COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 1 (2001), No. 3, pp. 265–284

THREE-LEVEL DIFFERENCE SCHEMES ON NON-UNIFORM IN TIME GRIDS

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Abstract — In this work, a stability of three-level operator-difference schemes on nonuniform in time grids in Hilbert spaces is studied. *A priori* estimates of a long time stability (for $t \to \infty$) in the sense of the initial data and the right-hand side are obtained in different energy norms without demanding the quasiuniformity of the grid. New difference schemes of the second order of local approximation on nonuniform grids both in time and space on standard stencils for parabolic and wave equations are adduced.

2000 Mathematics Subject Classification: 65M06; 65M12; 65M50.

Keywords: stability, three-level schemes, nonuniform in time grids.

Introduction

By now the main results on the theory of the stability of operator-difference schemes have been obtained using grids uniform in time [3, 4, 9–11]. Necessary and sufficient conditions of stability were already obtained in the sense of the initial data and the right-hand side in finite-dimensional Hilbert spaces. The results of later investigations along this lines are presented in [14].

For two-level difference schemes changeover to nonuniform grids is not fundamental in character [8], since the estimates obtained for each level easily yield estimates of stability at any time instant. The situation becomes more complicated for three-level schemes. There are only a few particular pertinent results (see, for example, [2, 5–8]). It should be note separately paper [13], in which basic canonical forms have been first introduced for the three-level difference schemes on nonuniform in time grids and important theorems concerning the stability in the sense of the initial data have formulated.

Note some features of the investigation of difference schemes on nonuniform in time grids. If in the initial differential problem the coefficients are constant, approximation on a nonuniform grid leads us to operator-difference schemes dependent on grid node t_n. If we require that these operators be Lipschitz–continuous, it would lead us to an the unnatural condition of the quasiuniformity of a time grid. One other problem is connected with a reduction of the order of local approximation in going over from a uniform grid to a nonuniform one. It turn out that we can raise the order of local approximation on nonuniform grids using here the standard stencils. In this paper the stability of new computational methods on nonuniform grids are investigated on the basis of general *a priori* estimates obtained for three-level operator-difference schemes.

The arising problems are discussed in more detail in the next section. Here, we note that the general conditions of stability and a long time stability of three-level operator-difference schemes on nonuniform in time grids are obtained in this paper without assuming the Lipschitz–continuity of operators on a time variable. Difference schemes of the second order of local approximation are built and investigated on nonuniform grids both in time and space on standard stencils for parabolic and wave equations.

1. Statement of the problem

At the present time the most thoroughly studied are three-level operator-difference schemes [9]

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1$$

on uniform in time grids $t_n = n\tau$, $n = 0, 1, \ldots$. Here $y = y_n = y(t_n) \in H$ is the sought function; $u_0, u_1, \varphi(t_n) \in H$ are given; H is the finite-dimensional Hilbert space; D, B, A are linear operators acting in H;

$$y_t = \frac{y_{n+1} - y_n}{\tau}\ , \quad y_{\bar{t}} = \frac{y_n - y_{n-1}}{\tau}\ , \quad y_{\bar{t}t} = \frac{y_t - y_{\bar{t}}}{\tau}, \quad y_{\overset{\circ}{t}} = \frac{y_{n+1} - y_{n-1}}{2\tau}.$$

The corresponding *a priori* estimates of the stability in the since of the initial data and the right-hand side are obtained on the following assumptions:

$$A(t) = A^*(t) > 0, \quad D(t) = D^*(t) > 0, \quad B(t) \geqslant 0, \tag{1.1}$$

$$A(t) \text{ and } D(t) \text{ are Lipschitz continuous in } t. \tag{1.2}$$

In order to understand the problems arising in the case of a nonuniform grid, $t_n = t_{n-1} + \tau_n$, $n = 1, 2, \ldots, t_0 = 0$, in the Hilbert space H we examine an abstract Cauchy problem for the evolutional equation of the second order with the constant self-adjoint positive operator $A = A^* > 0$:

$$\frac{d^2v}{dt^2} + Av = \varphi, \quad t > 0, \quad v(0) = u_0, \quad \frac{dv}{dt}(0) = \bar{u}_0\ . \tag{1.3}$$

Let us put into the correspondence for the Cauchy problem (1.3) the operator-difference scheme with weighting the solution:

$$y_{\bar{t}\hat{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1\ , \tag{1.4}$$

where

$$y_{\bar{t}\hat{t}} = \frac{y_t - y_{\bar{t}}}{\tau^*}, \quad y_t = \frac{y_{n+1} - y_n}{\tau_{n+1}}\ , \quad y_{\bar{t}} = \frac{y_n - y_{n-1}}{\tau_n}\ , \quad \tau^* = 0.5(\tau_n + \tau_{n+1}),$$
$$y^{(\sigma_1,\sigma_2)} = \sigma_1\hat{y} + (1 - \sigma_1 - \sigma_2)y + \sigma_2\check{y}, \quad \hat{y} = y_{n+1}, \quad \check{y} = y_{n-1}.$$

With the aid of the identity

$$y^{(\sigma_1,\sigma_2)} = y + (\sigma_1\tau_+ - \sigma_2\tau)y_{\overset{\circ}{t}} + \frac{\tau_+\tau}{2}(\sigma_1+\sigma_2)y_{\bar{t}\hat{t}},$$
$$\tau_+ = \tau_{n+1}, \quad y_{\overset{\circ}{t}} = \frac{y_{n+1}-y_{n-1}}{\tau_n+\tau_{n+1}},$$

we write down the operator scheme (1.4) in the natural canonical form:

$$Dy_{\bar{t}\hat{t}} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y_0 = u_0, \quad y_1 = u_1 \tag{1.5}$$

with the operators

$$D(t_n) = E + \frac{\tau_n\tau_{n+1}}{2}(\sigma_1+\sigma_2)A, \quad B(t_n) = (\sigma_1\tau_{n+1} - \sigma_2\tau_n)A\ . \tag{1.6}$$

From the structure of the operator D one can see that if the time grid is nonuniform, then it depends on t. Trying to generalize the sufficient conditions of stability (1.1), (1.2) in the case of the nonuniform grid, we come to the necessity to fulfill the requirement that the operator $D(t)$ is Lipschitz continuous:

$$|\,((D_n - D_{n-1})u_n, u_n)\,| \leqslant c_1\tau_n\,(D_{n-1}u_n, u_n)\ . \tag{1.7}$$

This requirement is equivalent to the condition of quasiuniformity of the grid:

$$|\tau_{n+1} - \tau_{n-1}| \leqslant c_1\tau_{n-1}\tau_n\ . \tag{1.8}$$

This is a fairly strong restriction on the time step.

Thus, the first actual and rather complicated theoretical and applied problem is formulated as follows:

- to find sufficient conditions of the stability of three-level operator-difference schemes on nonuniform in time grids without a requirement of Lipschitz continuity of the operator $D(t)$ or quasiuniformity of a grid.

The second problem deals with the construction of such computational algorithms which would be well compatible with the properties of the differential problem in the sense of stability. This is illustrated by the following example. In problem (1.3) we assume that $\varphi = 0$. On scalar multiplication of this equation by $2\partial u/\partial t$ and integration of the resulting identity from 0 to t and simple transformations we obtain the energy relationship

$$E^2(t) = E^2(0), \quad E(t) = \left\{\left\|\frac{\partial u}{\partial t}\right\|^2 + \|u(t)\|_A^2\right\},$$

which expresses a long time (i.e., for any $t \in (0, +\infty)$) stability of the solution of problem (1.3) in the sense of the initial data. The approximation of such a problem (1.4) on a nonuniform grid leads to the difference problem (1.5) with the operators dependent on the variable t_n. The standard assumptions of the Lipschitz continuity of operators with the subsequent application of Gronwall's lemma lead to *a priori* estimate such as

$$E_h(t) \leqslant e^{ct}E_h(0), \quad c = \text{const} > 0,$$

where $E_h(t)$ is the grid analogue of the norm $E(t)$. The latter inequality expresses the uniform or ρ-stability of the computational method only at the finite time. However, this very estimate does not express a long time stability of the solution. In other words, the solution of the difference problem is unstable when $t \to \infty$. The problem becomes too much complicated, if additionally we set a task to obtain *a priori* estimates of long time stability not only with respect to the initial data, but also to the right-hand side. Let us now formulate the second problem that would be solved in this paper:

- to receive *a priori* estimates of a long time stability in the sense of the initial data and the right-hand side of the solutions of three-level operator-difference schemes on nonuniform grids.

The third problem is connected with a rising order of local approximation of computational methods on nonuniform grids without an increment of standard grid stencils. We remind that while transition from a uniform to a nonuniform grid the order of the local approximation of the second derivative decreases:

$$\frac{d^2u}{dt^2}(t_n) - u_{\bar{t}\hat{t}} = O(\tau_{n+1} - \tau_n) + O(\tau_n^2) \ .$$

As a result, we formulate one more problem:

- to build stable three-level difference schemes of the second order of approximation on usual grid stencils without requiring the quasiuniformity of the time grid.

We will answer all of the above questions in this paper.

2. Intermediate results

Let the real finite-dimensional Hilbert space H and the nonuniform time grid

$$\hat{\bar{\omega}}_\tau = \{t_n = t_{n-1} + \tau_n,\ n \in 1, 2, \ldots, N,\ t_0 = 0,\ t_N = T\} = \hat{\omega}_\tau \cup \{0, T\}$$

be given. For any arbitrary functions $u, v \in H$ the Cauchy–Schwartz inequality and the ε-inequality hold true:

$$|(u,v)| \leqslant \|u\|\|v\| \leqslant \varepsilon\|u\|^2 + \frac{1}{4\varepsilon}\|v\|^2, \quad \varepsilon > 0 \ .$$

For the self-adjoint and nonnegative operator R we define a semi-norm of the grid function u:

$$\|u\|_R^2 = (Ru, u), \quad R = R^* \geqslant 0.$$

Let us examine the Cauchy problem for the operator-difference equation

$$B(t_n)\frac{y_{n+1} - y_n}{\tau_{n+1}} + A(t_n)y_n = \varphi_n, \quad t_n \in \hat{\omega}_\tau \ , \quad y_0 = u_0 \ , \tag{2.1}$$

where $B, A : H \to H$ are the linear operators in H, which, generally speaking, are dependent on τ_n, t_n; $y = y(t_n) \in H$ is the desired function, and $\varphi_n, u_0 \in H$ are given. Using index free notation, we write problem (2.1) in the form

$$By_t + Ay = \varphi, \quad t_n \in \hat{\omega}_\tau, \quad y_0 = u_0 \ . \tag{2.2}$$

The two-level scheme is taken to mean a variety of Cauchy problems (2.2) dependent on the parameter τ_n, and the representation in the form of (2.2) is called the canonical form of two-level schemes. Let us prove an auxiliary lemma about the stability of scheme (2.2) in the sense of the initial data and the right-hand side. In the case of the uniform grid ω_τ, the proof of this lemma using the method of extraction of stationary nonuniformities can be found in [9; p. 412]. The obtained us proof of Lemma 2.1 is very simple and beautiful proof even in the case of a uniform grid. Therefore we consider it should be represented in this work. We also note that the results on the stability of three-level operator-difference schemes will be based on this lemma.

Lemma 2.1. *In the equation* (2.2) *let A be a self-adjoint positive operator independent of n, and*

$$B \geqslant 0.5\tau_{n+1}A, \quad t \in \hat{\omega}_\tau.$$

Then, scheme (2.2) *is stable in the sense of the initial data, the right-hand side, and the following a priori estimate holds:*

$$\|y_{n+1}\|_A \leqslant \|u_0\|_A + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau_k \|\varphi_{\bar{t},k}\|_{A^{-1}}. \tag{2.3}$$

Proof. Let us scalarly multiply the scheme (2.2) by $2\tau_{n+1}y_t$:

$$2\tau_{n+1}(By_t, y_t) + 2\tau_{n+1}(Ay, y_t) = 2\tau_{n+1}(\varphi, y_t). \tag{2.4}$$

Since A is the self-adjoint operator, the scalar product (Ay, y_t) can be represented in the following form

$$2\tau_{n+1}(Ay, y_t) = (A\hat{y}, \hat{y}) - (Ay, y) - \tau_{n+1}^2\|y_t\|_A^2 \ . \tag{2.5}$$

Substituting (2.5) into (2.4) we obtain the energy relation

$$2\tau_{n+1}\|y_t\|^2_{B-0.5\tau_{n+1}A} + \|y_{n+1}\|_A^2 = \|y_n\|_A^2 + 2\tau_{n+1}(\varphi, y_t).$$

Taking into account the condition of the Lemma 2.1 we have

$$\|y_{n+1}\|_A^2 - 2(\varphi_n, y_{n+1}) \leqslant \|y_n\|_A^2 - 2(\varphi_n, y_n),$$

or

$$\|y_{n+1} - A^{-1}\varphi_n\|_A^2 \leqslant \|y_n - A^{-1}\varphi_n\|_A^2.$$

Extracting a square root of the last inequality, we get an estimate

$$\begin{aligned} \|y_{n+1} - A^{-1}\varphi_n\|_A &\leqslant \|y_n - A^{-1}\varphi_n\|_A \leqslant \|y_n - A^{-1}\varphi_{n-1}\|_A + \tau_n\|\varphi_{\bar{t},n}\|_{A^{-1}} \\ &\leqslant \ldots \leqslant \|y_1 - A^{-1}\varphi_0\|_A + \sum_{k=1}^{n} \tau_k \|\varphi_{\bar{t},k}\|_{A^{-1}}. \end{aligned} \tag{2.6}$$

The same inequality at $n = 0$ yields the estimate

$$\|y_1 - A^{-1}\varphi_0\|_A \leqslant \|y_0 - A^{-1}\varphi_0\|_A,$$

substituting which into (2.6), we obtain the following inequality:

$$\|y_{n+1} - A^{-1}\varphi_n\|_A \leqslant \|y_0 - A^{-1}\varphi_0\|_A + \sum_{k=1}^{n} \tau_k \|\varphi_{\bar{t},k}\|_{A^{-1}} \ . \tag{2.7}$$

Taking into consideration that

$$\|y_{n+1} - A^{-1}\varphi_n\|_A \geqslant \|y_{n+1}\|_A - \|\varphi_n\|_{A^{-1}}, \quad \|y_0 - A^{-1}\varphi_0\|_A \leqslant \|y_0\|_A + \|\varphi_0\|_{A^{-1}},$$

from (2.7) we can find the finite estimate (2.3). □

We give some useful identities that will be used in obtaining appropriate *a priori* estimates of stability:

$$y_n = \frac{y_n + y_{n+1}}{2} - \frac{\tau_{n+1}}{2} y_t, \quad y_n = \frac{y_{n-1} + y_n}{2} + \frac{\tau_n}{2} y_{\bar{t}} \ , \tag{2.8}$$

$$y_n = \frac{y_{n+1} + y_n}{4} + \frac{y_{n-1} + y_n}{4} - \frac{\tau_{n+1} - \tau_n}{4} \frac{y_{\bar{t}} + y_t}{2} - \frac{(\tau_n^*)^2}{4} y_{\bar{t}\hat{t}} \ , \tag{2.9}$$

$$y_{n-1} = \frac{y_{n-1} + y_n}{2} - \frac{\tau_n}{2} y_{\bar{t}}, \quad y_{n+1} = \frac{y_n + y_{n+1}}{2} + \frac{\tau_{n+1}}{2} y_t \ , \tag{2.10}$$

$$y^{(\sigma_1,\sigma_2)} = y_n + \frac{\sigma_1 - \sigma_2}{2}(y_{n+1} - y_{n-1}) + \frac{\sigma_1 + \sigma_2}{2}(y_{n+1} - 2y_n + y_{n-1}) \ ,$$

$$y^{(\sigma_1,\sigma_2)} = y_n + (\sigma_1 \tau_{n+1} - \sigma_2 \tau_n)\, y_{\overset{\circ}{t}} + \frac{\sigma_1 + \sigma_2}{2} \tau_n \tau_{n+1} y_{\bar{t}\hat{t}} \ , \tag{2.11}$$

$$y_{\bar{t}} = y_{\overset{\circ}{t}} - \frac{\tau_{n+1}}{2} y_{\bar{t}\hat{t}}, \quad y_t = \frac{y_t + y_{\bar{t}}}{2} + \frac{\tau_n^*}{2} y_{\bar{t}\hat{t}} \ , \tag{2.12}$$

$$y_{\overset{\circ}{t}} = y_t - \frac{\tau_n}{2} y_{\bar{t}\hat{t}}, \quad y_{\overset{\circ}{t}} = \frac{y_{\bar{t}} + y_t}{2} + \frac{\tau_{n+1} - \tau_n}{4} y_{\bar{t}\hat{t}} \ , \tag{2.13}$$

$$y_{\overset{\circ}{t}} = \frac{y_n + y_{n+1}}{2\tau_n^*} - \frac{y_{n-1} + y_n}{2\tau_n^*} \ , \tag{2.14}$$

$$y_{\bar{t}\hat{t}} = \frac{y_{n+1} - 2y_n + y_{n-1}}{\tau_n \tau_{n+1}} - \frac{(\tau_{n+1} - \tau_n)}{\tau_n \tau_{n+1}} \frac{(y_{n+1} - y_{n-1})}{2\tau^*} \ . \tag{2.15}$$

3. Canonical forms of three-level operator-difference schemes

We let $B_\alpha : H \to H, \alpha = 0, 1, 2$, be linear operators in H dependent on τ_n, t_n, and consider the Cauchy problem for the operator-difference equation:

$$B_2 y_{n+1} + B_1 y_n + B_0 y_{n-1} = \varphi_n, \quad y_0 = u_0, \quad y_1 = u_1. \tag{3.1}$$

Obtaining general criteria of stability is based on using some canonical form to write an operator-difference scheme. If the canonical form is appropriately selected, the stability criteria are written in the simplest operator inequalities most convenient for practical use.

The traditional canonical form for the three-level difference scheme (3.1) has the form (1.5)

$$D y_{\bar{t}\hat{t}} + B y_{\overset{\circ}{t}} + A y = \varphi, \quad t \in \hat{\omega}_\tau, \quad y_0 = u_0, \quad y_1 = u_1, \tag{3.2}$$

where

$$A = B_0 + B_1 + B_2, \quad B = \tau^*(B_2 - B_0), \quad D = \frac{\tau\tau_+}{2}(B_2 + B_0),$$

$$B_2 = \frac{1}{\tau_+\tau^*}D + \frac{1}{2\tau^*}B, \quad B_1 = A - \left(\frac{1}{\tau_+\tau^*} + \frac{1}{\tau\tau^*}\right)D, \quad B_0 = \frac{1}{\tau\tau^*}D - \frac{1}{2\tau^*}B.$$

The canonical form (3.2) can be directly associated with an approximation of evolution equation

$$D\frac{d^2u}{dt^2} + B\frac{du}{dt} + Au = f.$$

Below, it will be shown that it is convenient to carry out investigation of stability of three-level schemes in canonical form (3.2) when $\tau_{n+1} \geqslant \tau_n$.

We will also examine the three-level difference scheme (3.1) in the form

$$D\frac{y_{n+1} - 2y_n + y_{n-1}}{\tau_n\tau_{n+1}} + B\frac{y_{n+1} - y_{n-1}}{\tau_n + \tau_{n+1}} + Ay_n = \varphi_n,$$
$$y_0 = u_0, \quad y_1 = u_1, \tag{3.3}$$

where

$$A = B_0 + B_1 + B_2, \quad B = \tau^*(B_2 - B_0), \quad D = \frac{\tau\tau_+}{2}(B_2 + B_0),$$

$$B_2 = \frac{1}{\tau\tau_+}D + \frac{1}{2\tau^*}B, \quad B_1 = A - \frac{2}{\tau\tau_+}D, \quad B_0 = \frac{1}{\tau\tau_+}D - \frac{1}{2\tau^*}B.$$

The given canonical form is convenient in obtaining *a priori* estimates for the case $\tau_{n+1} \leqslant \tau_n$.

Let us define the space H^2 as a set of vectors:

$$Y = \left\{Y^{(1)}, Y^{(2)}\right\}, \quad Y^{(\alpha)} \in H, \quad \alpha = 1, 2,$$

in which addition and multiplication by a number are carried out coordinate-wise:

$$Y + V = \left\{Y^{(1)} + V^{(1)}, Y^{(2)} + V^{(2)}\right\}, \quad \alpha Y = \left\{\alpha Y^{(1)}, \alpha Y^{(2)}\right\}.$$

Since a scalar product $(\cdot, \cdot)$ is introduced in the space H, then in the space H^2 we also can introduce in the scalar product

$$(Y, V) = \sum_{\alpha=1}^{2} \left(Y^{(\alpha)}, V^{(\alpha)}\right).$$

The three-level scheme (3.2) can be reduced to a two-level scheme:

$$\mathcal{B}Y_t + \mathcal{A}Y = \Phi, \quad Y_1 = U_1 \ , \tag{3.4}$$

where

$$Y = Y_n \in H^2, \quad Y_{t,n} = (Y_{n+1} - Y_n)/\tau_{n+1}, \tag{3.5}$$

$\Phi = \Phi_n \in H^2$, $\mathcal{A}$ and $\mathcal{B}$ are operators acting in the space H^2. For this it is sufficient to define the vectors

$$Y_n = \left\{\frac{1}{2}\ (\check{y} + y)\ ,\ y_{\bar{t}}\right\}, \quad Y_t = \left\{\frac{\tau^*}{\tau_+}y_{\overset{\circ}{t}}, \frac{\tau^*}{\tau_+}y_{\bar{t}\hat{t}}\right\}, \tag{3.6}$$

$$\Phi = \{\varphi, 0\}, \quad U_1 = \left\{ \frac{u_0 + u_1}{2}, \frac{u_1 - u_0}{\tau_1} \right\} \tag{3.7}$$

and the operators $\mathcal{A}$ and $\mathcal{B}$ as operator matrices:

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & D - \frac{\tau\tau_+}{4}A \end{pmatrix}, \tag{3.8}$$

$$\mathcal{B} = \frac{\tau_+}{\tau^*} \begin{pmatrix} B + \frac{\tau}{2}A & D - \frac{\tau\tau_+}{4}A \\ -(D - \frac{\tau\tau_+}{4}A) & \frac{\tau_+}{2}\left(D - \frac{\tau\tau_+}{4}A\right) \end{pmatrix} \tag{3.9}$$

with elements being operators on H.

We calculate a norm of the vector Y_n in $H^2_{\mathcal{A}}$:

$$\|Y_n\|^2_{\mathcal{A}} = \frac{1}{4}\|y_{n-1} + y_n\|^2_A + \|y_{\bar{t},n}\|^2_{D - \frac{\tau_n\tau_{n+1}}{4}A}. \tag{3.10}$$

It can be easily seen that if

$$A = A^* > 0, \quad D^* = D > \frac{\tau_n\tau_{n+1}}{4}A,$$

then functional (3.10) satisfies all axioms of the norm, namely

$$\|\alpha Y_n\|_{\mathcal{A}} = |\alpha|\|Y_n\|_{\mathcal{A}},$$
$$\|Y_n\|_{\mathcal{A}} \geqslant 0 \qquad \text{for all } y_{n-1} \in H, \quad y_n \in H,$$

and

$$\|Y_n\|_{\mathcal{A}} = 0, \qquad \text{only when } y_{n-1} = y_n = 0;$$
$$\|\bar{Y}_n + Y_n\|_{\mathcal{A}} \leqslant \|\bar{Y}_n\|_{\mathcal{A}} + \|Y_n\|_{\mathcal{A}}.$$

We obtain some estimates for the norm of the vector Y in $H^2_{\mathcal{A}}$. They are proved similarly the case of the uniform time grid [9; pp. 449–451].

Lemma 3.1. *Let the following conditions be satisfied for the operators A, D:*

$$A = A^* > 0, \quad D = D^* \geqslant \frac{\tau_{n+1}(\tau_n + \varepsilon\tau_{n+1})}{4}A, \quad \varepsilon > 0.$$

Then the following inequality holds:

$$\|Y\|_{\mathcal{A}} \geqslant \sqrt{\frac{\varepsilon}{1+\varepsilon}}\|y\|_A. \tag{3.11}$$

Proof. Let us convert the norm $\|Y_{n+1}\|^2_{\mathcal{A}_{n+1}}$ with the aid of the identity $y_n = y_{n+1} - \tau_{n+1}y_t$ and ε-inequality:

$$\begin{aligned} \|Y_{n+1}\|^2_{\mathcal{A}_{n+1}} &= \frac{1}{4}\|2y_{n+1} - \tau_{n+1}y_t\|^2_A + \|y_t\|^2_{D - \frac{\tau_n\tau_{n+1}}{4}A} \\ &= \|y_t\|^2_{D - \frac{\tau_n\tau_{n+1}}{4}A + \frac{\tau^2_{n+1}}{4}A} + \|y_{n+1}\|^2_A - (Ay_{n+1}, \tau_{n+1}y_t) \\ &\geqslant \|y_t\|^2_{D - \frac{\tau_n\tau_{n+1}}{4}A + \frac{\tau^2_{n+1}}{4}A} + \|y_{n+1}\|^2_A - \varepsilon_1\|y_{n+1}\|^2_A - \frac{\tau^2_{n+1}}{4\varepsilon_1}\|y_t\|^2_A. \end{aligned}$$

Choosing $\varepsilon_1 = 1/(1+\varepsilon)$ in the last inequality and taking into consideration the condition of Lemma 3.1 we arrive at the required estimate (3.11). □

4. Stability on initial data

In what follows we will examine three-level schemes on a nonuniform in time grid in the following canonical form:

$$Dy_{\bar{t}\hat{t}} + By_{\overset{\circ}{t}} + Ay = 0, \quad t \in \hat{\omega}_\tau, \quad y_0 = u_0, \quad y_1 = u_1 \ . \tag{4.1}$$

Let us investigate the stability of scheme (4.1) in the sense of the initial data depending on the ratio between the steps τ_n and τ_{n+1}. Before formulating the main theorem, we prove auxiliary statements about the stability of scheme (4.1) when $\tau_{n+1} \geqslant \tau_n$ and $\tau_{n+1} \leqslant \tau_n$, respectively.

Theorem 4.1. *Let the operators of scheme* (4.1) *satisfy the conditions*

$$D_n = D(t_n) = D_n^* > 0, \quad B_n = B(t_n) > 0, \quad A_n = A(t_n) = A_n^* > 0,$$

$$R_n = D_n - \frac{\tau_n \tau_{n+1}}{4} A_n > 0,$$

$$B_n \geqslant \frac{\tau_{n+1} - \tau_n}{4} A_n, \tag{4.2}$$

$$\tau_{n+1} \geqslant \tau_n \ . \tag{4.3}$$

Then, the solution of problem (4.1) *is stable in sense of initial data, and the following a priori estimates hold:*

$$\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2 \leqslant \|y_{t,0}\|_R^2 + \|y_0^{(0.5)}\|_A^2, \tag{4.4}$$

if the operators R, A are constant;

$$\|y_{t,n}\|^2_{D-\frac{\tau_n\tau_{n+1}}{4}A} + \|y_n^{(0.5)}\|_A^2 \leqslant \|y_{t,0}\|^2_{D-\frac{\tau_1\tau_2}{4}A} + \|y_0^{(0.5)}\|_A^2, \tag{4.5}$$

if the operators D, A are constant;

$$\|y_{t,n}\|_{R_n}^2 + \|y_n^{(0.5)}\|_{A_n}^2 \leqslant M \left\{ \|y_{t,0}\|_{R_0}^2 + \|y_0^{(0.5)}\|_{A_0}^2 \right\}, \tag{4.6}$$

$$M = e^{c_0 t_{n+1}}, \quad c_0 = \max\{c_2, c_3\}, \quad y^{(\sigma)} = \sigma y_{n+1} + (1-\sigma) y_n,$$

if the operators R, A are Lipschitz continuous over the variable t with the constants c_2, c_3, respectively.

Proof. In order to get an energy identity let us scalarly multiply scheme (4.1) by $2\tau^* y_{\overset{\circ}{t}}$. By virtue of the second identity (2.13) we have the following representations of the scalar products $\left(Dy_{\bar{t}\hat{t}}, y_{\overset{\circ}{t}}\right)$, $\left(By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}\right)$:

$$2\tau^* \left(Dy_{\bar{t}\hat{t}}, y_{\overset{\circ}{t}}\right) = \|y_t\|_D^2 - \|y_{\bar{t}}\|_D^2 + \tau^* \frac{\tau_+ - \tau}{2} \|y_{\bar{t}\hat{t}}\|_D^2,$$

$$2\tau^* \left(By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}\right) = 2\tau^* \|y_{\overset{\circ}{t}}\|_B^2.$$

Using identities (2.9), (2.13), (2.14), for the scalar product $\left(Ay, y_{\overset{\circ}{t}}\right)$ we get the representation

$$2\tau^* \left(Ay, y_{\overset{\circ}{t}}\right) = \left\|\frac{\hat{y}+y}{2}\right\|_A^2 - \left\|\frac{y+\check{y}}{2}\right\|_A^2 + \|y_t\|^2_{-\frac{\tau\tau_+}{4}A} - \|y_{\bar{t}}\|^2_{-\frac{\tau\tau_+}{4}A}$$
$$+\,\tau^*\frac{\tau_+-\tau}{2}\|y_{\bar{t}t}\|^2_{-\frac{\tau\tau_+}{4}A} + 2\tau^*\|y_{\overset{\circ}{t}}\|^2_{-\frac{\tau_+-\tau}{4}A}\ .$$

Adding the obtained expressions, we find the energy identity:

$$\left\|\frac{\hat{y}+y}{2}\right\|_A^2 + \|y_t\|_R^2 + 2\tau^*\|y_{\overset{\circ}{t}}\|^2_{B-\frac{\tau_+-\tau}{4}A} + \tau^*\frac{\tau_+-\tau}{2}\|y_{\bar{t}t}\|^2_R = \left\|\frac{y+\check{y}}{2}\right\|_A^2 + \|y_{\bar{t}}\|^2_R\ .$$

Using conditions (4.2), (4.3) of the theorem, we obtain the energy inequality:

$$\|y_t\|_R^2 + \left\|\frac{\hat{y}+y}{2}\right\|_A^2 \leqslant \|y_{\bar{t}}\|_R^2 + \left\|\frac{y+\check{y}}{2}\right\|_A^2\ . \tag{4.7}$$

Let us examine three cases separately.

1. Let the operators R, A be constant. Then at once we arrive at the *a priori* estimate (4.4). We note immediately that for the three-level scheme with weights (1.4) the operator
$$R = D - \frac{\tau\tau_+}{4}A = E + \frac{\tau\tau_+}{4}(\sigma_1+\sigma_2-0.5)A$$
does not depend on t if $\sigma_1+\sigma_2 = 0.5$.

2. Let the operators R, A be variable. Using the condition of Lipschitz continuity of these operators, from (4.7) we get the relation
$$\|y_t\|^2_{R_n} + \|y_n^{(0.5)}\|^2_{A_n} \leqslant (1+c_0\tau_n)\left\{\|y_{\bar{t}}\|^2_{R_{n-1}} + \|y_{n-1}^{(0.5)}\|^2_{A_{n-1}}\right\}.$$
Continuing recurrently the latter inequality, we obtain estimate (4.6).

3. In the case of the constant operators D, A, from inequality (4.7) we have estimate (4.5).

□

In order to obtain *a priori* estimates on inverse relations of steps $\tau_{n+1} \leqslant \tau_n$, we reduce difference scheme (4.1) to the form (3.3). Using identity (2.15), we get the following representation:

$$D\frac{y_{n+1}-2y_n+y_{n-1}}{\tau_n\tau_{n+1}} + \left(B - \frac{\tau_{n+1}-\tau_n}{\tau_{n+1}\tau_n}D\right)\frac{y_{n+1}-y_{n-1}}{2\tau^*} + Ay_n = 0\ . \tag{4.8}$$

Theorem 4.2. *Assume that the operators $D(t), B(t), A$ satisfy the conditions*

$$D(t) = D^*(t) > 0, \quad B(t) > 0, \quad A = A^* > 0,$$

$$R = D - \frac{\tau_n\tau_{n+1}}{4}A > 0,$$

A, R are constant operators and

$$\tau_{n+1} \leqslant \tau_n \ .$$

Then, the operator-difference scheme (4.1) *is stable in the sense of the initial data and the following* a priori *estimate holds:*

$$\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2 \leqslant \left(\frac{\tau_1}{\tau_{n+1}}\right)^2 \left\{\|y_t(0)\|_R^2 + \|y_0^{(0.5)}\|_A^2\right\} \ . \tag{4.9}$$

Proof. Let us scalarly multiply scheme (4.8) by $\tau_n\tau_{n+1}(y_{n+1} - y_{n-1})$ and consider each term separately.

Since the operator D is self-adjoint, we obtain the representation

$$\begin{aligned}(D((y_{n+1} - y_n) - (y_n - y_{n-1})), (y_{n+1} - y_n) + (y_n - y_{n-1}))\\ = \|y_{n+1} - y_n\|_D^2 - \|y_n - y_{n-1}\|_D^2 = \tau_{n+1}^2\|y_t\|_D^2 - \tau_n^2\|y_{\bar{t}}\|_D^2.\end{aligned} \tag{4.10}$$

For the second item we have the expression

$$\left(\left(\frac{\tau_n\tau_{n+1}}{2\tau^*}B - \frac{\tau_{n+1} - \tau_n}{2\tau^*}D\right)(y_{n+1} - y_{n-1}), (y_{n+1} - y_{n-1})\right) \geqslant 0.$$

Using identities (2.8), (2.10) and taking into account that A is the self-adjoint operator, the third component can be represented in the form

$$\begin{aligned}\tau_n\tau_{n+1}(Ay_n, y_{n+1} - y_{n-1}) &= \tau_n\tau_{n+1}(Ay_n, y_{n+1}) - \tau_n\tau_{n+1}(Ay_n, y_{n-1})\\ &= \tau_n\tau_{n+1}\left(\|y_n^{(0.5)}\|_A^2 - \frac{\tau_{n+1}^2}{4}\|y_t\|_A^2\right)\\ &\quad - \tau_n\tau_{n+1}\left(\|y_{n-1}^{(0.5)}\|_A^2 - \frac{\tau_n^2}{4}\|y_{\bar{t}}\|_A^2\right).\end{aligned}$$

Summing the estimates obtained we arrive at the inequality

$$\tau_{n+1}^2\|y_t\|_R^2 + \tau_n\tau_{n+1}\|y_n^{(0.5)}\|_A^2 \leqslant \tau_n^2\|y_{\bar{t}}\|_R^2 + \tau_n\tau_{n+1}\|y_{n-1}^{(0.5)}\|_A^2 \ .$$

Taking into consideration the assumption of Theorem 4.2, the latter identity can be rewritten in the form

$$\tau_{n+1}^2\left\{\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2\right\} \leqslant \tau_n^2\left\{\|y_{t,n-1}\|_R^2 + \|y_{n-1}^{(0.5)}\|_A^2\right\}. \tag{4.11}$$

By virtue of the arbitrariness of n the required estimate follows from (4.11). □

5. Stability on arbitrary grids

Let us combine the results obtained and formulate the general theorem about uniform stability of three-level operator-difference schemes. We shall examine a more complicated situation, when the principle of refinement of the grid can be changed k times at the nodes $0 < t_{m_1} < t_{m_2} < \cdots < t_{m_k} \leqslant T$. Assume also that the time grid varies according to the following principle:

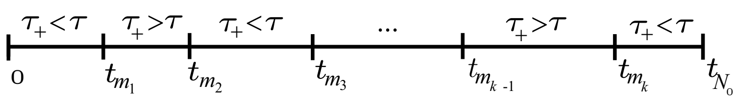

Figure 1.

We consider the conditions of Theorems 4.1, 4.2 are fulfilled, and operators R, A are constant.

Let $t \in [t_{m_k}, t_{N_0}]$ and the become finer to the end of the interval. Then, according to the theorem 4.2 the following estimate is true:

$$\|y_n\|_1^2 \leqslant \left(\frac{\tau_{m_k}}{\tau_{N_0}}\right)^2 \|y_{m_k}\|_1^2 \ , \quad n = m_k + 1, m_k + 2, \ldots, N_0 \ , \tag{5.1}$$

where

$$\|y_n\|_1^2 = \|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2 \ .$$

Now, at the time instant t_{m_k} the change occurred in the principle of the grid refinemet, i.e., the time steps became related as $\tau_+ > \tau$. Then, according to Theorem 4.1

$$\|y_{m_k}\|_1^2 \leqslant \|y_{m_{k-1}}\|_1^2.$$

Substituting the last inequality to (5.1) we receive an estimate

$$\|y_n\|_1^2 \leqslant \left(\frac{\tau_{m_k}}{\tau_{N_0}}\right)^2 \|y_{m_{k-1}}\|_1^2 \ , \quad n = m_{k-1} + 1, \ldots, N_0.$$

Reasoning similarly for all $t \in [0, t_{N_0}]$, we note the correctness of the following theorem.

Theorem 5.1. *Assume that the operators $D(t)$, $B(t)$, A satisfy the conditions*

$$D(t) = D^*(t) > 0, \quad A = A^* > 0,$$

$$R = D - \frac{\tau_n \tau_{n+1}}{4} A > 0, \quad B \geqslant \max\left\{\frac{\tau_{n+1} - \tau_n}{4} A, 0\right\},$$

and A, R are constant operators. Let the time steps be interrelated as

$$\frac{\tau_{m_j}}{\tau_{m_{j+1}}} \leqslant c_{m_{j+1}} \leqslant c_4, \quad j = 0, 1, \ldots, k,$$
$$\tau_{m_0} = \tau_1,$$
$$\tau_{m_{k+1}} = \tau_{N_0},$$

where k is the finite number of changes of the principle of grid refinemet.

Then the solution of problem (4.1) *is stable in the sense of the initial data, and for any τ_n a priori estimate is valid (an absolute stability)*

$$\|y_{t,n}\|_R^2 + \|y_n^{(0.5)}\|_A^2 \leqslant c_4^k \left(\|y_{t,0}\|_R^2 + \|y_0^{(0.5)}\|_A^2\right). \tag{5.2}$$

Remark 5.1. According to Lemma 3.1, from the *a priori* estimate (5.2) the stability of the scheme in the sense of the initial data in the energy norm $\|y\|_A$ also follows.

6. Stability on the right-hand side

For the simplicity, in this section we limit our discussion to examination of the time grid refinemet to the coordinate origin:

$$\tau_{n+1} \geqslant \tau_n. \tag{6.1}$$

Therefore, we turn to the canonical form of three-level schemes in the form of (3.2):

$$Dy_{\bar{t}\hat{t}} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad t \in \hat{\omega}_\tau, \quad y_0 = u_0, \quad y_1 = u_1. \tag{6.2}$$

Theorem 6.1. *Assume that the operators D, B, A satisfy the conditions*

$$A = A^* > 0, \quad D = D^* \geqslant \frac{\tau_+(\tau + \varepsilon\tau_+)}{4} A, \quad B \geqslant \frac{\tau_+ - \tau}{4} A, \quad \tau_+ \geqslant \tau.$$

Operators A, $R = D - \frac{\tau\tau_+}{4} A$ are constant. Then, difference scheme (6.2) *is stable in the sense of the initial data, on the right-hand side and the a priori estimate holds:*

$$\|y_{n+1}\|_A \leqslant \sqrt{\frac{1+\varepsilon}{\varepsilon}} \left(\|y_0\|_A + \|y_t(0)\|_D + \|\varphi_0\|_{A^{-1}} + \|\varphi_n\|_{A^{-1}} + \sum_{k=1}^{n} \tau_k \|\varphi_{\bar{t},k}\|_{A^{-1}} \right). \tag{6.3}$$

Proof. Let us use the representation of a three-level scheme in the form of a two-level scheme (3.4)–(3.9):

$$\mathcal{B}Y_t + \mathcal{A}Y = \Phi, \quad Y_1 = U_1.$$

We shall apply Lemma 2.1 to investigate the stability. For that, it is sufficient to show the validity of the operator inequality in H^2:

$$\mathcal{B} - \frac{\tau_+}{2}\mathcal{A} \geqslant 0. \tag{6.4}$$

According to (3.8), (3.9), we have

$$\left(\mathcal{B} - \frac{\tau_+}{2}\mathcal{A}\right) = \frac{\tau_+}{\tau^*} \begin{pmatrix} B - \frac{\tau_+ - \tau}{4} A & R \\ -R & \frac{\tau_+ - \tau}{4} R \end{pmatrix}.$$

Since owing to the assumptions of the theorem

$$B - \frac{\tau_+ - \tau}{4} A \geqslant 0, \quad R = D - \frac{\tau\tau_+}{4} A \geqslant 0,$$

then the operating inequality (6.4) holds for any vector $Y \in H^2$.

According to the *a priori* estimate (2.3), we conclude that

$$\|Y_{n+1}\|_{\mathcal{A}} \leqslant \|Y_1\|_{\mathcal{A}} + \|\Phi_0\|_{\mathcal{A}^{-1}} + \|\Phi_n\|_{\mathcal{A}^{-1}} + \sum_{k=1}^{n} \|\Phi_k - \Phi_{k-1}\|_{\mathcal{A}^{-1}}. \tag{6.5}$$

Note that by virtue of definition (3.10) and conditions of the theorem

$$\begin{aligned} \|Y_1\|_{\mathcal{A}} &= \left\{ \|y_0^{(0.5)}\|_A^2 + \|y_{t,0}\|_R^2 \right\}^{1/2} \leqslant \left\{ \left\| y_0 + \frac{\tau_1}{2} y_{t,0} \right\|_A^2 + \|y_{t,0}\|^2_{D - \frac{\tau_1^2}{4}A} \right\}^{1/2} \\ &= \left\{ \|y_0\|_A^2 + \tau_1(Ay_0, y_{t,0}) + \|y_{t,0}\|_D^2 \right\}^{1/2} \leqslant \left\{ \|y_0\|_A^2 + 2\|y_0\|_A \|y_{t,0}\|_D + \|y_{t,0}\|_D^2 \right\}^{1/2} \\ &\leqslant \|y_0\|_A + \|y_{t,0}\|_D. \end{aligned}$$

According to Lemma 3.1

$$\|Y\|_{\mathcal{A}} \geqslant \sqrt{\frac{\varepsilon}{1+\varepsilon}}\, \|y\|_{A}.$$

Substituting these estimates into inequality (6.5) and using an explicit representation of the vector Φ and operator $\mathcal{A}$, we come to the required estimate (6.3). □

Remark 6.1. Assume that

$$\lim_{n\to\infty} \|\varphi_n\|_{A^{-1}}\,, \quad \sum_{k=1}^{\infty} \tau_k \|\varphi_{\bar{t},k}\|_{A^{-1}} < \infty.$$

Then the estimate (6.3) expresses estimate of a long time stability.

7. Examples

In this section the *a priori* estimates obtained is applied to study the stability of particular difference schemes with weights approximating initial boundary-value problems for parabolic and wave equations. It should be note separately results concerning construction and investigation of the schemes of the second order of local approximation on nonuniform in time and space grids using standard stensils of the difference schemes. We note also that raising the order of local approximation is very important requirement for numerical solution of the mathematical physics problems on coarse grids [12].

7.1. Schemes with weights

Let us consider the first boundary-value problem for one-dimensional hyperbolic equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{7.1}$$

$$u(0,t) = u(l,t) = 0, \quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \bar{u}_0(x). \tag{7.2}$$

On a uniform space grid $\bar{\omega}_h = \{x_i = ih, \quad i = \overline{0,N}, \quad hN = 1\}$ we replace the problem (7.1), (7.2) by the differential-difference problem (1.3)

$$\frac{d^2 v}{dt^2} + Av = \varphi(t), \quad v_0 = u_0, \quad \frac{dv(0)}{dt} = \bar{u}_0, \tag{7.3}$$

where

$$(Av)_i = -v_{\bar{x}x,i}, \quad i = 1,2,\ldots,N-1, \quad v_0 = v_N = 0,$$

$$v_{\bar{x}x,i} = (v_{i+1} - 2v_i + v_{i-1})/h^2, \quad \varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_{N-1}(t))^T, \quad \varphi_i(t) = f(x_i,t);$$

the operator $A : H \to H$, where the linear space $H = \Omega_h$ consists of a set of vectors $v = (v_1(t), v_2(t), \ldots, v_{N-1}(t))^T$; a scalar product and a norm in H are assigned as usual

$$(y,v) = \sum_{i=1}^{N-1} h y_i v_i, \quad \|y\| = \sqrt{(y,y)}.$$

It is easy to see that $A = A^* > 0$ is a constant operator, since according to homogeneous boundary conditions

$$\|v\|_A^2 = (Av, v) = -(v_{\bar{x}x}, v) = \|v_{\bar{x}}]|^2 = \sum_{i=1}^{N} h v_{\bar{x},i}^2,$$

$$v_{\bar{x},i} = (v_i - v_{i-1})/h, \quad v_{x,i} = (v_{i+1} - v_i)/h.$$

For numerical solution of problem (7.3) on the nonuniform in time grid let us consider the class of difference schemes with weights (1.4):

$$y_{\bar{t}\hat{t}} + Ay^{(\sigma_1,\sigma_2)} = \varphi, \quad y_0 = u_0, \quad y_1 = u_1. \tag{7.4}$$

Reduction of scheme (7.4) to canonical form (1.5), (6.2) gives (see (1.6))

$$D(t_n) = E + \frac{\sigma_1 + \sigma_2}{2}\tau_n\tau_{n+1}A, \quad B(t_n) = (\sigma_1\tau_{n+1} - \sigma_2\tau_n)A.$$

Let us check the conditions of Theorem 5.1. It is obvious that A is the constant operator by definition, and the operator

$$R = D - \frac{\tau_n\tau_{n+1}}{4}A = E + \frac{\tau_n\tau_{n+1}}{2}\left(\sigma_1 + \sigma_2 - \frac{1}{2}\right)A = E$$

will be constant only when $\sigma_1 + \sigma_2 = \frac{1}{2}$. Assume that $\sigma_1 \geqslant \sigma_2$. Then, the inequality (4.2)

$$B - \frac{\tau_{n+1} - \tau_n}{4}A = \left[(\sigma_1 - \sigma_2)\tau_n^* + \frac{\tau_{n+1} - \tau_n}{2}\left(\sigma_1 + \sigma_2 - \frac{1}{2}\right)\right]A \geqslant 0$$

will be satisfied under the above assumptions. Therefore the following theorem is true.

Theorem 7.1. *Assume that*

$$\sigma_1 \geqslant \sigma_2, \quad \sigma_1 + \sigma_2 = \frac{1}{2}. \tag{7.5}$$

Then, the difference scheme with weights (7.4) *at* $\varphi = 0$ *is stable in the sense of the initial data and a priori estimate holds*

$$\|y_{t,n}\|^2 + \|y_n^{(0.5)}\|_A^2 \leqslant c_4^k\left(\|y_{t,0}\|^2 + \|y_0^{(0.5)}\|_A^2\right). \tag{7.6}$$

7.2. Uniform stability of the three-level scheme of the second order of local approximation $O(\tau^2)$

Let

$$(x, \bar{t}) = \left(x, t + \frac{\tau_+ - \tau}{3}\right) = \left(x, \frac{1}{3}(t_{n-1} + t_n + t_{n+1})\right),$$

$$\sigma_1\tau_+ - \sigma_2\tau = \frac{\tau_+ - \tau}{3}. \tag{7.7}$$

Then, using Taylor's formula and identities (2.11), (2.13), it is easy to show the correctness of of the relationships

$$u_{\bar{t}\hat{t}} - \frac{\partial^2 u}{\partial t^2}(x,\bar{t}) = \frac{\partial^2 u}{\partial t^2}(x,t) + \frac{\tau_+ - \tau}{3}\frac{\partial^3 u}{\partial t^3}(x,t) + O(\tau^{*2}) - \frac{\partial^2 u}{\partial t^2}(x,\bar{t}) = O(\tau^{*2}),$$

$$v^{(\sigma_1,\sigma_2)} = v + (\sigma_1\tau_+ - \sigma_2\tau)v_{\overset{\circ}{t}} + \frac{\sigma_1+\sigma_2}{2}\tau\tau_+ v_{\bar{t}\hat{t}} = v + \frac{\tau_+ - \tau}{3}\left(v_t - \frac{\tau}{2}v_{\bar{t}\hat{t}}\right) + O(\tau^{*2})$$
$$= v(x,\bar{t}) + O(\tau^{*2}).$$

Consequently, choosing in (7.4) $\varphi = f(x,\bar{t})$, $u_1 = u_0 + \tau_1 u_1 + 0.5\tau_1^2 u_0''(x)$, we satisfy ourselves that difference scheme (7.4) approximates the initial problem (7.1) with the second order of local approximation:

$$\psi = \left\{-u_{\bar{t}\hat{t}} + u_{\bar{x}x}^{(\sigma_1,\sigma_2)} + \varphi\right\} - \left\{-\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + f\right\}(x,\bar{t}) = O(h^2 + \tau^{*2}).$$

Combining the conditions of uniform stability (7.5) and the condition of the second order of approximation (7.7), we find

$$\sigma_1 = \frac{2\tau_+ + \tau}{6(\tau_+ + \tau)}, \quad \sigma_2 = \frac{\tau_+ + 2\tau}{6(\tau_+ + \tau)}. \tag{7.8}$$

The condition

$$\sigma_1 - \sigma_2 = \frac{\tau_+ - \tau}{6(\tau_+ + \tau)} \geqslant 0$$

is fulfilled, when $\tau_+ \geqslant \tau$.

Thus, difference scheme (7.4), (7.8) with the conditions defined above is uniformly stable in the sense of the initial data, and for its solution the *a priori* estimate (7.6) holds.

7.3. Parabolic equation

Let us consider the first initial boundary-value problem for one dimensional parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right), \quad 0 < x < l, \quad t > 0, \tag{7.9}$$

$$u(0,t) = u(l,t) = 0, \quad t > 0, \quad u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant l, \tag{7.10}$$

where $0 < c_1 \leqslant k(x) \leqslant c_2$, $c_1, c_2 = \text{const}$. On a uniform, in space and time, variable grid $\bar{\omega} = \bar{\omega}_h \times \hat{\bar{\omega}}_\tau$ let us approximate differential problem (7.9), (7.10) by the following finite difference scheme

$$y_t + 0.5\tau_+ y_{\bar{t}\hat{t}} = (a\hat{y}_{\bar{x}})_x, \tag{7.11}$$

$$\hat{y}_0 = \hat{y}_N = 0, \quad y(x,0) = u_0(x). \tag{7.12}$$

Here $a = a_i = 0.5(k_{i-1} + k_i)$, $k_i = k(x_i)$.

It is easy to verify that at the node (x_i, t_{n+1}) the three level scheme (7.11), (7.12) approximates differential problem with the second order, that is

$$\psi_i^{n+1} = -u_{t,i} - 0.5\tau_{n+1}u_{\bar{t}\hat{t},i} + (au_{\bar{x}}^{n+1})_{x,i} = O(h^2 + \tau_{n+1}^2).$$

Scheme (7.11) is one generalization of the well-known asymptotically stable scheme:

$$\frac{3}{2}y_t - \frac{1}{2}y_{\bar{t}} = (a\hat{y}_{\bar{x}})_x$$

on a nonuniform time grid. Scheme (7.11), (7.12) can be transformed into operator finite difference scheme (6.2) by putting

$$B = E + \tau_+ A, \quad D = \tau^* E + 0.5\tau\tau_+ A.$$

It is evident that

$$R = D - \frac{\tau\tau_+}{4}A = \tau^* E + \frac{\tau\tau_+}{4}A > 0,$$

$$B - \frac{\tau_+ - \tau}{4}A = E + \left(\frac{3}{4}\tau_+ + \frac{\tau}{4}\right) A > 0.$$

Hence, if the grid is quasiuniform in time (condition (1.8) holds), then the operator R is Lipschitz continuous and from the *a priori* estimates (4.6), (4.11) the stability of the difference scheme (7.11), (7.12) in the sense of the initial data follows.

7.4. Finite difference schemes of a raised order of approximation on spatially and temporally nonuniform grids

Suppose that in the domain $\bar{Q}_T$ it is required to find the continuous function $u(x,t)$ satisfying the following initial boundary-value problem:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad 0 < t < T,$$

$$u(0,t) = u(1,t) = 0, \quad u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = \bar{u}_0(x).$$

Let us consider the following spatially and temporally nonuniform grid $\bar{\omega} = \hat{\bar{\omega}}_h \times \hat{\bar{\omega}}_\tau$:

$$\hat{\bar{\omega}}_h = \{x_i = x_{i-1} + h_i, \quad i = 1, 2, \ldots, N, \quad x_0 = 0, \quad x_N = 1\} = \hat{\omega}_h \cup \{x_0 = 0, \quad x_N = 1\}.$$

On this grid we approximate the differential problem by the finite difference one:

$$\begin{aligned} y_{\bar{t}\hat{t}} + \frac{h_+ - h}{3} y_{\bar{t}\hat{t}\hat{x}} &= y_{\bar{x}\hat{x}}^{(\sigma_1,\sigma_2)}, \\ y_0^{n+1} = y_N^{n+1} = 0, \quad y_0 &= u_0, \quad y_1 = u_1. \end{aligned} \tag{7.13}$$

Here the usual notation is used [9] :

$$h_+ = h_{i+1}, \quad h = h_i, \quad y_{\bar{x}} = (y_i^n - y_{i-1}^n)/h_i,$$
$$y_x = (y_+ - y)/h_+, \quad y_\pm = y(x_{i\pm1}, t_n), \quad y_{\bar{x}\hat{x}} = (y_x - y_{\bar{x}})/\hbar, \quad \hbar = 0.5(h + h_+).$$

It is easy to show [6] that at the supplementary node $(\bar{x}_i, \bar{t}_n)$:

$$\bar{x}_i = \frac{1}{3}(x_{i-1} + x_i + x_{i+1}) = x_i + \frac{h_{i+1} - h_i}{3},$$
$$\bar{t}_n = \frac{1}{3}(t_{n-1} + t_n + t_{n+1}) = t_n + \frac{\tau_{n+1} - \tau_n}{3},$$

with

$$\sigma_1 \tau_{n+1} - \sigma_2 \tau_n = \frac{\tau_{n+1} - \tau_n}{3} \tag{7.14}$$

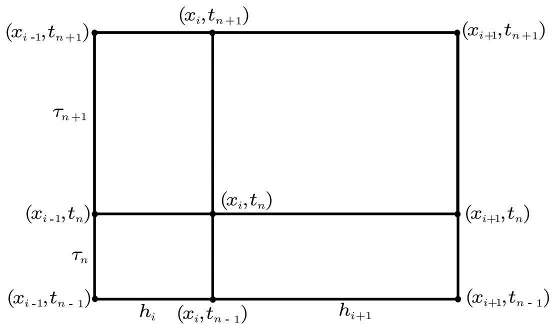

Figure 2.

finite difference scheme (7.13) approximates the differential problem with the second order $O(\hbar_i^2 + \tau_n^{*2})$ on the standard 9-points stencil (see Fig. 2).

For further investigation of finite difference scheme (7.13), (7.14) some known formulas and identities are required:

$$y = \frac{\hat{y} + y}{4} + \frac{y + \check{y}}{4} - \frac{\tau_+ - \tau}{4} y_{\overset{\circ}{t}} - \frac{\tau\tau_+}{4} y_{\bar{t}\hat{t}}, \tag{7.15}$$

$$y^{(\sigma_1,\sigma_2)} = y + (\sigma_1\tau_+ - \sigma_2\tau) y_{\overset{\circ}{t}} + \frac{\sigma_1 + \sigma_2}{2} \tau\tau_+ y_{\bar{t}\hat{t}}, \tag{7.16}$$

$$y_{\overset{\circ}{t}} = \frac{y_t + y_{\bar{t}}}{2} + \frac{\tau_+ - \tau}{4} y_{\bar{t}\hat{t}}. \tag{7.17}$$

Let us introduce scalar products and norms of the functions defined over a spatially nonuniform grid:

$$(y, v)_* = \sum_{i=1}^{N-1} \hbar_i y_i v_i, \quad \|y\|^2 = (y, y)_*, \quad (y, v] = \sum_{i=1}^{N} h_i y_i v_i, \quad \|y]|^2 = (y, y].$$

Lemma 7.1 (Green's first finite difference formula). *For any grid function $y(x)$, which is defined on the nonuniform grid $\hat{\omega}_h$ and vanish at $x = 0$ and $x = 1$ the following formula is valid*

$$(y, v_{\bar{x}\hat{x}})_* = (y_{\bar{x}}, v_{\bar{x}}]. \tag{7.18}$$

One has the following theorem.

Theorem 7.2 [6]. *Let us suppose that*

$$\|\hbar/h\|_C \leqslant c, \quad \|\cdot\|_C = \max_{x\in\hat{\omega}_h} |\cdot|, \quad \tau_{n+1} - \tau_n \geqslant \sqrt{\frac{2c}{3}} \|h_+ - h\|_C, \quad n = 1, 2, \ldots, N-1, \tag{7.19}$$

and

$$\sigma_1^n = \frac{2\tau_{n+1} + \tau_n}{6(\tau_{n+1} + \tau_n)}, \quad \sigma_2^n = \frac{\tau_{n+1} + 2\tau_n}{6(\tau_{n+1} + \tau_n)}. \tag{7.20}$$

Then, the finite difference scheme (7.13) of the second order of local approximation $O(\hbar_i^2 + \tau_n^{*2})$ is uniformly stable and one has the estimate

$$\|y_{t\bar{x}}^n]|^2 + \|y_{\bar{x}\hat{x}}^{(0.5)}(t_n)\|^2 \leqslant \|y_{t\bar{x}}^0]|^2 + \|y_{\bar{x}\hat{x}}^{(0.5)}(0)\|^2. \tag{7.21}$$

Proof. Let us note that σ_1^n, σ_2^n are defined by the formula (7.20) and satisfy relation (7.14), which is necessary for increasing the approximation order on a nonuniform grid. Let us multiply now the finite difference equation (7.13) by $-2\tau^*\hbar_i y_{\overset{\circ}{t}\bar{x}\hat{x},i}$ and sum at inner nodes of spatially nonuniform grid $\hat{\omega}_h$. After application the formula (7.18) we obtain the energy identity:

$$2\tau^*\left(y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}}\right] - 2\tau^*\left(\frac{h_+ - h}{3}y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* + 2\tau^*\left(y^{(\sigma_1,\sigma_2)}_{\bar{x}\hat{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* = 0. \tag{7.22}$$

Applying identity (7.17), one finds the equality

$$2\tau^*\left(y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}}\right] = ||y^n_{t\bar{x}}]|^2 - \left||y^{n-1}_{t\bar{x}}]\right|^2 + 0{,}5\tau^*_n\left(\tau_{n+1} - \tau_n\right)\left||y^n_{\bar{t}\hat{t}\bar{x}}]\right|^2. \tag{7.23}$$

Using now formulas (7.15), (7.16) and condition of the second order approximation (7.14), we obtain for $y^{(\sigma_1,\sigma_2)}$ the following representation:

$$y^{(\sigma_1,\sigma_2)} = \frac{1}{2}\left(y^{(0,5)} + \check{y}^{(0,5)}\right) + \frac{\tau_+ - \tau}{12}y_{\overset{\circ}{t}}. \tag{7.24}$$

In deriving formula (7.24), we used the property

$$\sigma_1^n + \sigma_2^n = \frac{1}{2}. \tag{7.25}$$

Let us note, that if the variable weight multipliers do not satisfy equality (7.25), then it is possible to prove stability of the finite difference scheme (7.13) only on a temporally quasiuniform grid (1.8). Taking into account that

$$y_{\overset{\circ}{t}} = (y^{(0,5)} - \check{y}^{(0,5)})/\tau^*$$

and using (7.24) for the third term in (7.22), we can find the following equality:

$$2\tau^*\left(y^{(\sigma_1,\sigma_2)}_{\bar{x}\hat{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* = \left\|y^{(0,5)}_{\bar{x}\hat{x}}(t_n)\right\|^2 - \left\|y^{(0,5)}_{\bar{x}\hat{x}}(t_{n-1})\right\|^2 + \tau^*_n\frac{\tau_{n+1} - \tau_n}{6}\left\|y_{\overset{\circ}{t}\bar{x}\hat{x}}\right\|^2. \tag{7.26}$$

Using the algebraic inequality $2ab \geqslant -a^2 - b^2$, we estimate the last remaining scalar product in (7.22):

$$-2\tau^*\left(\frac{h_+ - h}{3}y_{\bar{t}\hat{t}\bar{x}}, y_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_* \geqslant -\tau^*\frac{\tau_{n+1} - \tau_n}{2}||y_{\bar{t}\hat{t}\bar{x}}]|^2 - \frac{2}{9}\left(\frac{(h_+ - h)^2\hbar_i}{(\tau_{n+1} - \tau_n)h_i}, y^2_{\overset{\circ}{t}\bar{x}\hat{x}}\right)_*. \tag{7.27}$$

Substituting the obtained estimates (7.23), (7.26), (7.27) into energy identity (7.22), we arrive at the reccurent relation

$$||y^n_{t\bar{x}}]|^2 + \left\|y^{(0,5)}_{\bar{x}\hat{x}}(t_n)\right\|^2 \leqslant \left||y^{n-1}_{t\bar{x}}]\right|^2 + \left\|y^{(0,5)}_{\bar{x}\hat{x}}(t_{n-1})\right\|^2,$$

from which the estimate (7.21) follows. □

We express our enormous gratitude to Academician A.A. Samarskii and professor P. N. Vabishchevich for the statement of the problem and useful comments.

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Received 5 Oct. 2001