COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 1 (2001), No. 3, pp. 285–297

AN ANALYTICAL APPROACH FOR QUASI-LINEAR EQUATION IN SECOND ORDER

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Abstract — This paper is devoted to the studies of the properties of the solutions of non-linear partial differential equation $\operatorname{div}[\mu(|\nabla u|)\nabla u] = 0$, being basic ones in differential formulation of the magnetostatic problem of finding the magnetic field distribution. The question of the existence of solutions, possessing an unlimited gradient, for this equation is of particular interest. Previous works [3, 4] dealt with the linear equation type (for $\mu = \text{const}$), and in the paper [6] a boundary value problem was considered for certain requirements for the μ function, as well as a more general non-linear case was studied. It was shown that such solutions exist, and their properties will be investigated. The difference scheme for the boundary value problem was built in the domain with corner and numerical calculations were given.

2000 Mathematics Subject Classification: 35J60; 35J65.

Keywords: the solutions of non-linear partial differential equation $\operatorname{div}[\mu(|\nabla u|)\nabla u] = 0$, the question of the existence of solutions, possessing an unlimited gradient, a difference scheme for the boundary value problem.

1. Search for particular solutions

In this paragraph a method will be described, by means of which the particular solutions of a differential equation $\operatorname{div}[\mu(|\nabla u|)\nabla u] = 0$ can be found, and the existence of solutions with unrestrictedly growing $|\nabla u|$ will be shown.

1.1. Problem formulation

Let us consider the differential equation:

$$\operatorname{div}[\mu(|\nabla u(p)|)\nabla u(p)] = 0, \quad p \in R^2 \tag{1.1}$$

where the function $\mu(H)$ is an analytical function by H , that meets the following requirements:

$$\begin{aligned} \lim_{H\to+\infty}\mu(H)=1,\\ \lim_{H\to+\infty}\mu'(H)H=0,\\ \mu(H)\neq 0. \end{aligned} \tag{1.2}$$

We shall investigate the solution behavior of (1.1). In particular we are interested in the solutions meeting the following requirement:

$$\lim_{p\to Q}|\nabla u(p)|=+\infty \tag{1.3}$$

where the limit is taken by a contour γ, passing through a point Q, that is not lying on infinity. Ladyzhinskaya O. [5] considered equations of (1.1) type. The author estimated the growth of $|\nabla u(p)|$ in the case when the equation domain has a smooth boundary and the boundary conditions are also smooth "enough". Unlike the above case we considered the boundary of the equation domain to be piece-wise smooth and boundary conditions not to be necessarily smooth. For such conditions we shall demonstrate existence of solutions (1.1) meeting requirements (1.3). Also we shall describe a method which enables partial solutions of equation (1.1) to be found.

1.2. The Legendre transformation

Let us find out if the solutions of (1.1) exist, that meet (1.3). To do it let us try to find particular solutions of this equation. The initial equation is non-linear, therefore the Legendre transformation should be applied to reduce the equation to the linear form. The essence of the transformation is described in [1]. Suppose we have got a surface in space (x,y,u). There are two possible ways of defining this surface. It is possible either to define it as a multitude of points describing by a function $u(x,y)$, or to consider this surface as an envelope of the multitude of its tangent planes, i.e., to generate an equation describing a plane to be a tangent one to the surface. If $\bar{x},\bar{y},\bar{u}$ are some current coordinates on the surface defined by the equation:

$$\bar{u}-\xi\bar{x}-\eta\bar{y}+\omega=0, \tag{1.4}$$

then ξ,η,ω are called the coordinates of this plane. Since a plane equation, tangent to the surface $u(x,y)$ in (x,y,u), takes the following form

$$\bar{u}-u-(\bar{x}-x)u_x-(\bar{y}-y)u_y=0, \tag{1.5}$$

then its coordinates are

$$\xi=u_x\,,\eta=u_y\,,\omega=xu_x+yu_y-u. \tag{1.6}$$

The surface can be also defined if ω is defined as a function ξ and η, by which a two-parameter multitude of tangent planes is described. We can find the dependence of $\omega(\xi,\eta)$ on $u(x,y)$, defining x and y from:

$$\xi=u_x,\quad \eta=u_y, \tag{1.7}$$

and applying them into

$$\omega = xu_x + yu_y - u = x\xi + y\eta - u. \tag{1.8}$$

Backward, in order to find a point coordinates by the tangent plane coordinates we should find partial differentials of $\omega(\xi, \eta)$. Since $\xi = u_x$ and $\eta = u_y$, we have

$$\omega_\xi = x + \xi\frac{\partial x}{\partial \xi} + \eta\frac{\partial y}{\partial \xi} - u_x\frac{\partial x}{\partial \xi} - u_y\frac{\partial y}{\partial \xi} = x, \tag{1.9}$$

and, similarly,

$$\omega_\eta = y. \tag{1.10}$$

Thus we get

$$u_x = \xi, \quad u_y = \eta, \quad x = \omega_\xi, \quad y = \omega_\eta, \tag{1.11}$$

$$u(x, y) + \omega(\xi, \eta) = x\xi + y\eta. \tag{1.12}$$

Such a surface transformation from a point coordinates to a plane coordinates is named as the Legendre transformation for a function of two variables. It essentially differs by its nature from the conventional coordinate transformation, since it transforms not point to point, but surface element (x, y, u, u_x, u_y) to surface element $(\xi, \eta, \omega, \omega_\xi, \omega_\eta)$. The Legendre transformation is possible when equations $u_x = \xi$ and $u_y = \eta$ can be solved in respect to x and y ; they can if jacobian

$$\rho = u_{xx}u_{yy} - u_{xy}^2 \tag{1.13}$$

is not equal to zero in the points of the surface under consideration. At last, in order to apply the Legendre transformation to a second-order differential equation (1.1) we should find out how to transform the second order differentials $u(x, y)$ and $\omega(\xi, \eta)$. Let us consider the variables x and y in u_x , u_y are expressed through ξ and η with the help of $x = \omega_\xi$, $y = \omega_\eta$. Differentiating equations $\xi = u_x$, $\eta = u_y$ by ξ and η, we get:

$$\begin{aligned} 1 &= u_{xx}\omega_{\xi\xi} + u_{xy}\omega_{\xi\eta}, \\ 0 &= u_{xy}\omega_{\xi\xi} + u_{yy}\omega_{\xi\eta}, \\ 0 &= u_{xx}\omega_{\xi\eta} + u_{xy}\omega_{\eta\eta}, \\ 1 &= u_{xy}\omega_{\xi\eta} + u_{yy}\omega_{\eta\eta}, \end{aligned} \tag{1.14}$$

the matrix form of that is

$$\begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}\begin{pmatrix} \omega_{\xi\xi} & \omega_{\xi\eta} \\ \omega_{\xi\eta} & \omega_{\eta\eta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.15}$$

It can be reduced by

$$\rho = u_{xx}u_{yy} - u_{xy}^2, \quad \frac{1}{\rho} = \omega_{\xi\xi}\omega_{\eta\eta} - \omega_{\xi\eta}^2, \tag{1.16}$$

to

$$u_{xx} = \rho\omega_{\eta\eta}, \quad u_{xy} = -\rho\omega_{\xi\eta}, \quad u_{yy} = \rho\omega_{\xi\xi}. \tag{1.17}$$

1.3. Finding of particular solutions

Here we shall describe the method of finding the particular solutions of (1.1), based on the Legendre transformation. Equation (1.1) in the cartesian coordinate system is:

$$(1+u_x^2 f(|\nabla u|))u_{xx} + 2u_x u_y f(|\nabla u|)u_{xy} + (1+u_y^2 f(|\nabla u|))u_{yy} = 0, \tag{1.18}$$

where $f(H) = \mu'(H)/(\mu(H)H)$. Applying the Legendre transformation (1.12), (1.17) to equation (1.18), we get

$$\left[1+\xi^2 f\left(\sqrt{\xi^2+\eta^2}\right)\right]\omega_{\eta\eta} - 2\xi\eta f\left(\sqrt{\xi^2+\eta^2}\right)\omega_{\xi\eta} + \left[1+\eta^2 f\left(\sqrt{\xi^2+\eta^2}\right)\right]\omega_{\xi\xi} = 0. \tag{1.19}$$

Pay attention that equation (1.19) is linear unlike (1.18). Going to the polar coordinate system:

$$\begin{gathered}\xi = r'\cos\varphi', \quad \eta = r'\sin\varphi', \quad r' = \sqrt{\xi^2+\eta^2},\\ \omega(\xi,\eta) = \omega(r'\cos\varphi', r'\sin\varphi') = v(r',\varphi').\end{gathered} \tag{1.20}$$

We also shall use the polar coordinate system in the plane XOY:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r = \sqrt{x^2+y^2}. \tag{1.21}$$

Thus, the partial differentials are

$$\begin{aligned}
\omega_{\xi\xi} &= v_{r'r'}\cos^2\varphi' + v_{\varphi'\varphi'}\frac{\sin^2\varphi'}{r'^2} - v_{r'\varphi'}\frac{2\sin\varphi'\cos\varphi'}{r'} + v_{r'}\frac{\sin^2\varphi'}{r'} + v_{\varphi'}\frac{2\sin\varphi'\cos\varphi'}{r'^2},\\
\omega_{\xi\eta} &= v_{r'r'}\sin\varphi'\cos\varphi' - v_{\varphi'\varphi'}\frac{\sin\varphi'\cos\varphi'}{r'^2} + v_{r'\varphi'}\frac{\cos^2\varphi'-\sin^2\varphi'}{r'^2} - v_{r'}\frac{\sin\varphi'\cos\varphi'}{r'} +\\
&\quad + v_{\varphi'}\frac{\sin^2\varphi'-\cos^2\varphi'}{r'^2},\\
\omega_{\eta\eta} &= v_{r'r'}\sin^2\varphi' + v_{\varphi'\varphi'}\frac{\cos^2\varphi'}{r'^2} + v_{r'\varphi'}\frac{2\sin\varphi'\cos\varphi'}{r'} + v_{r'}\frac{\cos^2\varphi'}{r'} - v_{\varphi'}\frac{2\sin\varphi'\cos\varphi'}{r'^2}.
\end{aligned} \tag{1.22}$$

The substitution of these correlations in (1.19) yields

$$v_{r'r'} + a(r')\left[\frac{1}{r'}v_{r'} + \frac{1}{r'^2}v_{\varphi'\varphi'}\right] = 0, \tag{1.23}$$

where $a(r') = 1 + r'^2 f(r')$. We shall solve equation (1.23) by the method of factorization. Since we have $v(r',\varphi') = R(r')\Phi(\varphi')$, we get

$$R'' + \frac{1}{r'}a(r')R' - \frac{\lambda^2}{r'^2}a(r')R = 0, \tag{1.24}$$

$$\Phi'' + \lambda^2\Phi = 0. \tag{1.25}$$

Now we shall investigate the behavior of the solutions of equation (1.24) for the radial component $R(r')$. The behavior of the solutions of (1.24) is defined by the poles of the equation coefficients. The function $a(r')$ is analytical, since $f(r')$ is analytical because of (1.2). From the analytical theory of differential equations [2] the solution of (1.24) can be transformed into a series at the point, being the pole of the equation coefficients (1.24)

$$R_\lambda^{(1)}(r') = (r'-r_0)^{\vartheta_1}\sum_{k=0}^{+\infty} a_k^{(1)}(r'-r_0)^k, \tag{1.26}$$

where r_0 is the pole (in our case $r_0 = 0$), ϑ_1 is the root with the largest real part of characteristic equation

$$\vartheta^2 + \vartheta[p_1(r_0) - 1] + p_2(r_0) = 0, \tag{1.27}$$

where $p_1(r_0) = a(r_0)\,, p_2(r_0) = -\lambda^2 a(r_0)$. If ϑ_2 differs from ϑ_1 by not an integer number and they do not coincide, then the second linearly independent solution can be constructed as:

$$R_\lambda^{(2)}(r') = (r' - r_0)^{\vartheta_2} \sum_{k=0}^{+\infty} a_k^{(2)} (r' - r_0)^k, \tag{1.28}$$

otherwise it is possible to present the solution as:

$$R_\lambda^{(2)}(r') = C_0 R_\lambda^{(1)}(r') \ln(r' - r_0) + (r' - r_0)^{\vartheta_2} \sum_{k=0}^{+\infty} b_k (r' - r_0)^k. \tag{1.29}$$

Then the superposition

$$v(r', \varphi') = \sum_\lambda [C_\lambda^{(1)} R_\lambda^{(1)}(r') + C_\lambda^{(2)} R_\lambda^{(2)}(r')][C_\lambda^{(3)} \sin \lambda\varphi' + C_\lambda^{(4)} \cos \lambda\varphi'] \tag{1.30}$$

is a solution of (1.24). Then it is necessary to do the non-linear transformation in order to obtain the solution of (1.1). Thus, this is a method of finding the particular solutions of the non-linear differential equation (1.1). particular solutions is derived from (1.30) by defining the coefficients $C_\lambda^{(i)}\,, i = 1..4$.

1.4. The solution behavior

Now we can start finding solutions, meeting the requirement (1.3). On the ground of the Legendre transformation the function $v(r', \varphi')$ must meet:

$$r_Q = \lim_{p \to Q} r_p = \lim_{r' \to +\infty} |\nabla v(r', \varphi')| \leqslant C_0 \tag{1.31}$$

$$r_p = \sqrt{x_p^2 + y_p^2} \tag{1.32}$$

where C_0 - a constant. From now on we shall consider only the case when point Q coincides with the coordinate origin, i.e., $r_Q = 0$. This requirement is not principle. A way of extending the results, obtained for $r_Q = 0$, to the case of $r_Q \neq 0$ will be given below. Now the study will be made this way:

1. firstly, we'll find some particular solutions of (1.23);
2. then find out the features they possess;
3. and on the ground of these features we shall determine the features of the solutions of (1.1), corresponding to them.

Let's show that among the solutions of (1.1), meeting (1.3), there are solutions which do not have a limit in point Q and there are those which do. Let's consider them by order. In the simplest case when $\lambda = 0$ the the solution of (1.23) will be:

$$v(r', \varphi') = \left(c_1 + c_2 \int_0^{r'} \frac{dt}{t\mu(t)} \right) (c_3 \varphi' + c_4)\,. \tag{1.33}$$

Let's investigate a private case of (1.33). Let $c_2 = c_4 = 0$, and $c_1 c_3 = C$, then:

$$v(r', \varphi') = C\varphi', \tag{1.34}$$

$$|\nabla v| = C\frac{1}{r'}. \tag{1.35}$$

This solution meets (1.31). The Legendre transformation (1.12) yields

$$\begin{gathered} u(r,\varphi) = (\vec{r'}, \nabla v(r',\varphi')) - v(r',\varphi'), \\ \lim_{r\to 0} u(r,\varphi) = \lim_{r'\to+\infty} [(\vec{r'}, \nabla v(r',\varphi')) - v(r',\varphi')] = -C\varphi'. \end{gathered} \tag{1.36}$$

From this it is clear that $\lim\limits_{r\to 0} u(r,\varphi)$ depends on the contour by which it is taken. Indeed, let us derive function $u(r,\varphi)$ for solution (1.34), applying the Legendre transformation

$$u(x,y) = C_1 \arctan\frac{y}{x} = C_1\varphi, \tag{1.37}$$

$$|\nabla u| = C_1\frac{1}{r}. \tag{1.38}$$

It can be seen that $u(r,\varphi)$ is not defined at $r = 0$. Now let us consider the solutions of (1.1), which meet (1.3) and reach a limit at Q. Substitution of the variables in (1.24): $r' = 1/t$ and $R(r') = w(t)$, results in

$$w'' + \frac{1}{t}(2 - \bar{a}(t))w' - \frac{\lambda^2}{t^2}\bar{a}(t)w = 0, \tag{1.39}$$

where $\bar{a}(t) = a(r')$. The value of $\bar{a}(0)$ will be

$$\bar{a}(0) = \lim_{r'\to+\infty} a(r') = \lim_{r'\to+\infty} (1 + r'^2 f(r')) = \lim_{r'\to+\infty}\left(1 + \frac{r'\mu'(r')}{\mu(r')}\right) = 1. \tag{1.40}$$

Thus, the characteristic equation of (1.39) turns into

$$\vartheta^2 - \lambda^2 = 0, \quad \text{i.e.,} \quad \vartheta_1 = |\lambda|, \quad \vartheta_2 = -|\lambda|. \tag{1.41}$$

As a result the solution can be represented by series

$$w_1(t) = t^{|\lambda|}\sum_{k=0}^{+\infty} a_k^{(1)} t^k, \tag{1.42}$$

$$w_2(t) = t^{-|\lambda|}\sum_{k=0}^{+\infty} a_k^{(2)} t^k \tag{1.43}$$

or

$$w_2(t) = C_0 w_1(t)\ln t + t^{-|\lambda|}\sum_{k=0}^{+\infty} b_k t^k. \tag{1.44}$$

The solution $R_\lambda^{(1)}(r')$ of (1.24) corresponds to the first linearly independent solution $w_1(t)$ of (1.37)

$$R_\lambda^{(1)}(r') = r'^{-|\lambda|}\sum_{k=0}^{+\infty} a_k^{(1)} r'^{-k}. \tag{1.45}$$

This series will converge uniformly and absolutely if $r' > 1/r_0$, where r_0 is the series convergence radius (1.42). Now it can be seen that a series, generated by term-by-term differentiation of (1.45), also will converge under the same conditions. Thus, we get

$$\frac{d}{dr'}R_\lambda^{(1)}(r') = -\sum_{k=0}^{+\infty} a_k^{(1)}(k+|\lambda|)\left(\frac{1}{r'}\right)^{k+|\lambda|+1} \tag{1.46}$$

and

$$\lim_{r'\to+\infty}\frac{d}{dr'}R_\lambda^{(1)}(r') = 0. \tag{1.47}$$

Hence, there is a solution of (1.24) meeting (1.31). Such a solution can be built, for instance, in this way

$$v(r',\varphi') = \sum_\lambda R_\lambda^{(1)}(r')(C_\lambda^{(1)}\sin\lambda\varphi' + C_\lambda^{(2)}\cos\lambda\varphi') \tag{1.48}$$

(the sum in (1.48) is supposed to be finite here). Now let us do the reverse transformation and find out whether the obtained function $u(p)$ reaches its limit at Q. Similarly as for (1.36) we get

$$\lim_{r\to0} u(r,\varphi) = \lim_{r'\to+\infty}[(\vec{r'},\nabla v(r',\varphi')) - v(r',\varphi')] = 0. \tag{1.49}$$

Thus, there is a limit for function $u(p)$. It should be noted that if we build appropriate solutions for $w_2(t)$, then solutions, obtained from $R_\lambda^{(2)}(r')$, for function $u(p)$ will not reach their limits at point Q.

Remark 1.1. It should be noted that we can say nothing about the value of function $u(p)$ at the very point Q if it meets (1.3), since the Legendre transformation itself can lose its uniqueness at this point. However, if $u(p)$ meets (1.3) and in addition reaches its limit at point Q, then we can define $u(p)$ at Q , having assumed

$$u(Q) = \lim_{p\to Q} u(p). \tag{1.50}$$

In this case solution $u(p)$ will be continuous at point Q.

Remark 1.2. Let us note that from (1.49) it can not be concluded that all the solutions, reaching their limits at Q , will be equal to zero at Q. This results from the fact that the solution of (1.23) itself is defined to within the linear combination:

$$v(r',\varphi') = C_1 + C_2 v_1(r',\varphi') + C_3 r' + C_4\varphi', \tag{1.51}$$

where $v_1(r',\varphi')$ is the solution of equation (1.23). As a result, by selecting a constant C_1 it is possible (from (1.49) and assuming $C_3 = C_4 = 0$) to obtain any pre-determined limit value at point Q . From (1.51) a validity can be concluded of the generalizing of the obtained results in the case when $r_Q \neq 0$. Indeed, if $v_1(r',\varphi')$ is a solution in the case when $r_Q = 0$, then $v(r',\varphi')$ from (1.51) is the solution in the case of $r_Q \neq 0$ (if $C_3 \neq 0$), i.e., if $u(p)$ is the solution corresponding to $v(r',\varphi')$, then

$$\lim_{p\to Q} u(p) = -C_1 - C_4\varphi', \tag{1.52}$$

$$r_Q = \lim_{p\to Q} r_p = \lim_{r'\to+\infty}|\nabla v(r',\varphi')| = C_3. \tag{1.53}$$

In this case it was assumed that $v_1(r',\varphi')$ reaches its limit at point Q. Therefore, from now on in (1.51) it will be assumed that $C_1 = C_3 = C_4 = 0$, and $C_2 = 1$.

2. Boundary value problem for non-linear elliptic equation

In this paragraph the boundary value problem will be considered for eq. (1.1) in the domain with corner. An increase in $|\nabla u|$ in the region around the corner point will be assessed, and an algorithm for numerical calculation of this problem will be presented.

2.1. Assessment of $|\nabla u|$ for boundary value problem

Let us consider a boundary value problem for equation (1.1) in a domain with corner (Fig. 1)

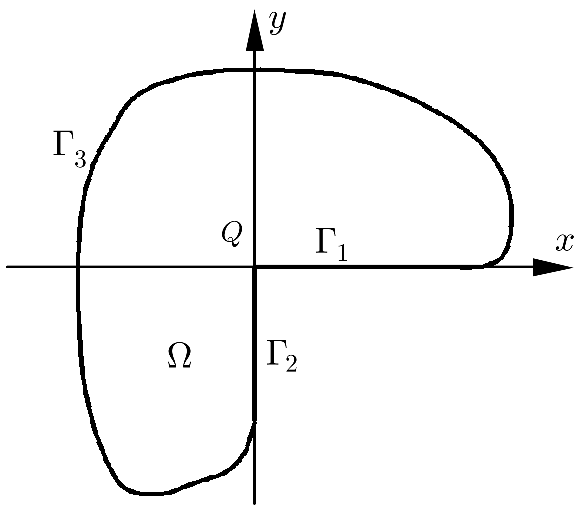

Figure 1.

$$\operatorname{div}[\mu(|\nabla u|)\nabla u] = 0, \qquad p \in \Omega, \tag{2.1}$$

$$u\,|_{\Gamma} = \Psi(p), \quad p \in \Gamma, \tag{2.2}$$

$$\Psi(p) = \begin{cases} 0, & p \in \Gamma_1 \cup \Gamma_2, \\ g(p), & p \in \Gamma_3, \end{cases} \tag{2.3}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, and $\Psi \in C^{(1)}(\Gamma)$. The function in equation (1.54) meets requirements (1.2) and $1 + r^2 f(r) > 0$, which renders equation (2.1) to be elliptic. Let $u(p)$ be a solution of (2.1-3), meeting (1.3). We are going to assess $|\nabla u|$ around point Q (coordinate origin). We assume that the solution $u(p)$ meets the following requirements:

$$\begin{aligned} &\exists \delta > 0 \\ &\qquad \forall p \in S_\delta(Q) \cap \Omega : \quad \rho(p) = u_{xx}(p)u_{yy}(p) - u_{xy}^2(p) \neq 0 \\ &\qquad \forall p_1,\ p_2 \in S_\delta(Q) \cap (\Omega \cup \Gamma),\ p_1 \neq p_2 : \quad \nabla u(p_1) \neq \nabla u(p_2) \\ &\qquad u_x(0,y),\ u_y(x,0) \quad \text{are monotonous functions} \\ &\qquad \text{if} \quad x \in \Gamma_1, \quad y \in \Gamma_2, \quad x < \delta,\ |y| < \delta \end{aligned} \tag{2.4}$$

It should be noted that such solutions exist, it can be seen from Section 1.4. Applying the Legendre transformation for function $u(p)$ under $p \in S_\delta(Q) \cap (\Omega \cup \Gamma)$, “boundary Γ_1”

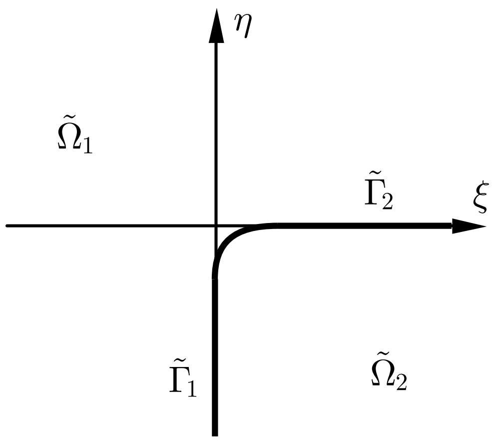

Figure 2.

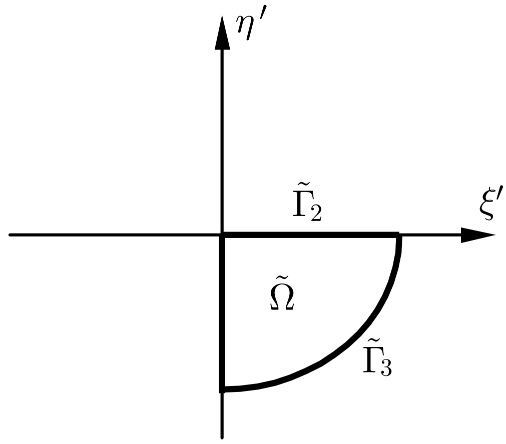

Figure 3.

($\omega_\eta = 0$, $\omega_\xi > 0$, $\xi = 0$) will be transformed in "boundary $\tilde{\Gamma}_1$", and "boundary Γ_2" ($\omega_\xi = 0$, $\omega_\eta < 0$, $\eta = 0$) in "boundary $\tilde{\Gamma}_2$". The signs of $\eta = u_y(x, 0)$ for "boundary Γ_1" and $\xi = u_x(0, y)$ for "boundary Γ_2" are unclear. Now we consider the case when $\eta < 0$ and $\xi > 0$ for boundaries $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ respectively. Applying the transformation we will get domain $\tilde{\Omega}_1$ or $\tilde{\Omega}_2$, illustrated in Fig. 2. Firstly, let us consider domain $\tilde{\Omega}_2$. Substituting variables $r' = 1/t$, $v(r', \varphi') = v(1/t, \varphi') = w(t, \varphi')$, we will come to the following boundary value problem (Fig. 3)

$$L_{\bar{a}} w(p) = 0, \qquad p \in \tilde{\Omega}, \tag{2.5}$$

$$w\,|_{\tilde{\Gamma}} = \tilde{\Psi}(p), \quad p \in \tilde{\Gamma}, \tag{2.6}$$

where

$$L_{\bar{a}} = \frac{\partial^2}{\partial t^2} + \frac{1}{t}(2 - \bar{a}(t))\frac{\partial}{\partial t} + \frac{\bar{a}(t)}{t^2}\frac{\partial^2}{\partial \varphi'^2}. \tag{2.7}$$

Since $\omega_\eta = 0$ on $\tilde{\Gamma}_1$ then $\omega\,|_{\tilde{\Gamma}_1} = 0$,and similarly $\omega\,|_{\tilde{\Gamma}_2} = 0$. Thus we get

$$\tilde{\Psi}(p) = \begin{cases} 0, & p \in \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2, \\ \tilde{g}(p), & p \in \tilde{\Gamma}_3, \end{cases} \tag{2.8}$$

where $\tilde{\Gamma} = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3$, and $\tilde{\Psi} \in C^{(1)}(\tilde{\Gamma})$. Function $\tilde{\Psi}$ is defined by $u(p)$ from the Legendre transformation. Now we will consider the solutions of (2.5), (2.6), possessing the limited gradients. Doing (2.5), (2.6) by the factorization method $w(p) = T(t)\Phi(\varphi')$, we come to

$$T'' + \frac{2 - \bar{a}(t)}{t}T' - \lambda^2 \frac{\bar{a}(t)}{t^2} T = 0, \qquad T(0) = 0, \tag{2.9}$$

$$\Phi'' + \lambda^2 \Phi = 0, \quad \Phi(\frac{3\pi}{2}) = \Phi(2\pi) = 0. \tag{2.10}$$

It results in

$$\Phi_k(\varphi') = \sin\left(\lambda_k\left(\varphi' - \frac{3\pi}{2}\right)\right), \quad \lambda_k = 2k,\ k = 1, \dots, \tag{2.11}$$

$$T_k^{(1)}(t) = t^{|\lambda_k|}\sum_{s=0}^{+\infty} a_s(\lambda_k)t^s, \tag{2.12}$$

$$T_k^{(2)}(t) = AT_k^{(1)}(t)\ln t + t^{-|\lambda_k|}\sum_{s=0}^{+\infty} d_s(\lambda_k)t^s. \tag{2.13}$$

Functions $T_k^{(2)}(t)$ will be absent when decomposing in a series of (2.5-6). Consequently, we get:

$$\bar{w}(t) = \sum_k \bar{C}_k T_k^{(1)}(t)\Phi_k(\varphi'). \tag{2.14}$$

Thus, for function $v(r', \varphi')$ under $r' > r_0$ (where $1/r_0 = t_0$) we get:

$$v(r', \varphi') = \sum_{k=1}^{+\infty} \bar{C}_k T_k^{(1)}(1/r')\Phi_k(\varphi'). \tag{2.15}$$

Now let us assess $|\nabla u|$. In order to do it we will assume v under $r' > r_0$ as:

$$v(r', \varphi') = \Phi_1(\varphi')\frac{\bar{C}_1}{r'^{|\lambda_1|}}a_0(\lambda_1) + O\left(\frac{1}{r'^{|\lambda_1|+1}}\right). \tag{2.16}$$

then we can derive:

$$\lim_{r'\to+\infty} r'^{|\lambda_1|+1}|\nabla v| = |\lambda_1||\bar{C}_1 a_0(\lambda_1)| = \text{const} \tag{2.17}$$

Since $r' = |\nabla u|$ and $r = |\nabla v| = |\nabla \omega|$, then

$$|\nabla u| \sim r^{-\frac{1}{1+|\lambda_1|}} \quad \text{under} \quad r \to 0 \tag{2.18}$$

Since $\lambda_1 = 2$ then the rate of the gradient increase for (2.1-3) around the corner point is $r^{-1/3}$, i.e., the same as that for the similar boundary value problem with the Laplace equation [3, 4]. In case of considering domain $\tilde{\Omega}_2$ instead of domain $\tilde{\Omega}_2$ (Fig. 2), the transformation loses its uniqueness. The case when $\eta > 0$ and $\xi < 0$ for boundaries $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ respectively is similar to the previous one and the same assessment on $|\nabla u|$ is valid. In other cases when:

1. $\eta < 0$ for Γ_1, $\xi < 0$ for Γ_2;
2. $\eta > 0$ for Γ_1, $\xi > 0$ for Γ_2 ;

the transformation also will lose its uniqueness.

2.2. Numerical algorithm for boundary value problem

Now a numerical method for the solution of the boundary value problem (2.1-3) will be described. An algorithm for the solution of the problem will be designed in the similar way that is described in [6] for the Laplace equation. A difference scheme will be built only for the neighborhood of the corner point $\tilde{\Omega}_Q$.

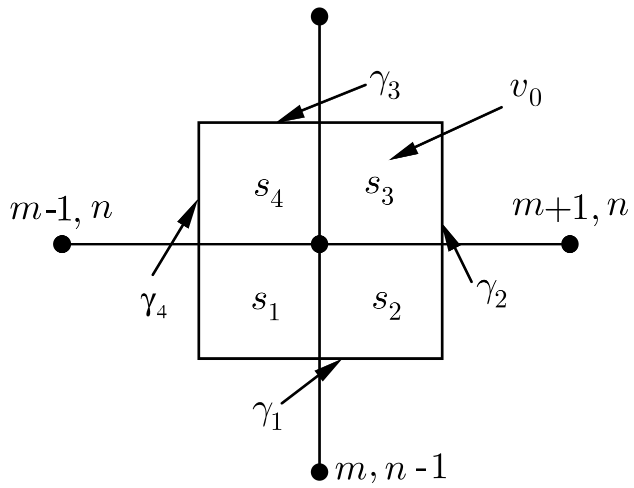

Figure 4.

Integration of equation (1.54) by volume v_0 (Fig. 4) yields

$$\int_{v_0} \operatorname{div}\,[\mu(|\nabla u|)\nabla u]dxdy = \oint_{\gamma} \mu\frac{\partial u}{\partial y}dx - \mu\frac{\partial u}{\partial x}dy = 0 \tag{2.19}$$

where γ is the boundary of domain v_0 and $\gamma = \bigcup_{m=1}^{4} \gamma_m$. Only approximation of $\int_{\gamma_1} \mu\ \partial u/\partial y dx$ will be considered, approximation for the other contours is similar. Function $P(r,\varphi)$ is assumed to be derived from function $T_1^{(1)}(1/r')\Phi_1(\varphi')$ applying the Legendre transformation. Let $G_\alpha = \partial P/\partial x_\alpha$ be for $\alpha = 1, 2$. From the above, function G_α contributes the main part in $|\nabla u|$ of (2.1-3) around the corner point, therefore we will apply the following approximation

$$\frac{\partial u(p)}{\partial x_\alpha} \approx C_1 G_\alpha(p),\ \alpha = 1, 2,\ p \in \tilde{\Omega}_Q \tag{2.20}$$

where $C_1 = \text{const}$ is to be determined. Mean values for G_α are assumed as

$$G_\alpha^\beta = \frac{1}{\Delta s_\beta}\int_{\Delta s_\beta} G_\alpha dx_\beta,\ \alpha = 1, 2\, \beta = 1, 2 \tag{2.21}$$

where Δs_β is the segment parallel to X- or Y- axes for $\beta = 1$ or $\beta = 2$ respectively. Let us assume function $\mu(|\nabla u|)$ to be piecewise constant in segments s_i (Fig. 4) and to be equal to μ_i, $i = 1..4$ accordingly.

$$\int_{\gamma_1} \mu\frac{\partial u}{\partial y}dx \approx \mu_1 \int_{x_m - h_m^x/2}^{x_m} \frac{\partial u}{\partial y}dx + \mu_2 \int_{x_m}^{x_m + h_{m+1}^x/2} \frac{\partial u}{\partial y}dx \tag{2.22}$$

We consider only the first integral term, the second one is considered in the similar way.

$$\int\limits_{x_m-h_m^x/2}^{x_m} \frac{\partial u}{\partial y}dx \approx C_1 \int\limits_{x_m-h_m^x/2}^{x_m} G_2 dx = \Delta s_1 G_2^1 C_1, \quad \Delta s_1 = \frac{h_m^x}{2} \tag{2.23}$$

$$u(x_m, y_n) - u(x_m, y_{n-1}) = \int\limits_{y_{n-1}}^{y_n} \frac{\partial u}{\partial y}(x_m, y)dy \approx C_1 \int\limits_{y_{n-1}}^{y_n} G_2 dy = C_1 \Delta s_2 G_2^2 \tag{2.24}$$

where $\Delta s_2 = h_n^y$, thus

$$C_1 \approx \frac{1}{G_2^2} \frac{u_{m,n} - u_{m,n-1}}{h_n^y} \tag{2.25}$$

As a result we get

$$\mu_1 \int\limits_{x_m-h_m^x/2}^{x_m} \frac{\partial u}{\partial y}dx \approx \frac{G_2^1}{G_2^2} \frac{h_m^x}{2} \frac{u_{m,n} - u_{m,n-1}}{h_n^y} = a_1(u_{m,n} - u_{m,n-1}) \tag{2.26}$$

$$\mu_2 \int\limits_{x_m}^{x_m+h_{m+1}^x/2} \frac{\partial u}{\partial y}dx \approx a_2(u_{m,n} - u_{m,n-1}) \tag{2.27}$$

$$\int\limits_{\gamma_1} \mu \frac{\partial u}{\partial y}dx \approx a_x^{(-)}(u_{m,n} - u_{m,n-1}), \quad \text{where} \quad a_x^{(-)} = a_1 + a_2 \tag{2.28}$$

Similar expressions can be derived for the other contours. Consequently, (2.19) takes the following form:

$$u_{m,n}\left(a_x^{(-)} + a_x^{(+)} + a_y^{(-)} + a_y^{(+)}\right) = a_x^{(-)} u_{m,n-1} + a_x^{(+)} u_{m,n+1} + a_y^{(+)} u_{m+1,n} + a_y^{(-)} u_{m-1,n} \tag{2.29}$$

Now we present a boundary value problem solved by the similar way. Function μ was assumed as:

$$\mu(H) = 1 + \frac{A}{H} - \frac{B}{H^2}, \quad \text{where} \quad H > H_0 \tag{2.30}$$

and function $\tilde{\Psi}(\varphi') = \tilde{C}_1 \sin(2(\varphi' - 3\pi/2)) + \tilde{C}_2 \sin(4(\varphi' - 3\pi/2))$ for (2.5-6). Values A, B, $\tilde{C}_1$, $\tilde{C}_2$ are constant. Comparative results of calculation of relative errors are presented below. These results were obtained with the use of the above algorithm (Fig. 5) and without it (Fig. 6).

As it can be seen from the figures, the relative errors around the corner point decrease by about 10 times.

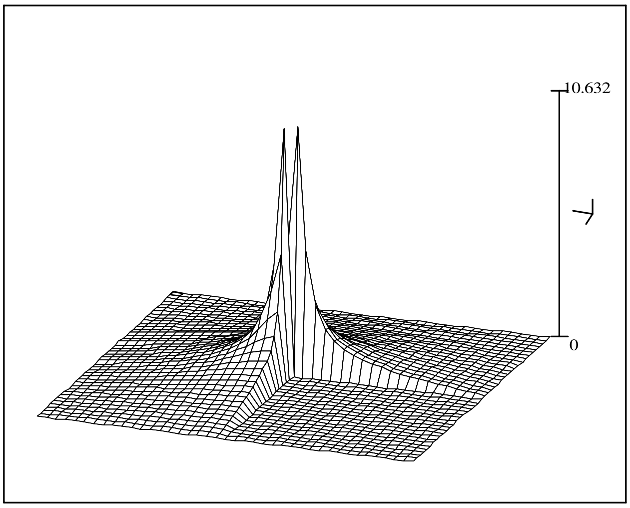

Figure 5.

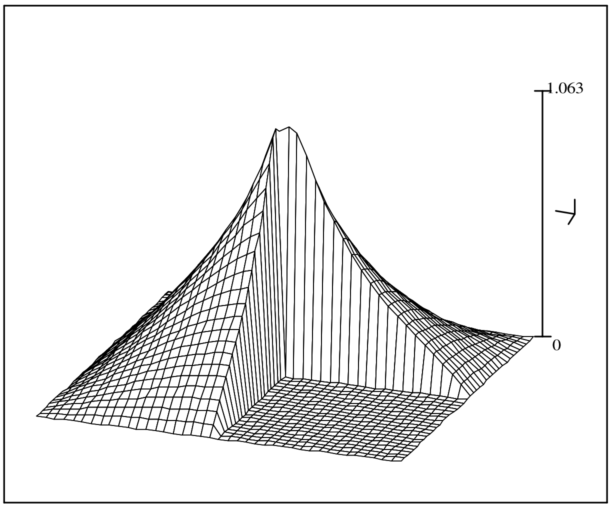

Figure 6.

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Received 25 May 2001
Revised 27 Sep. 2001