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SOLUTION OF A FINITE-DIMENSIONAL PROBLEM WITH M-MAPPINGS AND DIAGONAL MULTIVALUED OPERATORS [1]

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Abstract — The finite-dimensional problem: find a triple $(u, \gamma, \delta) \in (R^N)^3$ such that

$$Au + B\gamma + \delta = f, \;\; \gamma \in Cu, \;\; \delta \in Du \tag{0.1}$$

is studied. Here $A, B : R^N \to R^N$ are the continuous M-mappings and $C, D : R^N \to 2^{R^N}$ are the multivalued diagonal maximal monotone operators. The existence of a solution on an ordered interval, which is formed by the so-called subsolution and supersolution for problem (0.1) is proved. Under several additional assumptions on the operators A, B, C and D the monotone dependence of a solution upon the right-hand side is investigated. This result implies, in particular, the uniqueness of a solution and serves as a basis for the analysis of the convergence for a multisplitting iterative method. As an illustrative example, the finite difference scheme approximating a model variational inequality is studied by using the general results.

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1. Introduction

Mesh approximations of free and moving boundary problems with several unknown boundaries lead to (0.1), where the continuous operators A, B correspond to the mesh approximation (finite difference or finite element) of the nonlinear partial differential operators while multivalued operators $C = \operatorname{diag}(c_1, \ldots, c_N)$ and $D = \operatorname{diag}(d_1, \ldots, d_N)$ can be responsible for the constraints and nonlinear relations between the components of the solution (u, γ, δ) of the problem.

We can cite the following partial cases of problem (0.1):

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(i) $B = 0$ and δ is the subdifferential of the indicator function (or, normal cone) of a closed convex subset $K \subset R^N$. This is well-known finite-dimensional variational inequality

$$u \in K : \quad (Au, v - u) \geqslant (f, v - u) \quad \forall v \in K, \quad (u, v) = \sum_{i=1}^{N} u_i v_i.$$

The mesh schemes for the obstacle and two-side obstacle problems, dam problem after Baiocchi transformation, implicit mesh schemes at the fixed time level for one-phase Stefan problem are the examples of this variational inequality. Schwarz alternating methods for the variational inequalities have been studied in [5, 8–10, 13, 14, 16, 17, 25–27] for linear A and in [2, 13] for nonlinear M-mapping A.

(ii) c_i and d_i are the continuous functions for all $i \in R^N$. Now, problem (0.1) includes, for example, a mesh approximation of the classical two-phase Stefan problem at a fixed time level, when A is the identity matrix and B corresponds to a mesh approximation of Laplace operator, u stands for the mesh enthalpy function, while γ for the temperature. Various parallel iterative methods based on the multisplittings of the matrix pair (A, B) have been considered in [1, 3, 22–24] for this kind of problem (0.1).

(iii) A and B are M-matrices and c_i and d_i are the continuous functions. Problem (0.1) can correspond to an implicit in time mesh scheme for the steel continuous casting problem with nonlinear boundary flux, which is a particular case of the Stefan problem with prescribed convection. The additive and multiplicative Schwarz iterative algorithms for this problem have been investigated in [11, 12]. A mesh scheme for the general Stefan problem with prescribed convection and linear boundary conditions has the form (0.1) with multivalued maximal monotone graph γ_i and $D = 0$. The article [14] deals with the algebraic problem, including such kind of the mesh scheme.

In [15] the existence of a unique solution and the convergence of a multisplitting method has been investigated for the problem with several M-matrices and diagonal multivalued operators.

The iterative solutions of the systems of nonlinear equations with M-mappings and their applications have been thoroughly studied in many articles (cf., e.g., [7, 18, 20] and the bibliographies therein).

This article generalizes the results of [15] to problem (0.1) with nonlinear M-mappings A and B.

Because both operators C and D are, generally, multivalued, they can have the mutual points of multivalence, i.e., the points $u \in R^N$ such, that at least for one i both one-dimensional operators c_i and d_i have just the sets of values $c_i(u_i)$ and $d_i(u_i)$. In this case it is necessary to define correctly the unique sections of these sets when proving the uniqueness of a solution, as well as for studying the convergence of an iterative method. Moreover, if at least one of the sets $c_i(u_i)$ or $d_i(u_i)$ is unbounded at the mutual point u_i of their multivalence, then we construct the appropriate modification of the corresponding operator to receive the aforementioned results.

Dealing with the nonlinear M-mappings we also need to generalize the notions of weak and strict diagonal dominance, which we essentially use in the proof of the comparison theorem (Theorem 3.1).

The different notions of a diagonal dominance in rows for a nonlinear mapping A in finite-dimensional space have been defined in [18] and [7] by imposing some assumptions on the components a_i of the operator A for every i. Our notion is essentially different from those in [18] and [7], because we generalize the property of diagonal dominance in columns for matrices, thus, it concerns the properties of the components a_i of the operator A in their combination.

The comparison theorem (the monotone dependence of the solution on the right-hand side) is proved and used for proving the uniqueness of a solution as well as the convergence of a class of the iterative methods for problem (0.1).

The theoretical results are illustrated by applying to one model problem, namely, to the finite difference scheme for a variational inequality with nonlinear second order differential operator, nonlinear convective term and one-side constraint on the boundary of the domain.

2. Existence of a solution

Further we suppose the following basic assumptions being fulfilled:

$$A, B \text{ are continuous } M\text{-mappings,} \tag{2.1}$$

$$C, D \text{ are diagonal maximal monotone operators,} \tag{2.2}$$

there exist a subsolution $(\underline{u}, \underline{\gamma}, \underline{\delta})$ and a supersolution $(\bar{u}, \bar{\gamma}, \bar{\delta})$ for problem (0.1):

$$(\underline{u}, \underline{\gamma}, \underline{\delta}) \in (R^N)^3: \ \ A\underline{u} + B\underline{\gamma} + \underline{\delta} \ll f, \ \ \underline{\gamma} \in C\underline{u}, \ \ \underline{\delta} \in D\underline{u}, \tag{2.3}$$

$$(\bar{u}, \bar{\gamma}, \bar{\delta}) \in (R^N)^3: \ \ A\bar{u} + B\bar{\gamma} + \bar{\delta} \gg f, \ \ \bar{\gamma} \in C\bar{u}, \ \ \bar{\delta} \in D\bar{u}. \tag{2.4}$$

Here we use the notations $\gg$ and $\ll$ for the componentwise ordering of the vectors from R^N, namely, $u \gg 0 \Leftrightarrow u_i \geqslant 0 \quad \forall i$.

Below we recall these notions.

Let $Au = (a_1(u_1 \dots, u_N), \dots, a_N(u_1, \dots, u_N))$, $e^i = (0, \dots, \underbrace{1}_{i}, \dots, 0)$. Then

A is an M-mapping in R^N iff it is (cf. [20])

1) strictly diagonally isotone:

$$\forall i \text{ and } \forall u \in R^N \quad a_i(u + te^i) \text{ is strictly increasing function of } t,$$

2) off-diagonal antitone:

$$\forall i \text{ and } \forall u \in R^N \quad a_i(u + te^j) \text{ is decreasing function of } t \text{ for } j \neq i,$$

3) and inverse isotone:

$$\text{if } Au \ll Av \text{ then } u \ll v.$$

Further, domain of a multivalued operator $C : R^N \to 2^{R^N}$ is the subset $\operatorname{dom} C \subset R^N$ of those u that $C(u) \neq \varnothing$ and it is maximal monotone (cf., e.g., [6, 21]) iff

1) it is monotone, i.e. for each pair $u_1, u_2 \in \operatorname{dom} C$ we have

$$(\gamma_1 - \gamma_2, u_1 - u_2) \geqslant 0 \quad \forall \gamma_1 \in C(u_1) \text{ and } \forall \gamma_2 \in C(u_2),$$

2) the condition

$$(\gamma_1 - \gamma_2, u_1 - u_2) \geqslant 0 \quad \forall u_1 \in \operatorname{dom} C \text{ and } \forall \gamma_1 \in C(u_1)$$

implies $u_2 \in \operatorname{dom} C$ and $\gamma_2 \in C(u_2)$.

The assumption that C is a diagonal maximal monotone operator, i.e. $Cu = (c_1(u_1), c_2(u_2), \dots, c_N(u_N))$ implies that it is the subdifferential of a convex separable function: $Cu = \partial\big(\sum_{i=1}^{N} \phi_i(u_i)\big) \equiv \sum_{i=1}^{N} \partial\phi_i(u_i)$ with $c_i(u_i) = \partial\phi_i(u_i)$.

The result of lemma 2.1 below follows from the Kolodner-Tartar results on the existence of a fixed point for the monotone mappings in a partially ordered space (see, e.g., [4; p. 223]).

Lemma 2.1. *Let $\langle a, b\rangle$ be an ordered interval in R^N and F be a monotone operator, mapping this interval into itself, i.e., $x \gg y \Longrightarrow Fx \gg Fy$ and $F(\langle a, b\rangle) \subseteq \langle a, b\rangle$. Then F has a fixed point in $\langle a, b\rangle$.*

For the fixed vectors w and η we define the diagonal operators $A^0(w)$, $B^0(\eta)$ by the equalities

$$A^0(w)u = (a_1(u_1, w_2, \dots, w_N), \dots, a_N(w_1, \dots, w_{N-1}, u_N))$$

and

$$B^0(\eta)\gamma = (b_1(\gamma_1, \eta_2, \dots, \eta_N), \dots, b_N(\eta_1, \dots, \eta_{N-1}, \gamma_N))$$

and consider the auxiliary problem: for fixed w, η from the ordered interval $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ find $(u, \gamma, \delta) \in (R^N)^3$ such that

$$A^0(w)u + B^0(\eta)\gamma + \delta = f; \;\; \gamma \in Cu, \;\; \delta \in Du, \tag{2.5}$$

Lemma 2.2. *If the assumptions* (2.1)–(2.4) *are fulfilled and the operators C and D are bounded with the domains $dom\, C = dom\, D = R^N$, then there exists a unique solution (u, γ, δ) of the problem (2.5).*

Proof. Due to (2.1) the diagonal operators $A^0(w)$ and $B^0(\eta)$ are continuous and strictly monotone for any $w, \eta \in R^N$, i.e., $(A^0(w)u_1 - A^0(w)u_2, u_1 - u_2) > 0$ for all $u_1 \neq u_2$ and similar for $B^0(\eta)$. As a consequence, the operator $P \equiv A^0(w) + B^0(\eta) \circ C + D$ is the diagonal and strictly maximal monotone. Moreover,

$$A^0(w)\underline{u} + B^0(\eta)\underline{\gamma} + \underline{\delta} \ll A\underline{u} + B\underline{\gamma} + \underline{\delta} \ll f \ll A\bar{u} + B\bar{\gamma} + \bar{\delta} \ll A^0(w)\bar{u} + B^0(\eta)\bar{\gamma} + \bar{\delta}$$

for any w, η from $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ because of the off-diagonal antitone properties of *M*-mappings A and B.

The latter inequalities mean that the ordered interval $\langle \underline{u}, \bar{u}\rangle$ belongs to the domain dom P while $f \in \langle A^0(w)\underline{u} + B^0(\eta)\underline{\gamma} + \underline{\delta}, A^0(w)\bar{u} + B^0(\eta)\bar{\gamma} + \bar{\delta}\rangle$ belongs to the range of P. Thus, there exists a solution u of the inclusion $Pu \ni f$. Its uniqueness follows from the strict monotonicity of P. So, the existence of a vector (u, γ, δ) with $\gamma \in Cu$, $\delta \in Du$, which satisfies (2.5), is proved. Let us now prove that γ and δ are also defined uniquely.

First, we consider the case, when for a fixed i the component u_i of the solution u is not a point of the mutual multivalence of c_i and d_i – either c_i or d_i is continuous at this point. Let for the definiteness d_i be continuous. Then $\delta_i = d_i(u_i)$ and γ_i is also defined uniquely from the scalar equation $b_i(\eta_1, \dots, \eta_{i-1}, \gamma_i, \eta_{i+1}, \dots \eta_N) = f_i - \delta_i - a_i(w_1, \dots, w_{i-1}, u_i, w_{i+1}, \dots, w_N)$, because the function on left-hand side is strictly monotone.

Now, let u_i be a point, where both operators c_i and d_i are multivalued. Then, as we deal with bounded operators C and D, for t from a neighborhood of this point we have

$c_i(t) = \tilde{c}_i(t) + l_c H(t-u_i)$, $d_i(t) = \tilde{d}_i(t) + l_d H(t-u_i)$, where $\tilde{c}_i, \tilde{d}_i$ are the continuous functions, l_c, l_d are positive constants, and $H(t)$ is the Heaviside graph:

$$H(t) = \{0 \text{ for } t < 0, [0,1] \text{ for } t = 0, 1 \text{ for } t > 0\}.$$

The solution ξ of the equation $b_i(\eta_1, \dots, \tilde{c}_i(u_i) + l_c\xi, \dots \eta_N) + l_d\xi = f_i - \tilde{d}_i(u_i) - a_i(w_1, \dots, u_i, \dots, w_N)$ is unique due to the strict monotonicity of the function on the left-hand side. Thus, $\gamma_i = \tilde{c}_i(u_i) + l_c\xi \in c_i(u_i)$ and $\delta_i = \tilde{d}_i(u_i) + l_d\xi \in d_i(u_i)$ are defined uniquely. □

Theorem 2.1. *Let the assumptions* (2.1)–(2.4) *be fulfilled and the operators C and D be bounded with the domains $dom\, C = dom\, D = R^N$. Then there exists a solution $(u, \gamma, \delta) \in \langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$ of problem* (0.1).

Proof. Let the operator G define the correspondence $(w, \eta) \to (u, \gamma)$, where (u, γ) are the two first components of the solution for problem (2.5). This operator maps ordered interval $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ into itself.

Let us prove that it is monotone: if $w^1 \gg w^2$, $\eta^1 \gg \eta^2$, then $u^1 \gg u^2$ and $\gamma^1 \gg \gamma^2$. We argue by contradiction. First, let us suppose that $I_- = \{i : u_i^1 < u_i^2\} \neq \varnothing$. Then $\gamma_i^1 \leqslant \gamma_i^2$, $\delta_i^1 \leqslant \delta_i^2$ for $i \in I_-$ and because of the properties of M-mappings we have:

$$\begin{aligned}(A^0(w^1)u^1)_i = f_i - (B^0(\eta^1)\gamma^1)_i - \delta_i^1 \geqslant f_i - (B^0(\eta^2)\gamma^2)_i - \delta_i^2 \\ = (A^0(w^2)u^2)_i \geqslant (A^0(w^1)u^2)_i \quad \forall i \in I_-.\end{aligned}$$

As a consequence, $u_i^1 \geqslant u_i^2$ $\forall i \in I_-$, that is, the contradiction to our assumption and $I_- = \varnothing$.

Now, let $J_- = \{i : \gamma_i^1 < \gamma_i^2\} \neq \varnothing$. Because of the inequality $u^1 \gg u^2$ we have $u_i^1 = u_i^2 \equiv u_i$ for $i \in J_-$, that is, u_i is a point of multivalence of c_i. If it is not a point of multivalence for d_i then $\delta_i^1 = d_i(u_i) = \delta_i^2$. If both operators c_i and d_i are multivalued in u_i, then $\gamma_i^1 = \tilde{c}_i(u_i) + l_c\xi^1 < \gamma_i^2 = \tilde{c}_i(u_i) + l_c\xi^2$ implies $\xi^1 < \xi^2$ for the sections ξ^1, ξ^2 of the Heaviside graph $H(t - u_i)$ and, as a consequence, $\delta_i^1 = \tilde{d}_i(u_i) + l_d\xi^1 < \delta_i^2 = \tilde{d}_i(u_i) + l_d\xi^2$. So, in both cases $\delta_i^1 \leqslant \delta_i^2$.

Now, proceeding as above, for $i \in J_-$ we have

$$\begin{aligned}(B^0(\eta^1)\gamma^1)_i = f_i - (A^0(w^1)u^1)_i - \delta_i^1 \geqslant f_i - (A^0(w^2)u^2)_i - \delta_i^2 \\ = (B^0(\eta^2)\gamma^2)_i \geqslant (B^0(\eta^1)\gamma^2)_i,\end{aligned}$$

which leads to the inequality $\gamma_i^1 \geqslant \gamma_i^2$ $\forall i \in J_-$, or, to the contradiction with the supposition $J_- = \varnothing$.

Thus, the operator G is monotone. The property that it maps $\langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ into itself can be proved similarly. Owing to Lemma 2.1 it has a fixed point. It is easy to check that this fixed point (u, γ) is a solution for the problem

$$Au + B\gamma + Du \ni f; \quad \gamma \in Cu$$

and it means that there exists a section $\delta \in Du$ such that (u, γ, δ) is a solution for problem (0.1). □

Now we consider problem (0.1) without the additional assumption of the boundedness of the operators C and D.

Theorem 2.2. *Let the assumptions* (2.1)–(2.4) *be fulfilled. Then there exists a solution* $(u, \gamma, \delta) \in \langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$ *of problem* (0.1).

Proof. Because of the assumptions (2.3), (2.4), the domains $\operatorname{dom} C$ and $\operatorname{dom} D$ contain the ordered interval $\langle \underline{u}, \bar{u}\rangle$. So, for any i the operators c_i and d_i are bounded at the inner points of $[\underline{u}_i, \bar{u}_i]$ (if $\underline{u}_i \neq \bar{u}_i$) and can be unbounded only at $\underline{u}_i$ and/or at $\bar{u}_i$. As we look for a solution of problem (0.1) from the bounded interval $\langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$, we can change the unbounded components of the operators C and D by the bounded ones. Namely, if for example, $c_i(\bar{u}_i) = [\theta_i, +\infty)$, then we set $c_i(\bar{u}_i) = [\theta_i, \bar{\gamma}_i]$ and $c_i(t) = \bar{\gamma}_i$ for $t > \bar{u}_i$. Similarly, if $c_i(\underline{u}_i) = (-\infty, \xi_i]$, then we set $c_i(\underline{u}_i) = [\underline{\gamma}_i, \xi_i]$ and $c_i(t) = \underline{\gamma}_i$ for $t < \underline{u}_i$.

After these transformations we get the modified problem with the bounded maximal monotone operators C and D whose domains are R^N. Due to Theorem 2.1 there exists a solution (u, γ, δ) of this modified problem. As its first component u belongs to $\langle \underline{u}, \bar{u}\rangle$ and for this u the sets of values of nonmodified C, D contain the sets of values for modified, bounded, C, D, then a solution (u, γ, δ) of the modified problem is at the same time a solution for problem (0.1). □

3. Comparison theorem. Uniqueness of the solution

Definition 3.1. *Let u^1, u^2 be two vectors from R^N, the set $I = \{1, 2, \dots, N\}$ be divided into three nonintersecting subsets: $I_- = \{i \in I : u_i^1 < u_i^2\}$, $I_0 = \{i \in I : u_i^1 = u_i^2\}$, $I_+ = \{i \in I : u_i^1 > u_i^2\}$ and $L \subseteq I_0$ be any subset.*

We say that an M-mapping A has **weak diagonal dominance** *if*

1) for any pair u^1, u^2 and any vector η defined by

$$\eta_i = \{1 \textit{ for } i \in I_- \cup L,\ 0 \textit{ for } i \in I \setminus (I_- \cup L)\} \tag{3.1}$$

the following inequality holds:

$$(Au^1 - Au^2, \eta) \leqslant 0, \tag{3.2}$$

2) for any pair u^1, u^2 such that $I_- \neq \varnothing$ and

$$\eta_i = \{1 \textit{ for } i \in I_-,\ 0 \textit{ for } i \in I \setminus I_-\}$$

the strong inequality

$$(Au^1 - Au^2, \eta) < 0 \tag{3.3}$$

takes place.

M-mapping A has **strict diagonal dominance** *if for nonempty I_- and for any vector η defined by* (3.1) *the strong inequality* (3.3) *is valid.*

Remark 3.1. a) In the case of linear mapping, i.e., if A is M-matrix, the preceeding definitions mean that A is weakly (correspondingly, strictly) diagonal dominant in columns matrix.

In fact, let A be a weakly diagonal dominant in columns M-matrix with conjugate A^t. For η from (3.1) we have $(A^t\eta)_i \geqslant 0$ if $i \in I_- \cup L$ and $(A^t\eta)_i \leqslant 0$ if $i \in I_+$, so

$$(Au^1 - Au^2, \eta) = (u^1 - u^2, A^t\eta) \leqslant 0.$$

Let now $I_- \neq \varnothing$ and $\eta_i = \{1 \text{ for } i \in I_-,\ 0 \text{ for } i \in I \setminus I_-\}$. Then

$$(u^1 - u^2, A^t\eta) \leqslant ((u^1 - u^2)_{I_-}, (A^t)_{I_-I_-}\eta_{I_-}), \tag{3.4}$$

where the submatrix $(A^t)_{I_-I_-}$ of M-matrix A is also M-matrix with weak diagonal dominance, so, it is regular. It means that all coordinates of the vector $(A^t)_{I_-I_-}\eta_{I_-}$ are non-negative and at least one of them is strictly positive. Due to this the right-hand side of (3.4) is negative.

To prove the inverse statement, i.e., that an M-matrix A satisfying assumptions (3.2) and (3.3) has weak diagonal dominance in columns, we take for a fixed i the vectors $u_1 = e^i$, $u_2 = 0$ and η with the coordinates $(1, \dots, 1)$. Then from (3.2) we derive

$$(Au^1 - Au^2, \eta) = (u^1 - u^2, A^t\eta) = -\sum_{j=1}^{N} a_{ji} \leqslant 0.$$

The proof of (3.3) for a strictly diagonally dominant in columns M-matrix A and the inverse statement are straightforward.

b) If an M-mapping A is Frechet-differentiable, then using the formula

$$\begin{aligned}(Au^1 - Au^2, \eta) &= \int_0^1 (A'(tu^1 + (1-t)u^2)(u^1 - u^2), \eta)dt \\ &= \int_0^1 (u^1 - u^2, [A'(tu^1 + (1-t)u^2)]^t\eta)dt\end{aligned}$$

we find immediately that A is (strictly) diagonally dominant if its derivative $A'(u)$ is (strictly) diagonally dominant in columns for every u.

As in the previous section we first study problem (0.1) with the additional assumption on the boundedness of the operators C and D.

Let $u^* \in R$ be a mutual point of multivalence for the operators c_i and d_i for some non-empty set of indices $I(u^*) \subseteq I$. Then for $i \in I(u^*)$ we use the boundedness of C and D and set, as in the proof of Lemma 2.2, that

$$c_i(u_i) = \tilde{c}_i(u_i) + \alpha_i\xi_i, \ \ d_i(u_i) = \tilde{d}_i(u_i) + \beta_i\xi_i, \ \ \xi_i \in H(u_i - u^*).$$

Now, let $\Phi = \operatorname{diag}(\phi_1, \phi_2, \dots, \phi_n)$ be a diagonal maximal monotone operator with

$$\phi_i(u_i) = \{H(u_i - u^*) \ \text{ for } \ i \in I(u^*); \ \ 0 \ \text{ for } \ i \in I \setminus I(u^*)\}$$

and $P = \operatorname{diag}(p_{11}, \dots, p_{NN})$ and $G = \operatorname{diag}(g_{11}, \dots, g_{NN})$ be the diagonal positive definite matrices with entries

$$p_{ii} = \{\alpha_i \ \text{ for } \ i \in I(u^*); \ 1 \ \text{ for } \ i \in I \setminus I(u^*)\}$$

and

$$g_{ii} = \{\beta_i \ \text{ for } \ i \in I(u^*); \ 1 \ \text{ for } \ i \in I \setminus I(u^*)\}.$$

We suppose that $\{u_1^*, u_2^*, \ldots, u_m^*\}$ are all mutual points of multivalence for the operators C and D. Then, proceeding as before for every point $u_k^*,\ k = 1, 2, \ldots, m$, we get

$$C = \tilde{C} + \sum_{k=1}^{m} P_k \Phi_k; \quad D = \tilde{D} + \sum_{k=1}^{m} G_k \Phi_k,$$

where the operators $\tilde{C},\ \tilde{D},\ \Phi_k, k = 1, 2, \ldots, m$ have no mutual points of multivalence and the diagonal matrices $P_k,\ G_k$ are positive definite.

Theorem 3.1. *Let the assumptions* (2.1), (2.2) *be fulfilled, the operators C and D be bounded with the domains $dom\, C = dom\, D = R^N$, and A and B be the weakly diagonally dominant mappings. Let also one of the following assumptions hold:*

(a) either A or B is strictly diagonally dominant mapping

or

(b) C is either continuous monotone or strictly maximal monotone operator.

If $(u^1, \gamma^1, \delta^1)$ and $(u^2, \gamma^2, \delta^2)$ are the solutions of (0.1) *with the right-hand sides f^1 and f^2, respectively, then the inequality $f^1 \gg f^2$ implies the inequalities $u^1 \gg u^2,\ \gamma^1 \gg \gamma^2, \delta^1 \gg \delta^2$.*

Proof. We rewrite problem (0.1) in the form

$$\begin{cases} Au + B(\tilde{\gamma} + \sum\limits_{k=1}^{m} P_k \theta^k) + \tilde{\delta} + \sum\limits_{k=1}^{m} G_k \theta^k = f, \\ \tilde{\gamma} \in \tilde{C}u, \quad \tilde{\delta} \in \tilde{D}u, \quad \theta^k \in \Phi_k u \quad \forall k \end{cases} \tag{3.5}$$

and denote by $\theta^{k,1}$ and $\theta^{k,2}$ the corresponding components of the solutions for this problem with the right-hand sides f^1 and f^2.

Let us define the following subsets of I: $U_- = \{i \in I : u_i^1 < u_i^2\}$; $\Gamma_- = \{i \in I : \tilde{\gamma}_i^1 < \tilde{\gamma}_i^2\}$; $\Delta_- = \{i \in I : \tilde{\delta}_i^1 < \tilde{\delta}_i^2\}$; $\Theta_-^k = \{i \in I : \theta_i^{k,1} < \theta_i^{k,2}\}$ and

$$M = \cup_{k=1}^m \Theta_-^k \cup U_- \cup \Gamma_- \cup \Delta_-.$$

Note that

$$u_i^1 \leqslant u_i^2, \quad \tilde{\gamma}_i^1 \leqslant \tilde{\gamma}_i^2, \quad \tilde{\delta}_i^1 \leqslant \tilde{\delta}_i^2, \quad \theta_i^{k,1} \leqslant \theta_i^{k,2} \quad \forall\ k \ \text{ for all } i \in M, \tag{3.6}$$

because the operators $\tilde{C}, \tilde{D}, \Phi_k$ have no mutual points of multivalence, and

$$u_i^1 \geqslant u_i^2,\ \tilde{\gamma}_i^1 \geqslant \tilde{\gamma}_i^2, \quad \tilde{\delta}_i^1 \geqslant \tilde{\delta}_i^2,\ \theta_i^{k,1} \geqslant \theta_i^{k,2} \quad \forall\ k \ \text{ for } i \in I \setminus M. \tag{3.7}$$

We first prove that $\Delta_- = \varnothing, \Theta_-^k = \varnothing\ \forall\ k$ with any of the assumptions (a) or (b) of the theorem. We argue by contradiction.

Let the vector η be defined by

$$\eta_i = \{1 \ \text{ for } i \in M;\ 0 \ \text{ for } i \in I \setminus M\}. \tag{3.8}$$

From equation (3.5) with $f = f^1$ and $f = f^2$ we obtain

$$\begin{aligned} (Au^1 - Au^2, \eta) &+ \left(B\left(\tilde{\gamma}^1 + \sum_{k=1}^{m} P_k \theta^{k,1}\right) - B\left(\tilde{\gamma}^2 + \sum_{k=1}^{m} P_k \theta^{k,2}\right), \eta \right) \\ &+ (\tilde{\delta}^1 - \tilde{\delta}^2, \eta) + \left(\sum_{k=1}^{m} G_k \left(\theta^{k,1} - \theta^{k,2}\right), \eta \right) = (f^1 - f^2, \eta) \geqslant 0. \end{aligned} \tag{3.9}$$

We observe that all the terms on left-hand side of (3.9) are nonpositive for the chosen vector η because of the inequalities (3.6), (3.7) and of the weak diagonal dominance for A and B.

If now we suppose that $\Delta_- \neq \varnothing$, then

$$(\tilde{\delta}^1 - \tilde{\delta}^2, \eta) \leqslant \sum_{i \in \Delta_-} (\tilde{\delta}^1 - \tilde{\delta}^2)_i < 0,$$

which contradicts the nonnegativeness of the right-hand side in (3.9).

Similarly, if $\Theta^k_- \neq \varnothing$ for some k, then

$$(G_k(\theta^{k,1} - \theta^{k,2}), \eta) < 0$$

and once again we get the contradiction.

So, $\delta^1 \gg \delta^2$, $M = U_- \cup \Gamma_-$ and the inequality (3.9) can be rewritten as follows:

$$(Au^1 - Au^2, \eta) + \left(B\left(\tilde{\gamma}^1 + \sum_{k=1}^m P_k \theta^{k,1}\right) - B\left(\tilde{\gamma}^2 + \sum_{k=1}^m P_k \theta^{k,2}\right), \eta \right) \geqslant 0. \tag{3.10}$$

(a)

Let for the definiteness A be a strictly diagonally dominant mapping. If we suppose that $U_- \neq \varnothing$, then because of (3.3) the strict inequality $(Au^1 - Au^2, \eta) < 0$ holds for η defined by (3.8). As all the other terms on left-hand side of the inequality (3.10) are non-positive it leads to the contradiction. So, $U_- = \varnothing$ and $u^1 \gg u^2$.

Let us suppose now that $M = \Gamma_- \neq \varnothing$. As it is proved that $u^1 \gg u^2$, then, obviously, $u^1_i = u^2_i$ for $i \in \Gamma_-$. Further, the operators $\tilde{C}$, $\tilde{D}$, $\Phi_k, k = 1, 2, \ldots, m$ have no mutual points of multivalence and $\tilde{\delta}^1 \gg \tilde{\delta}^2$, $\theta^{k,1} \gg \theta^{k,2}$ $\forall k$. Owing to this $\tilde{\delta}^1_i = \tilde{\delta}^2_i$, $\theta^{k,1}_i = \theta^{k,2}_i$ $\forall k$ for $i \in \Gamma_-$. This implies: $\gamma^1_i < \gamma^2_i$ and $\delta^1_i = \delta^2_i$ for $i \in \Gamma_-$. The inequality (3.10) becomes

$$(B\gamma^1 - B\gamma^2, \eta) \geqslant (f^1 - f^2, \eta) \geqslant 0$$

for $\eta_i = \{1 \text{ for } i \in \Gamma_-;\ 0 \text{ for } i \in I \setminus \Gamma_-\}$. But it contradicts the property of weak diagonal dominance of M-mapping B, namely, the inequality $(B\gamma^1 - B\gamma^2, \eta) < 0$, if $\Gamma_- \neq \varnothing$ (cf. with (3.3)).

The case when B is strictly diagonally dominant mapping is studied similarly.

(b)

Let C be a strictly monotone operator. In this case $U_- \subset \Gamma_-$ and $M = \Gamma_-$. Proceeding as in the case of the assumption (a), we get $\gamma^1_i < \gamma^2_i$, $u^1_i \leqslant u^2_i$ while $\delta^1_i = \delta^2_i$ for $i \in \Gamma_-$. Inequality (3.10) is transformed to

$$(B\gamma^1 - B\gamma^2, \eta) \geqslant (f^1 - f^2, \eta) \geqslant 0$$

for $\eta_i = \{1 \text{ for } i \in \Gamma_-;\ 0 \text{ for } i \in I \setminus \Gamma_-\}$, which contradicts the property of weak diagonal dominance of M-mapping B.

Let now C be a continuous monotone operator. Then $\Gamma_- \subset U_-$, $M = U_-$ and the inequality (3.10) takes the form

$$(Au^1 - Au^2, \eta) \geqslant (f^1 - f^2, \eta) \geqslant 0$$

for $\eta_i = \{1 \text{ for } i \in U_-;\ 0 \text{ for } i \in I \setminus U_-\}$ and the same arguments as in previous case lead to contradiction. □

Remark 3.2. The result of Theorem 3.1 is still valid in the case of the unbounded operators C and D, if they are bounded at all the mutual points of their multivalence, because also in this case we can rewrite problem (0.1) in the form of (3.5).

Obviously, the comparison Theorem 3.1 implies the uniqueness of the solution $(u,\gamma,\delta) \in \langle(\underline{u},\underline{\gamma},\underline{\delta}),\ (\bar{u},\bar{\gamma},\bar{\delta})\rangle$ for problem (0.1).

We can expect the nonuniqueness of the solution for problem (0.1) in the case when at least one of the operators C or D is unbounded at some mutual points of multivalence of these operators, because, for example, we can construct different modifications of the problem (0.1) (one modification was constructed in the proof of Theorem 2.2). In the following theorem we prove that also in this case the solution of problem (0.1) is unique up to the possible different sections of C and D only at the mutual points of their multivalence.

Theorem 3.2. *Let all the assumptions of Theorem 3.1 be fulfilled except the assumption of the boundedness of the operators C and D. Let*

$$\begin{cases} Au + B(\tilde{\gamma} + \sum_{k=1}^{m} P_k^1 \theta^k) + \tilde{\delta} + \sum_{k=1}^{m} G_k^1 \theta^k = f \\ \tilde{\gamma} \in \tilde{C}u, \quad \tilde{\delta} \in \tilde{D}u, \quad \theta^k \in \Phi_k u \quad \forall k, \end{cases} \tag{3.11}$$

$$\begin{cases} Au + B(\tilde{\gamma} + \sum_{k=1}^{m} P_k^2 \theta^k) + \tilde{\delta} + \sum_{k=1}^{m} G_k^2 \theta^k = f \\ \tilde{\gamma} \in \tilde{C}u, \quad \tilde{\delta} \in \tilde{D}u, \quad \theta^k \in \Phi_k u \quad \forall k \end{cases} \tag{3.12}$$

be two different modifications of (0.1) *to the problems with bounded operators C and D.*

If $(u^1, \tilde{\gamma}^1, \tilde{\delta}^1, \theta^{1,1}, \dots, \theta^{m,1})$ and $(u^2, \tilde{\gamma}^2, \tilde{\delta}^2, \theta^{1,2}, \dots, \theta^{m,2})$ are the unique solutions of (3.11) *and* (3.12)*, respectively, then*

$$(u^1, \tilde{\gamma}^1, \tilde{\delta}^1) = (u^2, \tilde{\gamma}^2, \tilde{\delta}^2).$$

Proof. We prove the formulated result only in the case when the operators C and D are unbounded from above at the mutual points of their multivalence, because all other cases can be studied similarly.

Let $u^* \in R$ be a mutual point of multivalence for the operators c_i and d_i for some nonempty set of indices $i \in I(u^*)$. Then for $i \in I(u^*)$ the corresponding components of the multivalued operators in the problems of (3.11) and (3.12) are defined by

$$c_i^1(u_i) = \tilde{c}_i(u_i) + \alpha_i^1 \xi_i^1, \quad d_i^1(u_i) = \tilde{d}_i(u_i) + \beta_i^1 \xi_i^1, \quad \xi_i^1 \in H(u_i - u^*)$$

and

$$c_i^2(u_i) = \tilde{c}_i(u_i) + \alpha_i^2 \xi_i^2, \quad d_i^2(u_i) = \tilde{d}_i(u_i) + \beta_i^2 \xi_i^2, \quad \xi_i^2 \in H(u_i - u^*).$$

Along with the problems (3.11) and (3.12) we consider one more modified problem, which in some sense imbeds both previous ones. To construct it, we set

$$\tilde{\alpha}_i = \max\{\alpha_i^1, \alpha_i^2\}, \quad \tilde{\beta}_i = \max\{\beta_i^1, \beta_i^2\}$$

and define the matrices $\tilde{P}$ and $\tilde{G}$ with the entries

$$\tilde{p}_{ii} = \{\tilde{\alpha}_i \ \text{ for } i \in I(u^*);\ 1 \ \text{ for } i \in I \setminus I(u^*)\},$$
$$\tilde{g}_{ii} = \{\tilde{\beta}_i \ \text{ for } i \in I(u^*);\ 1 \ \text{ for } i \in I \setminus I(u^*)\}.$$

Proceeding similarly for all mutual points $\{u_1^*, u_2^*, \dots, u_m^*\}$ of multivalence of the operators C and D and choosing the same operators Φ_k as in (3.11) and (3.12), we get the new modified problem:

$$\begin{cases} Au + B(\tilde{\gamma} + \sum_{k=1}^{m} \tilde{P}_k \theta^k) + \tilde{\delta} + \sum_{k=1}^{m} \tilde{G}_k \theta^k = f, \\ \tilde{\gamma} \in \tilde{C}u, \quad \tilde{\delta} \in \tilde{D}u, \quad \theta^k \in \Phi_k u \quad \forall k. \end{cases} \tag{3.13}$$

The vectors $(u^1, \tilde{\gamma}^1, \tilde{\delta}^1, \theta^{1,1}, \dots, \theta^{m,1})$ and $(u^2, \tilde{\gamma}^2, \tilde{\delta}^2, \theta^{1,2}, \dots, \theta^{m,2})$ are the solutions of (3.13) with different sections of the sets $\Phi_k(u_k^*)$ in the following sense. For the first solution $(u^1, \tilde{\gamma}^1, \tilde{\delta}^1, \theta^{1,1}, \dots, \theta^{m,1})$ we have

$$Au^1 + B(\tilde{\gamma}^1 + \sum_{k=1}^{m} \tilde{P}_k \chi^k) + \tilde{\delta}^1 + \sum_{k=1}^{m} \tilde{G}_k \zeta^k = f.$$

Here $\chi_i^k = (\alpha_i^1/\tilde{\alpha}_i)\, \theta_i^{k,1} \in H(u_i - u_i^*)$ because of the inequality $\alpha_i^1/\tilde{\alpha}_i \leqslant 1$ at the points of the modifications of the multivalued operators while $\chi_i^k = \theta_i^{k,1}$ at other points. It means that $\chi^k \in \Phi_k(u_k^*)$. Similarly, $\zeta_i^k = (\beta_i^1/\tilde{\beta}_i)\, \theta_i^{k,1} \in H(u_i - u_i^*)$ at the points of the modifications of the multivalued operators while $\zeta_i^k = \theta_i^{k,1}$ in other points, thus $\zeta^k \in \Phi_k(u_k^*)$.

But due to Theorem 3.1 the problem (3.13) has the unique solution $(u^*, \tilde{\gamma}^*, \tilde{\delta}^*, \theta^{1,*}, \dots, \theta^{m,*})$ with $\theta^{k,*} \in \Phi_k(u_k^*)$. As a consequence,

$$(u^1, \tilde{\gamma}^1, \tilde{\delta}^1) = (u^2, \tilde{\gamma}^2, \tilde{\delta}^2) = (u^*, \tilde{\gamma}^*, \tilde{\delta}^*).$$

□

4. Iterative methods

In this section we study problem (0.1) with the bounded operators C and D. On the basis of the comparison result of Theorem 3.1 we prove the convergence of the coordinate relaxation-type iterative methods for problem (0.1). We consider two variants of a multisplitting method (cf., e.g., [1, 3, 19] for the case of systems of linear and nonlinear algebraic equations).

Let I_l for $l = 1, 2, \dots, p \leqslant N$ be the subsets of $I = \{1, \dots, N\}$, $J_l = I \setminus I_l$ and $I = \bigcup_{l=1}^p I_l$. We denote by $N_{1,l} = \mathrm{card} I_l$, $N_{2,l} = \mathrm{card} J_l$, $N_{1,l} + N_{2,l} = N$. Let further the coordinates of the vectors from R^N are partitioned for every l in the corresponding way: $u = (u_{I_l}, u_{J_l})$, where, for example, u_{I_l} contains the coordinates with indices $i \in I_l$.

We define for all l and for fixed vectors $z_{J_l} \in R^{N_{2,l}}$ the mappings $A_{I_l}(., z_{J_l})$ from $R^{N_{1,l}}$ to $R^{N_{1,l}}$ by collecting the functions a_i with $i \in I_l$ from the definition of A: $A_{I_l}(u_{I_l}, z_{J_l}) = (a_i(u_{I_l}, z_{J_l}))_{i \in I_l}$. The mappings $B_{I_l}(., z_{J_l})$ are defined similarly. Let also $E_l \gg 0$ be the diagonal $N \times N$ matrices whose entries satisfy the property $e_{ii}^l = 1$ for $i \in I_l \setminus \bigcup_{j \neq l} I_j$ and $\sum_{l=1}^p E_l = Id$, where Id is the identity matrix. We consider the following iterative method for solving problem (0.1):

$$\begin{cases} A_{I_l}(v_{I_l}^{l,k+1}, u_{J_l}^k) + B_{I_l}(\eta_{I_l}^{l,k+1}, \gamma_{J_l}^k) + \delta_{I_l}^{l,k+1} = f_{I_l}, \\ \eta_{I_l}^{l,k+1} \in C_{I_l} v_{I_l}^{l,k+1}, \quad \delta_{I_l}^{l,k+1} \in D_{I_l} v_{I_l}^{l,k+1}, \\ v_{J_l}^{l,k+1} = u_{J_l}^k, \quad \eta_{J_l}^{l,k+1} = \gamma_{J_l}^k, \quad \delta_{J_l}^{l,k+1} = \delta_{J_l}^k, \qquad l = 1, 2, \dots, p, \\ u^{k+1} = \sum_{l=1}^p E_l v^{l,k+1}, \quad \gamma^{k+1} = \sum_{l=1}^p E_l \eta^{l,k+1}, \quad \delta^{k+1} = \sum_{l=1}^p E_l \delta^{l,k+1} \end{cases} \tag{4.1}$$

for $k = 0, 1, 2, \dots$ with initial guess $(u^0, \gamma^0, \delta^0)$.

Lemma 4.1. *Let the mapping A satisfy the assumptions* (2.1) *and have the property of weak (strict) diagonal dominance. Then all $A_{I_l}(., z_{J_l})$, $l = 1, \dots, p$ with the fixed z_{J_l} keep these properties.*

Proof. Obviously, the mappings $A_{I_l}(., z_{J_l})$ are continuous, diagonally isotone and off-diagonally antitone. Let us prove that they are inverse isotone. We fix an index l and consider the problem

$$A_{I_l}(u_{I_l}, z_{J_l}) = f_{I_l} \tag{4.2}$$

Along with problem (4.2) we consider an auxiliary equivalent problem by defining the affine subset $K = \{u \in R^N : u_{J_l} = z_{J_l}\}$ of R^N and denoting by N_K the subdifferential of the indicator function for K (the normal cone for K). Let also $f = (f_{I_l},\ f_{J_l})$ with any f_{J_l}. The auxiliary problem is

$$Az + \gamma = f, \quad \gamma \in N_K z. \tag{4.3}$$

From the definitions of (4.2) and (4.3) it is easy to deduce that if u_{I_l} is a solution of (4.2), then (u_{I_l}, z_{J_l}) is a solution of (4.3) and vice versa. On the other hand, (4.3) is a partial case of (0.1) with $D = 0$ and the identity matrix B. It means that all assumptions of Theorem 2, part (a), are satisfied for this problem. Therefore, if

$$A_{I_l}(u^1_{I_l}, z_{J_l}) \equiv f^1_{I_l} \geqslant A_{I_l}(u^2_{I_l}, z_{J_l}) \equiv f^2_{I_l},$$

then $f^1 = (f^1_{I_l},\ f_{J_l}) \geqslant f^2 = (f^2_{I_l},\ f_{J_l})$ (with any f_{J_l}) and $w^1 \geqslant w^2$ for the solutions of (4.3) with corresponding right-hand sides. The latter implies the inequality $u^1_{I_l} \geqslant u^2_{I_l}$, i.e., the property of $A_{I_l}(., z_{J_l})$ to be inverse isotone.

Let now suppose that A is weakly diagonally dominant mapping and prove the same property for $A_{I_l}(., z_{J_l})$. To this end we denote by $f^k_{I_l} \equiv A_{I_l}(u^k_{I_l}, z_{J_l})$, $f^k_{J_l} = A_{J_l}(u^k_{I_l}, z_{J_l})$, $k = 1, 2$ and set $I_- = \{i \in I_l : u^1_{I_l,i} < u^2_{I_l,i}\}$, $I_0 = \{i \in I_l : u^1_{I_l,i} = u^2_{I_l,i}\}$, $L \subset I_0$. Let also $\tilde{u}^k = (u^k_{I_l}, z_{J_l})$, $k = 1, 2$. Obviously, $\tilde{u}^1_i < \tilde{u}^2_i$ for $i \in I_-$, $\tilde{u}^1_i = \tilde{u}^2_i$ for $i \in I_0 \cup L$ and $\tilde{u}^1_i \geqslant \tilde{u}^2_i$ for other coordinates.

Let now the vector η_{I_l} have the coordinates $\eta_i = 1$ for $i \in I_- \cup L$ and 0 otherwise and the vector $\eta \in R^N$ is equal to $(\eta_{I_l}, 0)$. Due to the weak diagonal dominance of A we have

$$(A_{I_l}(u^1_{I_l}, z_{J_l}) - A_{I_l}(u^2_{I_l}, z_{J_l}), \eta_{I_l}) = (A\tilde{u}^1 - A\tilde{u}^2, \eta) \leqslant 0.$$

Further, if $I_- \neq \varnothing$ and η_{I_l} is such that $\eta_i = 1$ for $i \in I_-$ and 0 otherwise, then from (3.3) we obtain

$$(A_{I_l}(u^1_{I_l}, z_{J_l}) - A_{I_l}(u^2_{I_l}, z_{J_l}), \eta_{I_l}) = (A\tilde{u}^1 - A\tilde{u}^2, \eta) < 0.$$

Thus, $A_{I_l}(., z_{J_l})$ is a weakly diagonally dominant mapping.

In a similar way it is easy to prove that $A_{I_l}(., z_{J_l})$ inherits from A the property of strict diagonal dominance. □

Theorem 4.1. *Let the assumptions of Theorem 3.1 for problem* (0.1) *be fulfilled. Then:*

(i) iterative method (4.1) *is convergent for any initial guess* $(u^0, \gamma^0, \delta^0)$ *from ordered interval* $\langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$;

(ii) if $(u^0, \gamma^0, \delta^0) = (\bar{u}, \bar{\gamma}, \bar{\delta})$ $((u^0, \gamma^0, \delta^0) = (\underline{u}, \underline{\gamma}, \underline{\delta}))$, *then the sequence of the iterations* $\{(u^k, \gamma^k, \delta^k)\}$ *converges monotonically decreasing (increasing) to the unique solution* $(u^*, \gamma^*, \delta^*)$ *of the problem* (0.1).

Proof. (i)

First, we note that mappings $A_{I_l}(\cdot,u)$, $B_{I_l}(\cdot,\gamma)$, $l=1,2,\ldots,p$ for any fixed $(u,\gamma)\in\langle(\underline{u},\underline{\gamma}),(\bar u,\bar\gamma)\rangle$ keep all the properties of A and B due to Lemma 4.1.

Further, the vectors $(\underline{u},\underline{\gamma},\underline{\delta})$ and $(\bar u,\bar\gamma,\bar\delta)$ are the sub- and supersolutions for all problems

$$\begin{gathered} A_{I_l}(v^l_{I_l},u_{J_l})+B_{I_l}(\eta^l_{I_l},\gamma_{J_l})+\delta^l_{I_l}=f_{I_l},\\ \eta^l_{I_l}\in C_{I_l}v^l_{I_l},\quad \delta^l_{I_l}\in D_{I_l}v^l_{I_l},\\ v^l_{J_l}=u_{J_l},\quad \eta^l_{J_l}=\gamma_{J_l},\ \delta^l_{J_l}=\delta_{J_l} \end{gathered}$$

with (u,γ,δ) from the ordered interval $\langle(\underline{u},\underline{\gamma},\underline{\delta}),(\bar u,\bar\gamma,\bar\delta)\rangle$. In fact, due to the off-diagonal antitonicity of A, B the following inequalities hold:

$$\begin{aligned} A_{I_l}(\bar u_{I_l},\bar u_{J_l})+B_{I_l}(\bar\gamma_{I_l},\bar\gamma_{J_l})+\bar\delta_{I_l}\gg f_{I_l}&=A_{I_l}(v^l_{I_l},u_{J_l})+B_{I_l}(\eta^l_{I_l},\gamma_{J_l})+\delta^l_{I_l}\\ &\gg A_{I_l}(v^l_{I_l},\bar u_{J_l})+B_{I_l}(\eta^l_{I_l},\bar\gamma_{J_l})+\delta^l_{I_l}, \end{aligned}$$

and from the comparison result of Theorem 3.1, applied to the equations with mappings $A_{I_l}(.,\bar u_{J_l})$, $B_{I_l}(.,\bar\gamma_{J_l})$, we derive the result: $(\bar u_{I_l},\bar\gamma_{I_l},\bar\delta_{I_l})\gg(v^l_{I_l},\eta^l_{I_l},\delta^l_{I_l})$. Similar arguments are used to prove that $(\underline{u}_{I_l},\underline{\gamma}_{I_l},\underline{\delta}_{I_l})\ll(v^l_{I_l},\eta^l_{I_l},\delta^l_{I_l})$.

Now we can use the results of Theorem 2.1 and Theorem 3.1 for proving by induction in k the existence and the uniqueness of the solutions $(v^{l,k},\eta^{l,k},\delta^{l,k})\in\langle(\underline{u},\underline{\gamma},\underline{\delta}),(\bar u,\bar\gamma,\bar\delta)\rangle$ for problems (4.1) for all k and l.

(ii)

We take $(u^0,\gamma^0,\delta^0)=(\bar u,\bar\gamma,\bar\delta)$. To study the convergence of method (4.1) we consider along with it the Jacobi method:

$$\left\{\begin{array}{l} A^0(w^k)w^{k+1}+B^0(\beta^k)\beta^{k+1}+\varepsilon^{k+1}=f,\\ \beta^{k+1}\in Cw^{k+1},\ \varepsilon^{k+1}\in Dw^{k+1} \end{array}\right. \tag{4.4}$$

with initial guess $(w^0,\beta^0)=(\bar u,\bar\gamma)$ and diagonal operators $A^0(w)$, $B^0(w)$ for fixed w. We put also $\epsilon^0=\bar\delta$.

The following statements are valid:

$$\begin{aligned} &(\mathbf{a})\quad (v^{l,k+1},\eta^{l,k+1},\delta^{l,k+1})\ll(v^{l,k},\eta^{l,k},\delta^{l,k}),\\ &\qquad\quad (u^{k+1},\gamma^{k+1},\delta^{k+1})\ll(u^k,\gamma^k,\delta^k),\\ &\qquad\quad (w^{k+1},\beta^{k+1},\epsilon^{k+1})\ll(w^k,\beta^k,\varepsilon^k)\quad\forall k,\ \forall l;\\ &(\mathbf{b})\qquad (u^*,\gamma^*,\delta^*)\ll(u^k,\gamma^k,\delta^k)\ll(w^k,\beta^k,\varepsilon^k)\quad\forall k;\\ &(\mathbf{c})\qquad (w^k,\beta^k,\varepsilon^k)\downarrow(u^*,\gamma^*,\delta^*)\ \text{when}\ k\to\infty. \end{aligned}$$

(a) We prove only a part of the inequalities from (**a**) , namely, those of them which contain the vector $(v^{l,k},\eta^{l,k},\delta^{l,k})$ for fixed l. The inequality $(u^{k+1},\gamma^{k+1},\delta^{k+1})\ll(u^k,\gamma^k,\delta^k)$ follows from them as the consequence due to the definition of the matrices E_l, and the inequality $(w^{k+1},\beta^{k+1},\epsilon^{k+1})\ll(w^k,\beta^k,\varepsilon^k)$ $\forall k$, $\forall l$ can be proved similarly.

We proceed by induction. For $k=0$ the statements of (**a**) follow from the definition of supersolution. Let (**a**) be valid for some k. From the inequality $(u^k,\gamma^k,\delta^k)\ll(u^{k-1},\gamma^{k-1},\delta^{k-1})$ $(k>0)$ we derive

$$\begin{aligned} A_{I_l}(v^{l,k+1}_{I_l},u^k_{J_l})+B_{I_l}(\eta^{l,k+1}_{I_l},\gamma^k_{J_l})+\delta^{l,k+1}_{I_l}=f_{I_l}&=A_{I_l}(v^{l,k}_{I_l},u^{k-1}_{J_l})+B_{I_l}(\eta^{l,k}_{I_l},\gamma^{k-1}_{J_l})+\delta^{l,k}_{I_l}\\ &\ll A_{I_l}(v^{l,k}_{I_l},u^k_{J_l})+B_{I_l}(\eta^{l,k}_{I_l},\gamma^k_{J_l})+\delta^{l,k}_{I_l},\\ \eta^{l,k+1}_{I_l}\in C_{I_l}v^{l,k+1}_{I_l},\ \delta^{l,k+1}_{I_l}\in D_{I_l}v^{l,k+1}_{I_l},\quad &\eta^{l,k}_{I_l}\in C_{I_l}v^{l,k}_{I_l},\ \delta^{l,k}_{I_l}\in D_{I_l}v^{l,k}_{I_l}. \end{aligned}$$

Applying the comparison result of Theorem 3.1 to the equations with mappings $A_{I_l}(\cdot, u^k_{J_l})$, $B_{I_l}(\cdot, \gamma^k_{J_l})$ and taking into account that $(v^{l,k+1}_{J_l}, \eta^{l,k+1}_{J_l}, \delta^{l,k+1}_{J_l})$ is equal to $(v^{l,k}_{J_l}, \eta^{l,k}_{J_l}, \delta^{l,k}_{J_l})$, we obtain the inequality $(v^{l,k+1}, \eta^{l,k+1}, \delta^{l,k+1}) \ll (v^{l,k}, \eta^{l,k}, \delta^{l,k})$.

(b) We prove that $(v^{l,k}_{I_l}, \eta^{l,k}_{I_l}, \delta^{l,k}_{I_l}) \ll (w^k_{I_l}, \beta^k_{I_l}, \epsilon^k_{I_l})$ for a fixed l and that $(u^k, \gamma^k, \delta^k) \ll (w^k, \beta^k, \varepsilon^k)$ by induction in k. For $k = 0$ we have $(v^{l,0}, \eta^{l,0}, \delta^{l,0}) \equiv (u^0, \gamma^0, \delta^0) = (w^0, \beta^0, \varepsilon^0) = (\bar{u}, \bar{\gamma}, \bar{\delta})$. Let now the desired inequalities hold for some $k \geqslant 0$ and let us prove them for $k+1$. For a fixed l and a fixed $i \in I_l$ because of the inequalities $v^{l,k+1} \ll v^{l,k} \ll w^k$, $u^k \ll w^k$ and off-diagonal antitonicity of A we have

$$(A^0(v^{l,k+1}, w^k))_i = a_i(w^k_1, \ldots, v^{l,k+1}_i, \ldots, w^k_N) \leqslant a_i(v^{l,k+1}_{I_l}, u^k_{J_l})) = (A_{I_l}(v^{l,k+1}_{I_l}, u^k_{J_l}))_i$$

and similarly

$$(B^0(\eta^{l,k+1}, \beta^k))_i \leqslant (B_{I_l}(\eta^{l,k+1}_{I_l}, \gamma^k_{J_l}))_i.$$

Thus,

$$\begin{aligned} A^0_{I_l}(v^{l,k+1}, w^k) + B^0_{I_l}(\eta^{l,k+1}, \beta^k) + \delta^{l,k+1}_{I_l} &\ll A_{I_l}(v^{l,k+1}, u^k_{J_l}) + B_{I_l}(\eta^{l,k+1}_{I_l}, \gamma^k_{J_l}) + \delta^{l,k+1}_{I_l} \\ &= f_{I_l} = A^0_{I_l}(w^{k+1}, w^k) + B^0_{I_l}(\beta^{k+1}, \beta^k) + \varepsilon^{k+1}_{I_l} \end{aligned}$$

with the corresponding inclusions $\eta^{l,k+1}_{I_l} \in C_{I_l} v^{l,k+1}_{I_l}$, $\delta^{l,k+1}_{I_l} \in D_{I_l} v^{l,k+1}_{I_l}$ and $\beta^{k+1}_{I_l} \in C_{I_l} w^{k+1}_{I_l}$, $\varepsilon^{k+1}_{I_l} \in D_{I_l} w^{k+1}_{I_l}$.

Applying Theorem 3.1 for the equations with mappings $A^0_{I_l}(\cdot, w^k)$, $B^0_{I_l}(\cdot, \beta^k)$, we obtain the inequalities $(v^{l,k+1}_{I_l}, \eta^{l,k+1}_{I_l}, \delta^{l,k+1}_{I_l}) \ll (w^{k+1}_{I_l}, \beta^{k+1}_{I_l}, \varepsilon^{k+1}_{I_l})$ for all l. Because of the definition of the matrices E_l these inequalities imply $(u^{k+1}, \gamma^{k+1}, \delta^{k+1}) \ll (w^{k+1}, \beta^{k+1}, \varepsilon^{k+1})$.

To prove that $(u^*, \gamma^*, \delta^*) \ll (u^{k+1}, \gamma^{k+1}, \delta^{k+1})$ we argue similarly.

(c) The sequence $\{(w^k, \beta^k, \epsilon^k)\}$ is monotonically decreasing due to **(a)** and bounded below by $(\underline{u}, \underline{\gamma}, \underline{\delta})$, so it converges to a vector $(u_*, \gamma_*, \delta_*) \gg (\underline{u}, \underline{\gamma}, \underline{\delta})$. Passing to the limit in (4.4), we derive the equality

$$Au_* + B\gamma_* + \delta_* = f.$$

The inclusions $\gamma_* \in Cu_*$, and $\delta_* \in Du_*$ follow from the closeness of maximal monotone operators. It means that $(u_*, \gamma_*, \delta_*) = (u^*, \gamma^*, \delta^*)$ is just the unique solution of the problem (0.1).

As the sequence $\{(u^k, \gamma^k, \delta^k)\}$ satisfies the inequalities

$$(u^*, \gamma^*, \delta^*) \ll (u^k, \gamma^k, \delta^k) \ll (w^k, \beta^k, \epsilon^k) \ \forall k,$$

it also converges to the solution $(u^*, \gamma^*, \delta^*)$ of problem (0.1).

The proof of the monotone convergence (namely, increasing) of the iterations (4.1) starting from the subsolution is similar to the previous one.

Now, let the initial guess $(u^0, \gamma^0, \delta^0)$ for (4.1) belong to the ordered interval $\langle\!\langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle\!\rangle$. We use the notation $(u^k, \gamma^k, \delta^k)$ and $(v^{l,k}, \eta^{l,k}, \delta^{l,k})$, $l = 1, \ldots, p$ for these iterations and their counterparts, while by $(\underline{u}^k, \underline{\gamma}^k, \underline{\delta}^k)$ and $(\bar{u}^k, \bar{\gamma}^k, \bar{\delta}^k)$ (respectively, by $(\underline{v}^{l,k}, \underline{\gamma}^{l,k}, \underline{\delta}^{l,k})$ and $(\bar{v}^{l,k}, \bar{\gamma}^{l,k}, \bar{\delta}^{l,k})$) we denote the iterations (and their counterparts) constructed starting from the sub- and supersolution. The convergence of the sequence $\{(u^k, \gamma^k, \delta^k)\}$ to the solution $(u^*, \gamma^*, \delta^*)$ of the problem (0.1) follows from the inequalities

$$(\underline{u}^k, \underline{\gamma}^k, \underline{\delta}^k) \leqslant (u^k, \gamma^k, \delta^k) \leqslant (\bar{u}^k, \bar{\gamma}^k, \bar{\delta}^k) \quad \forall k, \tag{4.5}$$

which can be proved as above by using the comparison result of Theorem 3.1. Namely, we prove, first, that

$$(\underline{v}^{l,1}, \underline{\gamma}^{l,1}, \underline{\delta}^{l,1}) \ll (v^{l,k}, \eta^{l,k}, \delta^{l,k}) \ll (\bar{v}^{l,1}, \bar{\gamma}^{l,1}, \bar{\delta}^{l,1})$$

for all $l = 1, \ldots, p$, so, the inequalities (4.5) for $k = 1$. After that we proceed by induction. □

Now we study the convergence of the iterative method which can be viewed as a Schwarz multiplicative method for the problem (0.1). We keep the notations for the subsets I_l, $l = 1, \ldots, p$ and for the mappings $A_{I_l}(., v_{J_l})$ and $B_{I_l}(., v_{J_l})$.

We consider the following iterative method: for $l = 1 \ldots, p$ and for $k = 0, 1, 2, \ldots$ starting from the initial guess $(u^0, \gamma^0, \delta^0)$ solve

$$\begin{cases} A_{I_l}(u_{I_l}^{k+1}, \tilde{u}_{J_l}^{k+1}) + B_{I_l}(\gamma_{I_l}^{k+1}, \tilde{\gamma}_{J_l}^{k+1}) + \delta_{I_l}^{k+1} = f_{I_l}, \\ \gamma_{I_l}^{k+1} \in C_{I_l} u_{I_l}^{k+1}, \ \delta_{I_l}^{k+1} \in D_{I_l} u_{I_l}^{k+1}, \end{cases} \tag{4.6}$$

where $(\tilde{u}_{J_l}^{k+1})_i = u_i^{k+1}$, $(\tilde{\gamma}_{J_l}^{k+1})_i = \gamma_i^{k+1}$ for $i \in I_1, \ldots, I_{l-1}$ and $(\tilde{u}_{J_l}^{k+1})_i = u_i^k$, $(\tilde{\gamma}_{J_l}^{k+1})_i = \gamma_i^k$ for $i \in I_{l+1}, \ldots, I_p$.

Theorem 4.2. *Let the assumptions of Theorem 3.1 for problem* (0.1) *be fulfilled. Then:*

(i) iterative method (4.6) *is convergent for any initial guess* $(u^0, \gamma^0, \delta^0)$ *from ordered interval* $\langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$;

(ii) if $(u^0, \gamma^0, \delta^0) = (\bar{u}, \bar{\gamma}, \bar{\delta})$ $((u^0, \gamma^0, \delta^0) = (\underline{u}, \underline{\gamma}, \underline{\delta}))$ *then the sequence* $\{(u^k, \gamma^k, \delta^k)\}$ *converges monotonically decreasing (increasing) to the unique solution* $(u^*, \gamma^*, \delta^*)$ *of problem* (0.1).

Proof. Similar to Theorem 4.1 we establish that the mappings $A_{I_l}(\cdot, u)$ and $B_{I_l}(\cdot, \gamma)$ for $l = 1, 2, \ldots, p$ and for any fixed $(u, \gamma) \in \langle(\underline{u}, \underline{\gamma}), (\bar{u}, \bar{\gamma})\rangle$ keep all the properties of A and B and that $(\underline{u}, \underline{\gamma}, \underline{\delta})$ and $(\bar{u}, \bar{\gamma}, \bar{\delta})$ are the sub- and supersolutions for all problems in (4.6). Thus, we can use the results of Theorem 2.1 and Theorem 3.1 for proving by induction in l and in k the existence and the uniqueness of the solutions $(u^k, \gamma^k, \delta^k) \in \langle(\underline{u}, \underline{\gamma}, \underline{\delta}), (\bar{u}, \bar{\gamma}, \bar{\delta})\rangle$ for problems (4.6) for all k.

We prove the convergence of the iterations, when $(u^0, \gamma^0, \delta^0) = (\bar{u}, \bar{\gamma}, \bar{\delta})$, and to do this we compare the iterations of method (4.6) with those of method (4.1). Namely, we will prove by induction in l and in k that

$$(u^*, \gamma^*, \delta^*) \ll (w^k, \beta^k, \varepsilon^k) \ll (u^k, \gamma^k, \delta^k), \tag{4.7}$$

where by $(w^k, \beta^k, \varepsilon^k)$ we denote here the k-th iteration of method (4.6), while $(u^k, \gamma^k, \delta^k)$ stands for the k-th iteration of method (4.1). In both methods we take as the intial guess the supersolution $(\bar{u}, \bar{\gamma}, \bar{\delta})$ and $(u^*, \gamma^*, \delta^*)$ is as before the exact solution for problem (0.1). We restrict ourselves to the brief proof of the right inequality in (4.7) as the arguments for proving these inequalities are very similar to that we used in the proof of Theorem 4.1.

For $k = 0$ the inequalities (4.7) are valid because of the definition of the initial guess for both iterative methods. Now let they be valid for some k. First, we prove that $w_{I_1}^{k+1} \ll v_{I_1}^{1,k+1}$, $\beta_{I_1}^{k+1} \ll \eta_{I_1}^{1,k+1}$, $\varepsilon_{I_1}^{k+1} \ll \delta_{I_1}^{1,k+1}$, where $(v_{I_1}^{1,k+1}, \eta_{I_1}^{1,k+1}, \delta_{I_1}^{1,k+1})$ are defined by the first equation $(l = 1)$ in (4.1). To do this, we compare this equation with the first equation $(l = 1)$ in (4.6).

Because of the equalities $(\tilde{w}_{J_1}^{k+1}) = w_{J_1}^k$, $(\tilde{\beta}_{J_1}^{k+1}) = \beta_{J_1}^k$, the supposition of the induction and the off-diagonal antitonicity of the mappings A, B we have

$$\begin{aligned}
&A_{I_1}(w_{I_1}^{k+1}, \tilde{w}_{J_1}^{k+1}) + B_{I_1}(\beta_{I_1}^{k+1}, \tilde{\beta}_{J_1}^{k+1}) + \varepsilon_{I_1}^{k+1} \\
&\quad \equiv A_{I_1}(w_{I_1}^{k+1}, w_{J_1}^k) + B_{I_1}(\beta_{I_1}^{k+1}, \beta_{J_1}^k) + \varepsilon_{I_1}^{k+1} \\
&\quad = f_{I_1} = A_{I_1}(v_{I_1}^{1,k+1}, u_{J_1}^k) + B_{I_1}(\eta_{I_1}^{1,k+1}, \gamma_{J_1}^k) + \delta_{I_1}^{l,k+1} \\
&\quad \ll A_{I_1}(v_{I_1}^{1,k+1}, w_{J_1}^k) + B_{I_1}(\eta_{I_1}^{1,k+1}, \beta_{J_1}^k) + \delta_{I_1}^{l,k+1}
\end{aligned}$$

with corresponding inclusions $\beta_{I_1}^{k+1} \in C_{I_1} w_{I_1}^{k+1}$, $\varepsilon_{I_1}^{1,k+1} \in D_{I_1} w_{I_1}^{k+1}$ and $\eta_{I_1}^{1,k+1} \in C_{I_1} v_{I_1}^{1,k+1}$, $\delta_{I_1}^{1,k+1} \in D_{I_1} v_{I_1}^{1,k+1}$. Applying the comparison result from Theorem 3.1 to the equation with the mappings $A_{I_1}(\cdot, w_{J_1}^k)$ and $B_{I_1}(\cdot, \beta_{J_1}^k)$, we derive the inequalities $w_{I_1}^{k+1} \ll v_{I_1}^{1,k+1}$, $\beta_{I_1}^{k+1} \ll \eta_{I_1}^{1,k+1}$, $\varepsilon_{I_1}^{k+1} \ll \delta_{I_1}^{l,k+1}$.

As by the definition $(u_i^{k+1}, \gamma_i^{k+1}, \delta_i^{k+1}) = ((u_{I_1}^{1,k+1})_i, (\gamma_{I_1}^{1,k+1})_i, (\delta_{I_1}^{l,k+1})_i)$ for $i \in I_1 \setminus \cup_{j \neq 1} I_j$, then the inequalities $w_i^{k+1} \leqslant u_i^{k+1}$, $\beta_i^{k+1} \leqslant \gamma_i^{k+1}$, $\varepsilon_i^{k+1} \leqslant \delta_i^{k+1}$ hold for these indices.

Proceeding further by induction in l and keeping in mind the inequality $(u^{k+1}, \gamma^{k+1}, \delta^{k+1}) \ll (u^k, \gamma^k, \delta^k)$ (see proof of Theorem 3.2), we find that for all l the inequalities $\tilde{w}_{J_l}^{k+1} \ll u^k$ and $\tilde{\beta}_{J_l}^{k+1} \ll \gamma^k$ take place. It allows us to use for every l the procedure similar to the previous one. Namely, we use the off-diagonal antitonicity properties of A and B and then the comparison result of Theorem 3.1 for proving the inequalities (4.7).

Obviously, the inequalities (4.7) and convergence of the sequence $\{(u^k, \gamma^k, \delta^k)\}$ of the iterations for method (4.1) ensure the convergence of the iterations for method (4.6). The monotone decreasing of this sequence is proved by using arguments similar to the previous one.

The proof of all other statements of the theorem is the same as in Theorem 4.1. □

Now we discuss the possible implementation of iterative methods (4.1) and (4.6).

First, we note that the point Jacobi and Gauss-Seidel methods are their particular cases. Namely, if $p = N$, $I_l = \{l\}$ for every l and $e_{ii}^l = 1$ for $i = l$, while $e_{ii}^l = 0$ for $i \neq l$, then method (4.1) is the point Jacobi method. If $p = N$, $I_l = \{1, \ldots, l\}$ for every l and $e_{ii}^l = 1$ for $i \in I_l \setminus \bigcup_{j \neq l} I_j \equiv \{l\}$, while $e_{ii}^l = 0$ for $i \neq l$, then iterative method (4.6) becomes the Gauss-Seidel method.

In both these cases an implementation of (4.1) consists of the sequential solution of N one-dimensional problems, each of them is equivalent to a minimization one.

In fact, let for the simplicity $l = 1$. Then the corresponding subproblem in (4.1) for the Jacobi method or in (4.6) for the Gauss-Seidel method reads as

$$\left\{ \begin{array}{l} a_1(v_1^{k+1}, u_2^k, \ldots, u_N^k) + b_1(\eta_1^{k+1}, \gamma_2^k, \ldots, \gamma_N^k) + \delta_1^{k+1} = f_1, \\ \eta_1^{k+1} \in c_1(v_1^{k+1}), \;\; \delta_1^{k+1} \in d_1(v_1^{k+1}). \end{array} \right. \tag{4.8}$$

The scalar continuous and increasing functions $a(t) \equiv a_1(t, u_2^k, \ldots, u_N^k)$ and $b(t) \equiv b_1(t, \gamma_2^k, \ldots, \gamma_N^k)$ are the gradients of the convex differentiable functions $\psi_a(t)$ and $\psi_b(t)$. On the other hand, $c_1(t)$ and $d_1(t)$ are the subdifferentials of the convex functions $\phi_c(t)$ and $\phi_d(t)$, respectively. As it follows from the theory of convex functions, the scalar convex function

$$\Psi(t) = \psi_a(t) + \psi_b(\phi_c(t)) + \phi_d(t) - f_1 t$$

is subdifferentiable and its subdifferential is equal to $\partial\psi(t) = a(t) + b(c_1(t)) + d_1(t) - f_1$. It means that the first component v_1^{k+1} of the solution for problem (4.8) is the minimum point of $\Psi(t)$. In the concrete applications this function can be easily constructed. The problem of minimization for a scalar convex and continuous function can be solved by using the methods of convex optimization.

Now the components η_1^{k+1} and δ_1^{k+1} of the solution to (4.8) can be found as the corresponding sections of $c_1(v_1^{k+1})$ and $d_1(v_1^{k+1})$. In the most general case, when v_1^{k+1} is a point of the mutual multivalence for c_1 and d_1, we proceed as it was described in the proof of Lemma 2.2. Namely, we split $c_1(t) = \tilde{c}_1(t) + \alpha\xi$ and $d_1(t) = \tilde{d}_1(t) + \beta\xi$ with the continuous functions $\tilde{c}_1(t)$ and $\tilde{d}_1(t)$, $\alpha > 0$, $\beta > 0$ and with $\xi \in H(t - v_1^{k+1})$. For ξ we get the problem

$$a\left(v_1^{k+1} + b\left(\tilde{c}_1(v_1^{k+1}) + \alpha\xi\right) + \tilde{d}_1(v_1^{k+1}) + \beta\xi\right) = f_1.$$

As above, this problem is equivalent to the minimization of the strictly convex function

$$\psi_b\left(\tilde{c}_1(v_1^{k+1}) + \alpha(t - v_1^{k+1})^+\right) + \phi_d\left(\tilde{d}_1(v_1^{k+1}) + \beta(t - v_1^{k+1})^+\right) - \left(f_1 - a(v_1^{k+1})\right)t$$

with $t^+ = 1/2(|t| + t)$.

In a general case for the implementation of iterative methods (4.1) and (4.6) we need to solve each subproblem by an inner iterative algorithm. The Jacobi or the Gauss-Seidel method, described above, can be chosen as an inner iterative method.

5. Mesh scheme for the variational inequality

Let $\Omega = (0, l_1) \times (0, l_2)$ with the boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_s$, where $\Gamma_D = \{x \in \partial\Omega :\ x_1 = 0 \vee x_1 = l_1\}$, $\Gamma_N = \{x :\ 0 < x_1 < l_1;\ x_2 = 0\}$ and $\Gamma_s = \{x :\ 0 < x_1 < l_1;\ x_2 = l_2\}$. Let $V^0 = \{u \in W_p^1(\Omega) :\ u(x) = 0 \text{ on } \Gamma_D\}$, $V^z = \{u \in W_p^1(\Omega) :\ u(x) = z(x) \text{ on } \Gamma_D\}$, where $p \geqslant 3/2$, $z(x)$ be a given continuous function, such that $z(x) > 0$ for $x \in \Gamma_D : x_1 = 0$ and $z(x) \leqslant 0$ for $x \in \Gamma_D : x_1 = l_1$. We define also a closed convex subset of V^z by $K = \{u \in V^z : u(x) \geqslant 0 \text{ on } \Gamma_s\}$. Let $g \in C(R)$ be such that $g(0) = 0$ and for all $t_1, t_2 \in R^1$

$$c_0|t_1 - t_2|^p \leqslant (g(t_1) - g(t_2))(t_1 - t_2) \leqslant c_1|t_1 - t_2|^p, \quad c_0 > 0. \tag{5.1}$$

Further, let $H(\cdot)$ be the maximal monotone graph in $R^1 \times R^1$ defined by

$$H(t) = \{\alpha_1 t \text{ for } t < 0; \quad [0,1] \text{ for } t = 0; \quad 1 + \alpha_2 t \text{ for } t > 0\}, \qquad \alpha_1 > 0, \alpha_2 > 0.$$

We consider the problem: find $u \in K$, $\gamma \in L_\infty(\Omega)$ such that

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \sum_{i=1}^2 k(x) \cdot g\left(\frac{\partial u}{\partial x_i}\right) \cdot \frac{\partial(v - u)}{\partial x_i}\, dx - \int_\Omega \gamma \cdot \frac{\partial(v - u)}{\partial x_1}\, dx \geqslant 0 \quad \forall v \in K, \\ \gamma(x) \in H(u(x)) \text{ for a.a. } x \in \Omega. \end{array}\right. \tag{5.2}$$

Here $k \in C(\bar{\Omega})$, $k(x) \geqslant k_0 > 0\ \forall x \in \bar{\Omega}$.

Formally, we can write this problem in the following pointwise form:

$$-\sum_{i=1}^2 \frac{\partial}{\partial x_i}\left(k(x) \cdot g\left(\frac{\partial u}{\partial x_i}\right)\right) + \frac{\partial \gamma}{\partial x_1} = 0, \quad \gamma(x) \in H(u(x)), \text{ in } \Omega,$$

$$u(x) = z(x) \text{ on } \Gamma_D, \quad Q_n(x) \equiv \sum_{i=1}^2 k(x) \cdot g\left(\frac{\partial u}{\partial x_i}\right) \cdot n_i(x) = 0 \text{ on } \Gamma_N,$$

$$u(x) \geqslant 0, \quad Q_n \geqslant 0, \quad u(x) \cdot Q_n(x) = 0 \text{ on } \Gamma_s$$

with unit vector $n = (n_1, n_2)$ of outward normal to $\partial\Omega$.

We approximate problem (5.2) by a finite difference scheme on a uniform square mesh of size h, constructing it via finite element approximation with quadrature formulas.

Let T_h be a partitioning of Ω into squares Δ of dimensions $h \times h$, $V_h = \{u_h(x) \in C(\bar{\Omega}) : u_h(x) \in Q_1 \quad \forall \delta \in T_h\}$, where Q_1 is the space of bilinear functions, $V_h^z = \{u_h(x) \in V_h : u_h(x) = z_h(x),\ x \in \Gamma_D\}$, where $z_h \in V_h$ is the interpolant of z, and $K_h = \{u \in V_h^z : u_h(x) \geqslant 0 \text{ in } \Gamma_s\}$. By $\bar{\omega}$ we denote the set of all mesh nodes – vertices of $\delta \in T_h$.

We use the quadrature formulas:

$$\int\limits_{\Delta} u_h(x)dx \approx S_\Delta(u_h) = 1/4h^2 \sum_{i=1}^{4} u_h(d_j), \qquad S_h(u_h) = \sum_{\Delta\in\Omega} S_\Delta(u_h),$$
$$\int\limits_{\Delta} u_h(x)dx \approx E_\Delta(u_h) = \frac{1}{2}h^2(u_h(d_1) + u_h(d_2)), \quad E_h(u_h) = \sum_{\Delta\in\Omega} E_\Delta(u_h),$$

where d_j are vertices of $\Delta \in T_h : d_1 = (x_1, x_2), d_2 = (x_1, x_2 + h), d_3 = (x_1 + h, x_2), d_4 = (x_1 + h, x_2 + h)$.

The finite difference scheme for (5.2) with the up-wind approximation of the nonlinear convective term can be written in the following implicit form:

find $u_h \in K_h, \gamma_h(x) \in V_h$ such that

$$\begin{cases} S_h\left(\sum_{i=1}^{2} k(x)\cdot g\left(\frac{\partial u_h}{\partial x_i}\right)\cdot\frac{\partial(v_h - u_h)}{\partial x_i}\right) - E_h\left(\gamma_h \cdot \frac{\partial(v_h - u_h)}{\partial x_i}\right) \geqslant 0 & \forall v_h \in K_h, \\ \gamma_h(x) \in H(u_h(x)) & \forall x \in \bar{\omega}. \end{cases} \tag{5.3}$$

Let $v \in R^N$ be the vector of the nodal values of a function $v_h \in V_h^0$ for the nodes lying in $\bar{\Omega} \setminus \Gamma_D$. Further we use the notation $v_h \Leftrightarrow v$ for this bijection.

Let $w_h(x)$ and $\tilde{\gamma}_h(x)$ from V_h be the auxiliary functions:

$$\begin{aligned} w_h(x) &= z_h(x) && \text{for } x \in \bar{\Gamma}_D; \quad w_h(x) = 0 \quad \text{for } x \in \Omega \setminus \bar{\Gamma}_D, \\ \tilde{\gamma}_h(x) &= H(z_h(x)) && \text{for } x \in \bar{\Gamma}_D; \quad \tilde{\gamma}_h(x) = 0 \quad \text{for } x \in \Omega \setminus \bar{\Gamma}_D. \end{aligned}$$

We note that the values of $\tilde{\gamma}_h(x)$ in the points $x \in \Gamma_D : x_1 = l_1$ are not used in the mesh scheme (5.3), so, we can formally take any section of $H(0)$ as the values of $\tilde{\gamma}_h(x)$ in these points if $z_h(x) = 0$.

We define nonlinear operator $A : R^N \to R^N$, $N \times N$ matrix B and vector $f \in R^N$ by the equalities

$$\begin{aligned} (Au, v) &= S_h\left(\sum_{i=1}^{2} k(x)\cdot g\left(\frac{\partial(u_h + w_h)}{\partial x_i}\right)\cdot\frac{\partial v_h}{\partial x_i}\right), \\ (Bu, v) &= -E_h\left(u_h \cdot \frac{\partial v_h}{\partial x_i}\right), \\ (f, v) &= E_h\left(\tilde{\gamma}_h \cdot \frac{\partial v_h}{\partial x_1}\right) \end{aligned}$$

for all $u_h, v_h \in V_h^0$, where $(\cdot,\cdot)$ is the inner product in Euclidean space R^N and $v_h \Leftrightarrow v \in R^N,\ u_h \Leftrightarrow u \in R^N$.

Let also $K_0 \in R^N$ be the subset of vectors in R^N which correspond to the mesh functions $u_h \in K_h$, i.e., $K_0 \ni u \Leftrightarrow u_h \in K_h$. Below $D = \partial I_{K_0}$ is the subdifferential of the indicator function for the set K_0 and $Cu = (H(u_1), H(u_2), \ldots, H(u_N))$. In all these notations the mesh scheme becomes a partial case of problem (0.1).

Now we study the properties of A and B.

We pay the greatest attention to the study of the properties of A. For brevity we denote by $\tilde{u}_h = u_h + w_h$.

Let us prove that A is an off-diagonally antitone mapping. Let $e^i = (0, \ldots, \underbrace{1}_{i}, \ldots, 0)$, $e_i \Leftrightarrow \phi_i(x) \in V_h$, where $\phi_i(x)$ is the corresponding basis function. We have to prove that the function $t \to (A(u + te_j))_i$ is decreasing for $j \neq i$.

By definition

$$(A(u + te_j))_i = \sum_{k=1}^{2} S_h \left(k(x) \cdot g \left(\frac{\partial}{\partial x_k} (\tilde{u}_h + t\phi_j) \right) \cdot \frac{\partial \phi_i}{\partial x_k} \right). \tag{5.4}$$

If $\operatorname{supp} \phi_j \cap \operatorname{supp} \phi_i$ has zero measure, then the right-hand side in (5.4) is zero. It is nonzero, if the nodes x_i and x_j are neighbor and in this case it is easy to check that $\partial \phi_i / \partial x_k$ and $\partial \phi_j / \partial x_k$ have the opposite signs for all points in $\operatorname{supp} \phi_j \cap \operatorname{supp} \phi_i$ (corresponding nodes among them) for $k = 1, 2$. The function $g(t)$ is strictly monotone, so, the function $t \to g\left(\partial(\tilde{u}_h + t\phi_j)/\partial x_k\right) \cdot \partial \phi_i / \partial x_k$ is decreasing for such i and j in the corresponding nodes and, as a consequence, the function $t \to (A(u + te_j))_i$ is dereasing for $j \neq i$.

Using the same arguments, we prove that $t \to (A(u + te_i))_i$ is strictly increasing, i.e., A is strictly diagonally isotone.

Now, we prove that A is inverse isotone, i.e., $Au^1 \gg Au^2$ implies $u^1 \gg u^2$. Let $f^k = Au^k$ and $f^1 \gg f^2$. Below we use the notations a^+ and a^- for positive and negative parts of $a:\ a = a^+ - a^-$. Let $z = u^1 - u^2 = z^+ - z^-$ and $(z^+)_h$, $(z^-)_h$ be the V_h-interpolants of the vectors z^+, z^-, i.e. $z^+ \Leftrightarrow (z^+)_h \in V_h^0$, $z^- \Leftrightarrow (z^-)_h \in V_h^0$. We have

$$\begin{aligned} 0 &\leqslant (f^1 - f^2, z^-) = (Au^1 - Au^2, z^-) \\ &= (A(u^2 - z^- + z^+) - A(u^2 - z^-) + A(u^2 - z^-) - Au^2, z^-) \end{aligned} \tag{5.5}$$

Let now x be a fixed point in a finite element δ. It is easy to check that the terms $\partial (z^+)_h(x)/\partial x_k$ and $\partial (z^-)_h(x)/\partial x_k$ have the opposite signs (if both are nonzero). Due to the strict monotonicity of g it ensures that

$$\left(g\left(\frac{\partial \tilde{u}_h^2}{\partial x_k} - \frac{\partial (z^-)_h}{\partial x_k} + \frac{\partial (z^+)_h}{\partial x_k} \right) - g\left(\frac{\partial \tilde{u}_h^2}{\partial x_k} - \frac{\partial (z^-)_h}{\partial x_k} \right) \right) \cdot \frac{\partial (z^-)_h}{\partial x_k} \leqslant 0.$$

These inequalities are valid for every $x \in \delta$ (nodes among them) and every $k = 1, 2$. Using the definition of A we derive

$$\begin{aligned} &(A(u^2 - z^- + z^+) - A(u^2 - z^-), z^-) \\ &\quad = \sum_{k=1}^{2} S_h \Bigg(k(x) \cdot \Bigg(g\left(\frac{\partial \tilde{u}_h^2}{\partial x_k} - \frac{\partial (z^-)_h}{\partial x_k} + \frac{\partial (z^+)_h}{\partial x_k} \right) \\ &\qquad - g\left(\frac{\partial \tilde{u}_h^2}{\partial x_k} - \frac{\partial (z^-)_h}{\partial x_k} \right) \Bigg) \cdot \frac{\partial (z^-)_h}{\partial x_k} \Bigg) \leqslant 0. \end{aligned} \tag{5.6}$$

On the other hand

$$(A(u^2 - z^-) - Au^2, z^-) = \sum_{k=1}^{2} S_h\left(k(x)\cdot\left(g\left(\frac{\partial\tilde{u}_h^2}{\partial x_k} - \frac{\partial(z^-)_h}{\partial x_k}\right) - g\left(\frac{\partial\tilde{u}_h^2}{\partial x_k}\right)\right)\cdot\frac{\partial(z^-)_h}{\partial x_k}\right)$$

$$\leqslant -k_0c_0\sum_{k=1}^{2} S_h\left(\left|\frac{\partial(z^-)_h)}{\partial x_k}\right|^p\right)$$

From this inequality and from (5.5), (5.6)

$$\sum_{k=1}^{2} S_h\left(\left|\frac{\partial(z^-)_h)}{\partial x_k}\right|^p\right) = 0.$$

It means that $(z^-)_h = ((u^1 - u^2)^-)_h = 0$ (there is the Dirichlet boundary condition on the part of the boundary), i.e., $u^1 \gg u^2$.

Thus, it is proved that A is M-mapping.

The last statement that we prove is weak diagonal dominance of A.

Let u^1, u^2 be two vectors with nonempty set $I_- = \{i:\ u_i^1 < u_i^2\}$ and $I_0 = \{i:\ u_i^1 = u_i^2\}$, $L \subset I_0$. We take the vector η with coordinates $\eta_i = 1$ for $i \in I_- \cup L$ and $\eta_i = 0$ for other indices i. Then

$$(Au^1 - Au^2, \eta) = \sum_{k=1}^{2} S_h\left(k(x)\cdot\left(g\left(\frac{\partial\tilde{u}_h^1}{\partial x_k}\right) - g\left(\frac{\partial\tilde{u}_h^2}{\partial x_k}\right)\right)\cdot\frac{\partial\eta_h}{\partial x_k}\right). \tag{5.7}$$

If for some point $x \in \delta$ both $\partial\eta_h/\partial x_k$ and $\partial\tilde{u}_h^1/\partial x_k - \partial\tilde{u}_h^2/\partial x_k$ are nonzero, then they have the opposite signs. Due to the strict monotonicity of g it ensures that all the terms in the sum on right-hand side of (5.7) are zero or negative, so, it is nonpositive. Moreover, if $I_- \neq \varnothing$ ($I_0 \neq \varnothing$ because of the Dirichlet boundary conditions) and vector η has coordinates $\eta_i = 1$ for $i \in I_-$ and $\eta_i = 0$ for other indices i, then at least one term on right-hand side of (5.7) is negative while others are nonpositive. So, in this case we have strong inequality $(Au^1 - Au^2, \eta) < 0$, and weak diagonal dominance of A is proved.

Direct calculations show that the matrix B has entries equal to h on its main diagonal and exactly one nonzero entry in each row and column equal to $-h$. So, it is M-matrix with weak diagonal dominance both in rows and in columns (we note that just using the quadrature formula E instead of S leads to up-wind approximation of convective term, so to M-matrix B).

Obviously, the operators C and D are diagonal and maximal monotone and C is strictly monotone.

The only assumption we have to check is the existence of a sub- and a supersolution.

Below we prove that the following vectors can be chosen as a sub- and a supersolution for problem (5.3):

$$\bar{u} \equiv \max_{x\in\Gamma_D} z(x), \quad \bar{\gamma} \equiv H(\bar{u}), \quad \bar{\delta} \equiv 0$$

and

$$\underline{u}_i = \min_{x\in\Gamma_D} z(x), \quad \underline{\gamma}_i = H(\underline{u}_i), \quad \underline{\delta}_i = 0 \ \text{ for the indices } i \ \text{ corresponding to the nodes in } \Omega\cup\Gamma_N,$$

$$\underline{u}_i = 0, \quad \underline{\gamma}_i = 0, \quad \underline{\delta}_i = \min\{0; f_i - A\underline{u}_i - B\underline{\gamma}_i\} \ \text{ for the indices } i \text{ corresponding to the nodes in } \Gamma_s.$$

First, by the definition $\bar{\gamma} \in C\bar{u}$, $\bar{\delta} \in D\bar{u}$. Moreover, because of the definition of $\bar{\delta}$ we need only to prove that

$$A\bar{u}_i + B\bar{\gamma}_i - f_i \geqslant 0 \tag{5.8}$$

for all i.

Let $\phi_i(x) \in V_h^0$ be the basis function corresponding to the node $x_i \notin \Gamma_D$ of the triangulation T_h. Then

$$A\bar{u}_i + B\bar{\gamma}_i - f_i = \sum_{k=1}^{2} S_h\left(k(x) \cdot g\left(\frac{\partial}{\partial x_k}(\bar{u}_h + w_h)\right) \cdot \frac{\partial \phi_i}{\partial x_k}\right) - E_h\left((\bar{\gamma}_h + \tilde{\gamma}_h) \cdot \frac{\partial \phi_i}{\partial x_1}\right),$$

where $\bar{u}_h$ and $\bar{\gamma}_h$ are V_h^0-interpolants of the vectors $\bar{u}$ and $\bar{\gamma}$, i.e., these mesh functions vanish on the boundary Γ_D.

The function $\bar{u}_h + w_h \in V_h$ is constant and equal to $\bar{u}$ in the finite elements which are not adjacent to the boundary Γ_D, while it has values less than $\bar{u}$ at the points of Γ_D. It means that the functions $g(\partial(\bar{u}_h + w_h)/\partial x_k)$, $k = 1, 2$, are equal to 0 in the finite elements which are not adjacent to the boundary Γ_D. On the other hand, $g(\partial(\bar{u}_h + w_h)/\partial x_1)$ has the positive values at the points of the finite elements which are adjacent to the left part of the boundary Γ_D, while they are negative at the points of finite elements which are adjacent to the right part of the boundary Γ_D. Now, using these properties and taking into account the signs of $\partial \phi_i/\partial x_k$ in the different finite elements, we can easily calculate that

$$S_h\left(k(x) \cdot g\left(\frac{\partial}{\partial x_2}(\bar{u}_h + w_h)\right) \cdot \frac{\partial \phi_i}{\partial x_2}\right) = 0,$$
$$S_h\left(k(x) \cdot g\left(\frac{\partial}{\partial x_1}(\bar{u}_h + w_h)\right) \cdot \frac{\partial \phi_i}{\partial x_1}\right) > 0.$$

Similar reasoning leads to the inequality

$$-E_h\left((\bar{\gamma}_h + \tilde{\gamma}_h) \cdot \frac{\partial \phi_i}{\partial x_1}\right) \geqslant 0.$$

Thus, the statement (5.8) is proved.

Further, $\underline{\gamma} \in C\underline{u}$ and if we prove thats

$$A\underline{u}_i + B\underline{\gamma}_i - f_i \leqslant 0 \tag{5.9}$$

for all i corresponding to the nodes in $\Omega \cup \Gamma_N$, then because of the definition of $\underline{\delta}$, first, it will be a selection of $C\underline{u}$ and, second, the inequality

$$A\underline{u}_i + B\underline{\gamma}_i + \underline{\delta}_i - f_i \leqslant 0,$$

will be valid for all indices i.

We omit the proof of (5.9), which is similar to the previous one.

As we checked the validity of all the assumptions of Theorems 2.2 and 3.1 for the mesh scheme (5.3), it has a unique solution.

Now we decompose the domain Ω into p overlapping subdomains Ω_i, all interfaces $\partial\Omega_i \cap \partial\Omega_j$ consist of the sides of finite elements from T_h. We arrange the set of indices of the vectors $u \Leftrightarrow u_h \in V_h^0$ in such a way that I_l contains the indices corresponding to the nodes x_i which belong to $\bar{\Omega}_l$. Then we construct the iterative methods (4.1) and (4.6).

Owing to Theorems 4.1 and 4.2 these iterative methods converge for any initial guess and monotonically if starting from the subsolution or from the supersoultion of the problem.

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