q}(\Omega)}. \tag{20}$$

Setting $3 - q = s$, we obtain the estimate

$$\|\varphi_{1,\bar{x}_1}]|_1 \leqslant C h_{2,\max}^{s-1} \|u\|_{W_2^s(\Omega)}, \quad 2 < s < 3. \tag{21}$$

By the same technique we obtain fractional-order estimates for other terms:

$$\|\varphi_{2,\bar{x}_2}]|_2 \leqslant C h_{1,\max}^{s-1} \|u\|_{W_2^s(\Omega)}, \qquad 2 < s < 3, \tag{22}$$

$$\|\psi_{1,\hat{x}_2}]|_1 \leqslant Cc_1 h_{1,\max}^{s-1} \|u\|_{W_2^s(\Omega)}, \quad 2 < s < 3, \tag{23}$$

$$\|\psi_{2,\hat{x}_1}]|_2 \leqslant Cc_1 h_{2,\max}^{s-1} \|u\|_{W_2^s(\Omega)}, \quad 2 < s < 3. \tag{24}$$

Finally, from (7), (8)–(16), and (21)–(24) we deduce the following theorem.

Theorem 1. *Let the mesh $\hat{\omega}$ satisfies the condition* (14). *Then for $\theta < 1/4$ the finite-difference scheme* (4) *converges in the $W_2^1(\hat{\omega})$ norm and the following estimate holds true:*

$$\|u - v\|_{W_2^1(\hat{\omega})} \leqslant C(h_{1,\max}^{s-1} + h_{2,\max}^{s-1}) \|u\|_{W_2^s(\Omega)}, \quad 2 \leqslant s \leqslant 3. \tag{25}$$

The estimate (25) is consistent with the smoothness of the solution. Here $C = C(c_1) = C_0 c_1$, where C_0 is an absolute constant. For $\theta = 1/6$ the terms $\psi_{j,\hat{x}_{3-j}}$ vanish and result holds on arbitrary nonuniform mesh, without condition (14). Note that in this case the finite difference scheme (4) reduces to the scheme proposed in [8].

Remark 1. In such a way, the convergence rate estimate obtained in [4] is extended to the case of solutions from Sobolev–Slobodetskiĭ spaces. Analogous result can be obtained for $1 \leqslant s \leqslant 2$ and, also, in other discrete norms (e. g., L_2-norm).

4. Monotonous scheme

We assume that $h_{j+} > h_j$, $j = 1, 2$, in the mesh $\hat{\bar{\omega}}$. We approximate the problem (3) with the following finite-difference scheme [4]:

$$-\tilde{\Delta}_h^\theta v = S_2^{(1)} S_2^{(2)} f \quad \text{in} \quad \hat{\omega}, \qquad v = 0 \quad \text{on} \quad \hat{\gamma}, \tag{26}$$

where

$$\tilde{\Delta}_h^\theta v = v_{\bar{x}_1\hat{x}_1} + v_{\bar{x}_2\hat{x}_2} + 2\theta\big[(h_{1+} - h_1)v_{x_1\bar{x}_2\hat{x}_2} + (h_{2+} - h_2)v_{x_2\bar{x}_1\hat{x}_1}\big].$$

The error $z = u - v$ satisfies the conditions

$$-\tilde{\Delta}_h^\theta z = \varphi_{1,\bar{x}_1\hat{x}_1} + \varphi_{2,\bar{x}_2\hat{x}_2} + \psi_{1,\hat{x}_1\hat{x}_2} + \psi_{2,\hat{x}_1\hat{x}_2} + \chi_{1,\bar{x}_1\hat{x}_1} + \chi_{2,\bar{x}_2\hat{x}_2} \quad \text{in} \quad \hat{\omega}, \qquad z = 0 \quad \text{on} \quad \hat{\gamma},$$

where

$$\chi_1 = \theta h_2^2 u_{\bar{x}_2\hat{x}_2}, \qquad \chi_2 = \theta h_1^2 u_{\bar{x}_1\hat{x}_1},$$

and the terms φ_j and ψ_j have the same form as in the previous case.

Now we investigate the stability of the scheme (26) in the discrete W_2^1 norm. Applying Lemma 1 and Cauchy–Schwartz inequality with $\varepsilon > 0$, we have

$$\begin{aligned}(-\tilde{\Delta}_h^\theta z, z)_* &= \|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2 + 2\theta\left\{((h_{2+} - h_2)z_{x_2\bar{x}_1}, z_{\bar{x}_1}]_1 + ((h_{1+} - h_1)z_{x_1\bar{x}_2}, z_{\bar{x}_2}]_2\right\} \\ &\geqslant \|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2 - |\theta|\varepsilon\left(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\right) \\ &\quad - \frac{|\theta|}{\varepsilon}\left\{\|(h_{2+} - h_2)z_{x_2\bar{x}_1}]|_1^2 + \|(h_{1+} - h_1)z_{x_1\bar{x}_2}]|_2^2\right\}\end{aligned}$$

Further, from the previous inequality and Lemma 6 follows

$$(-\tilde{\Delta}_h^\theta z, z)_* \geqslant \left(1 - |\theta|\varepsilon - \frac{4|\theta|}{\varepsilon}\right)\left(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\right).$$

The term $1 - |\theta|\varepsilon - 4|\theta|/\varepsilon$, for $\varepsilon > 0$, have the maximum $1 - 4|\theta|$ for $\varepsilon = 2$. Setting $\varepsilon = 2$, we obtain

$$(-\tilde{\Delta}_h^\theta z, z)_* \geqslant (1 - 4|\theta|)\|z\|_*^2.$$

Further,

$$\begin{aligned}&(\varphi_{1,\bar{x}_1\hat{x}_1} + \varphi_{2,\bar{x}_2\hat{x}_2} + \psi_{1,\hat{x}_1\hat{x}_2} + \psi_{2,\hat{x}_1\hat{x}_2} + \chi_{1,\bar{x}_1\hat{x}_1} + \chi_{2,\bar{x}_2\hat{x}_2}, z)_* \\ &\quad = -(\varphi_{1,\bar{x}_1}, z_{\bar{x}_1}]_1 - (\varphi_{2,\bar{x}_2}, z_{\bar{x}_2}]_2 - (\psi_{1,\hat{x}_2}, z_{\bar{x}_1}]_1 - (\psi_{2,\hat{x}_1}, z_{\bar{x}_2}]_2 - (\chi_{1,\bar{x}_1}, z_{\bar{x}_1}]_1 - (\chi_{2,\bar{x}_2}, z_{\bar{x}_2}]_2 \\ &\quad \leqslant \sqrt{3}\big(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2 + \|\chi_{1,\bar{x}_1}]|_1^2 + \|\chi_{2,\bar{x}_2}]|_2^2\big)^{1/2}\big(\|z_{\bar{x}_1}]|_1^2 + \|z_{\bar{x}_2}]|_2^2\big)^{1/2}.\end{aligned}$$

From the previous inequalities, for $|\theta| < 1/4$ we have

$$|z|_{W_2^1(\hat{\omega})} \leqslant \frac{\sqrt{3}}{1 - 4|\theta|}\left(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2 + \|\chi_{1,\bar{x}_1}]|_1^2 + \|\chi_{2,\bar{x}_2}]|_2^2\right)^{1/2}.$$

Applying Lemma 4 we obtain the estimate

$$\|z\|_{W_2^1(\hat{\omega})} \leqslant C_{1,\theta}\big(\|\varphi_{1,\bar{x}_1}]|_1^2 + \|\varphi_{2,\bar{x}_2}]|_2^2 + \|\psi_{1,\hat{x}_2}]|_1^2 + \|\psi_{2,\hat{x}_1}]|_2^2\|\chi_{1,\bar{x}_1}]|_1^2 + \|\chi_{2,\bar{x}_2}]|_2^2\big)^{1/2}, \tag{27}$$

where $C_{1,\theta} = 3\sqrt{3/2}/(2(1 - 4|\theta|))$.

In the paper [4] next estimate of the term $\chi_{j,\bar{x}_j}$ is derived:

$$\|\chi_{j,\bar{x}_j}]|_j \leqslant Ch^2_{3-j,max}\|u\|_{W_2^3(\Omega)}, \quad j=1,2. \tag{28}$$

The value of $\chi_{1,\bar{x}_1}$ at the mesh point $(x_1,x_2)\in\hat{\omega}_1$ can be represented in the form

$$\chi_{1,\bar{x}_1} = \frac{\theta h_2^2}{\hbar_2}\left\{\frac{1}{h_1h_{2+}}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2}^{x_2+h_{2+}}\frac{\partial^2 u(\xi_1,\xi_2)}{\partial x_1\partial x_2}d\xi_2 d\xi_1 - \frac{1}{h_1h_2}\int\limits_{x_1-h_1}^{x_1}\int\limits_{x_2-h_2}^{x_2}\frac{\partial^2 u(\xi_1,\xi_2)}{\partial x_1\partial x_2}d\xi_2 d\xi_1\right\}$$

Analogous representation holds true for the term $\chi_{2,\bar{x}_2}$. Applying Cauchy–Schwartz inequality and summing over the mesh $\hat{\omega}_j$ we obtain

$$\|\chi_{j,\bar{x}_j}]|_j \leqslant Ch_{3-j,max}\|u\|_{W_2^2(\Omega)}, \quad j=1,2. \tag{29}$$

By the same technique as before we can interpolate estimates (28) and (29), and obtain

$$\|\chi_{j,\bar{x}_j}]|_j \leqslant Ch^{s-1}_{3-j,max}\|u\|_{W_2^s(\Omega)}, \quad 2<s<3, \quad j=1,2. \tag{30}$$

Finally, from (27), (28)–(30), and (21)–(24) we deduce the following result.

Theorem 2. *Let the mesh $\hat{\omega}$ satisfies the condition* (14). *Then the finite-difference scheme* (26) *converges in the $W_2^1(\hat{\omega})$ norm and, under conditions $h_{j+}>h_j$, $j=1,2$, and $|\theta|<1/4$, the following estimate holds true:*

$$\|u-v\|_{W_2^1(\hat{\omega})} \leqslant C(h^{s-1}_{1,\max}+h^{s-1}_{2,\max})\|u\|_{W_2^s(\Omega)}, \quad 2\leqslant s\leqslant 3. \tag{31}$$

The estimate (31) is consistent with the smoothness of the solution. As in the previous case, $C=C(c_1)=C_0c_1$, where C_0 is an absolute constant. For $\theta=1/6$ the result holds true on arbitrary nonuniform mesh. In this case the finite difference scheme (26) reduces to the monotonous scheme proposed in [8]. Analogous result can be obtained for $1\leqslant s\leqslant 2$ and, also, in other discrete norms (e. g., L_2-norm).

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