COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 1 (2001), No. 4, pp. 333–355

EXPONENTIALLY CONVERGENT PARALLEL DISCRETIZATION METHODS FOR THE FIRST ORDER EVOLUTION EQUATIONS [1]

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Abstract — We propose a new discretization of an initial value problem for differential equations of the first order in a Banach space with a strongly P-positive operator coefficient. Using the strong positiveness, we represent the solution as a Dunford-Cauchy integral along a parabola in the right half of the complex plane, then transform it into real integrals over $(-\infty, \infty)$, and finally apply an exponentially convergent Sinc quadrature formula to this integral. The integrand values are the solutions of a finite set of elliptic problems with complex coefficients, which are independent and may be solved in parallel.

Keywords: evolution equations, unbounded operator coefficients, strongly P-positive operators, quadrature rule.

1. Introduction

We consider the initial value problem

$$\dot{u} + Au = 0, \quad t \in (0, T], \quad u(0) = u_0, \tag{1.1}$$

where $u : R_+ \to E$ is a vector-valued function, A is a strongly P-positive densely defined closed operator with the spectrum lying in a parabola Γ and with a domain $D(A)$ in a Banach space E (see [7] and below for definitions and examples). In particular, equation (1.1) with the Laplace operator $A = -\Delta$ is the well-known heat equation . Using the improper Dunford-Cauchy integral (see [9, 11] for details) we will show that one can represent the solution of (1.1) by

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-zt} (z - A)^{-1} u_0 dz. \tag{1.2}$$

Denoting by $\hat{u}(z)$ the solution of the stationary equation

$$(z - A)\hat{u}(z) = u_0, \tag{1.3}$$

[1]Partially supported by DFG

we get

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-zt} \hat{u}(z) dz. \tag{1.4}$$

The key steps that allow us to find an approximate solution to (1.1) are:

1. Choose N points $z_1, z_2, ..., z_N$ on the parabola and find the solutions $\hat{u}(z_i)$ of (1.3).
2. Find the approximation $u_N(t)$ for the solution of (1.1)by

$$u_N(t) = \sum_{p=-N}^{N} \alpha_p e^{-z_p t} \hat{u}(z_p) \tag{1.5}$$

with appropriate coefficients α_i.

The next question is how to choose z_i and α_i. This will be discussed in what follows. Note that there are many possibilities of choosing the integration path, and this choice can decide on the quality of discretisation. An analogous idea to use an integral representation for the solution of equation (1.1) was used in [12]. A common feature of [12] and of our paper is only the very general idea of integral representation of the solution. But the realization of this idea, the way to an algorithm, and the algorithms are principally different. We underline these differences in detail: 1. The choice of the integration path depends decidedly on the "geometrical" properties of the spectrum and the behavior of the resolvent. Paper [12] considers only self-adjoint spatial operators, and the choice of the integration path leads to a singular integrand which implies that a part of the algorithm deals with the overcoming of this difficulties. At the end the algorithm possesses only a polynomial (of degree 2 or 4) convergence rate. 2. We choose a parabola as an integration path (which is defined by the coefficients of the elliptic spatial part), get an a n a l y t i c a l integrand, and use a Sinc quadrature. At the end we get a principally other algorithm with an e x p o n e n t i a l convergence rate which works for a significant larger class of spatial operators as self-adjoint ones, namely for general elliptic operators and other P-positive operators.

2. Representation of solutions of the first order differential equations with a strongly P-positive operator coefficients

In order to motivate the next definitions, we begin with the following examples.

Example 2.1. Let us consider the one-dimensional operator $A : L_1(0,1) \to L_1(0,1)$ with the domain $D(A) = \{u | u \in H_0^2(0,1)\}$ in the Sobolev space $H_0^2(0,1)$ defined by

$$Au = -u'' \quad \forall\, u \in D(A).$$

The eigenvalues $\lambda_k = k^2\pi^2, k = 1, 2, ...$ of A lie on the real axis inside of the path

$$\Gamma = \begin{cases} z = \eta^2 \pm i\eta, & \eta \geqslant 1\ , \\ z = 1 \pm i\eta^2, & |\eta| \leqslant 1. \end{cases}$$

The Green function for the problem

$$(zI - Au) \equiv u''(x) + zu(x) = -f(x),\ x \in (0,1); u(0) = u(1) = 0$$

is

$$G(x,\xi) = \frac{1}{\sqrt{z}\sin\sqrt{z}} \begin{cases} \sin\sqrt{z}x \sin\sqrt{z}(1-\xi) & x \leqslant \xi, \\ \sin\sqrt{z}\xi \sin\sqrt{z}(1-x) & x \geqslant \xi, \end{cases}$$

i.e., we have

$$u(x) = (zI - A)^{-1} f = \int_0^1 G(x,\xi) f(\xi) d\xi.$$

Let us estimate the Green function on the parabola $z = \eta^2 \pm i\eta = \sqrt{\eta^4 + \eta^2}(\cos\phi \pm i\sin\phi)$ for $|z|$ large enough, where

$$\cos\phi = \frac{\eta}{\sqrt{\eta^2+1}}, \ \sin\phi = \frac{1}{\sqrt{\eta^2+1}}.$$

Actually, we have $\sqrt{z} = \sqrt[4]{\eta^4+\eta^2}(\cos\frac{\phi}{2} \pm i\sin\frac{\phi}{2}) = a \pm ib$ with

$$\begin{aligned}
&\cos\frac{\phi}{2} = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}}, \quad \sin\frac{\phi}{2} = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}\sqrt[4]{\eta^4+\eta^2}}, \\
&a = \frac{\sqrt{\eta^2 + \sqrt{\eta^4+\eta^2}}}{\sqrt{2}}, \\
&b = \frac{\sqrt{\sqrt{\eta^4+\eta^2} - \eta^2}}{\sqrt{2}} = \frac{|\eta|}{\sqrt{2}\sqrt{\sqrt{\eta^4+\eta^2}+\eta^2}} = \frac{1}{\sqrt{2}\sqrt{\sqrt{1+1/\eta^2}+1}}.
\end{aligned}$$

The above function increases monotonically for $\eta \in [1,\infty)$, i.e.,

$$\min_{|\eta|} b(\eta) = b(1) = \frac{1}{\sqrt{2}\sqrt{1+\sqrt{2}}}. \tag{2.1}$$

It follows from

$$\begin{aligned}
&-\left(\eta^4+\eta^2\right) + (3|\eta|-1)^4 = 80\eta^4 - 108|\eta|^3 + 53\eta^2 - 12|\eta| + 1 \\
&= 80\eta^2\left(|\eta| - \frac{27}{40}\right)^2 + 331\left(|\eta| - \frac{6}{331}\right)^2 + 1 - \frac{6^2}{331} > 0
\end{aligned}$$

that

$$\frac{1}{|\eta|} \leqslant \frac{3}{1+\sqrt{|z|}} = \frac{3}{1+\sqrt[4]{\eta^4+\eta^2}}. \tag{2.2}$$

Now, taking into account that

$$\coth\frac{b(\eta)}{2} \leqslant \coth\frac{b(1)}{2}.$$

and estimates (2.1),(2.2),we get that the following estimates hold for $x \leqslant \xi$ and for $|\eta|$ large enough:

$$\begin{aligned}
\left|\frac{\sin\sqrt{z}x\sin\sqrt{z}(1-\xi)}{\sqrt{z}\sin\sqrt{z}}\right| &= \frac{[\sin^2 ax+\sinh^2 bx]^{\frac12}[\sin^2 a(1-\xi)+\sinh^2 b(1-\xi)]^{\frac12}}{\sqrt[4]{\eta^4+\eta^2}[\sin^2 a+\sinh^2 b]^{\frac12}}\\
&\leqslant \frac{\sqrt{1+\sinh^2 bx}\sqrt{1+\sinh^2 b(1-\xi)}}{|\eta|\sinh b} = \frac{\cosh bx\cosh b(1-\xi)}{|\eta|\sinh b}\\
&\leqslant \frac{\cosh^2(b/2)}{|\eta|\sinh b} = \frac{\coth(b/2)}{2|\eta|} \leqslant \frac{3}{2}\coth\frac{b(1)}{2}\frac{1}{1+\sqrt{|z|}}.
\end{aligned}$$

The case $\xi \leqslant x$ can be considered analogously. The above estimate implies that $\|(zI-A)^{-1}f\|_{L_1} \leqslant \frac{M}{1+\sqrt{|z|}}\|f\|_{L_1}\forall f \in L_1(0,1),\ \forall z \in \mathbb{C}\setminus\Omega_\Gamma$ where Ω_Γ is the domain inside the parabola. The same estimates for the Green function imply the analogous estimate in the norm of $L_\infty(0,1)$.

Example 2.2. This example deals with a differential operator which being considered in the Hilbert space $L_2(0,1)$ is not symmetric.

Let $D(A)=\{v(x)\in W^2_\infty(0,1)|\ v(0)=0, v'(0)=v'(1)\}$ be the domain of the operator A defined by

$$Au \equiv -u''(x)+u(x), \qquad \forall u \in D(A)$$

It is easy to find that the spectrum of A consists of the eigenvalues $\lambda_k=(2k\pi)^2+1, k=0,1,2,...$ which are enveloped by the parabola $\Gamma=\{z=\xi+i\eta : \xi=\eta^2\}$. Each eigenvalue corresponds to one eigenfunction and one joint function.We denote by Ω_Γ the domain inside of the parabola. The solution of the problem $(zI-A)u=-f(x), x\in(0,1)$ can be represented by

$$\begin{aligned}
u(x) = \frac{1}{\sqrt{z-1}(1-\cos\sqrt{z-1})}\Bigg\{\int_0^x [\sin(\sqrt{z-1}(\xi-x))-\\
-\cos(\sqrt{z-1}(1-x))\sin(\sqrt{z-1}\xi)]f(\xi)d\xi-\\
-\int_x^1 (\sin(\sqrt{z-1}x)\cos(\sqrt{z-1}(1-\xi))f(\xi)d\xi\Bigg\}.
\end{aligned}$$

Analogously, as in Example 2.1, one can show that for all $z\in\mathbb{C}\setminus\Omega_\Gamma$ the estimate

$$\|(zI-A)^{-1}f\|_\infty = \|u\|_\infty \leqslant \frac{M}{\sqrt{|z-1|}}\|f\|_\infty,\ \forall f(x)\in L_\infty(0,1)$$

holds with a positive constant M. Using the estimate

$$\sqrt{|z-1|} \geqslant \frac{1+\sqrt{|z|}}{4}, \qquad \forall z \in \mathbb{C}\setminus\Omega_\Gamma,$$

we get

$$\|(Iz-A)^{-1}f\|_{L_\infty(0,1)\to L_\infty(0,1)} \leqslant \frac{M_1}{1+\sqrt{|z|}}.$$

Example 2.3. Let Ω be a bounded domain with a Lipschitz boundary in $\mathbf{R}^d, d = 2, 3$ and

$$\begin{aligned} L(u) &= -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{d} a_i(x)\frac{\partial u}{\partial x_i} \\ &+ a_0(x)u, x \in \Omega \subset \mathbf{R}^d,\ d = 2, 3 \end{aligned} \tag{2.3}$$

be a strongly elliptic differential operator, i.e.,

$$\sum_{i,j=1}^{d} a_{ij} y_i \overline{y}_j \geqslant c_1 \sum_{i=1}^{d} |y_i|^2 \tag{2.4}$$

and $a_{pq} = a_{qp}$.

In the usual way (see for example [6, 11]) the differential expression (2.3) defines the sesquilinear form $a(u, v)$ and the unbounded operator $A : L_2(\Omega) \to L_2(\Omega)$ with the domain $D(A) = \{u \mid u \in H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)\}$. Let $|u|_1^2$ be the semi-norm of the Sobolev space $H^1(\Omega)$, $\| \cdot \|_k$ be the norm of the Sobolev space $H^k(\Omega),\ k = 0, 1, ..., H^0(\Omega) = L_2(\Omega)$, c_* be the edge of the cube containing the domain Ω, and $c_F = 1/(4c_*^2)$ be the constant from the Friedrichs inequality.

Denoting

$$c_2 = \min_x |\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a_0|$$

and using the well-known Gårdings inequality [11], one can obtain (see also [6]) for the numerical range $\{a(u, u) = \xi + i\eta\ \forall u \mid \|u\|_{L_2} = 1\}$ of the operator A

$$\begin{aligned} \xi &= \Re e\ a(u, u) \geqslant c_1 |u|_1^2 - c_2 \\ |\eta| &= |\Im m\ a(u, u)| \leqslant c_3 |u|_1, \end{aligned} \tag{2.5}$$

from where

$$\begin{aligned} \xi &> c_1 c_F - c_2,\ |u|_1^2 \leqslant \frac{\xi + c_2}{c_1}, \\ & |\eta| < c_3 \sqrt{\frac{\xi + c_2}{c_1}} \end{aligned} \tag{2.6}$$

with $c_3 = \sqrt{d} \max_{x,p} |a_p(x)|$. It follows from the first and the second inequalities that the numerical range (and the spectrum) of A are enveloped by the parabola $\eta^2 = k(\xi - \xi_0)$ with

$$\begin{aligned} k &= \frac{c_3^2}{c_1} = \frac{\sqrt{d} \max_{x,p} |a_p(x)|}{c_1}, \\ \xi_0 &= -c_2 = -\min_x |\frac{1}{2}\sum_p \frac{\partial a_p}{\partial x_p} - a_0|, \end{aligned} \tag{2.7}$$

i.e., the parabola is completely determined by the coefficients of the differential equation. It was proved in [6] that A possesses only a discrete spectrum. Supposing that the spectrum of A lies in the right half-plane, one can easily see that there exists a parabola $\Gamma = \{z = \xi + i\eta |\ \xi = a\eta^2 + b, \}$ with $a, b > 0$ enveloping the spectrum of A. Analogously to [7] one can prove that

$$\|(z-A)^{-1}\| \leqslant \frac{M}{1+\sqrt{|z|}},\ \forall z \in \mathbb{C} \setminus \Omega_\Gamma \tag{2.8}$$

holds with a positive constant M.

Note that the inequality $c_1 c_F - c_2 > 0$ is a sufficient one to guarantee that the spectrum of A lies in the right half-plane.

This example gives a motivation for the following generalization.

Let $V \subset H \subset V^*$ be a triple of Hilbert spaces and let $a(\cdot,\cdot)$ be a sesquilinear form on V. We denote by c_e the constant from the imbedding inequality $\|u\|_X \leqslant c_e \|u\|_V$. Assume that $a(\cdot,\cdot)$ is bounded, i.e.,

$$|a(u,v)| \leqslant c\|u\|_V \|v\|_V,\ u, v \in V. \tag{2.9}$$

The boundedness of $a(\cdot,\cdot)$ implies the well-posedness of a bounded operator $A : V \to V^*$ through the identity

$$a(u,v) =< Au, v >, u, v \in V., \tag{2.10}$$

where $< \cdot, \cdot >$ denotes the duality relation between V and V^*. One can restrict A to a domain $D(A) \in V$ and consider this operator (perhaps as unbounded) acting in X. The assumptions

$$\Re e\ a(u,u) \geqslant \delta_0 \|u\|_V^2 - \delta_1 \|u\|_X^2\ \forall u \in X \tag{2.11}$$

and

$$|\Im m\ a(u,u)| \leqslant \kappa \|u\|_V \|u\|_X \tag{2.12}$$

guarantee that the numerical range $\{a(u,u),\ \forall u \mid \|u\|_X = 1\}$ of A (and the spectrum $\Sigma(A)$) lies inside of the parabola determined by the constants $\delta_0, \delta_1, \kappa$. Actually, if $a(u,u) = \xi + i\eta$, then we get

$$\begin{aligned} \xi &= \Re e\ a(u,u) \geqslant \delta_0 N_V - \delta_1, \\ |\eta| &= |\Im m\ a(u,u)| \leqslant \kappa \sqrt{N_V}, \end{aligned} \tag{2.13}$$

where $N_V = \|u\|_V^2$. It implies

$$\begin{aligned} \xi &> \delta_0 c_e^{-2} - \delta_1,\ N_V \leqslant \frac{1}{\delta_0}(\xi + \delta_1), \\ & |\eta| \leqslant \kappa \sqrt{\frac{\xi + \delta_1}{\delta_0}}. \end{aligned} \tag{2.14}$$

The first and the second inequalities mean that the parabola $\Gamma_\delta = \{z = \xi + i\eta : \xi = \frac{\delta_0}{\kappa^2}\eta - \delta_1\}$ envelopes the numerical range of A . It is easy to see that under the assumption that $\Re e\ \Sigma(A) > 0$ there exists another parabola $\Gamma_0 = \{z = \xi + i\eta : \xi = a\eta^2 + \gamma_0\}$ with $a, \gamma_0 > 0$ containing the spectrum of A. We denote by Ω_{Γ_0} the domain inside of this parabola. Now, we are in a position to give the following definition generalizing the common properties of the three operators above.

Definition 2.1. *We say that an operator $A : E \to E$ is strongly P-positive if its spectrum $\Sigma(A)$ lies in the domain Ω_{Γ_0} enveloped by the parabola Γ_0 and on Γ_0 and outside of Γ_0 the estimate*

$$\|(z - A)^{-1}\|_{E \to E} \leqslant \frac{M}{1 + \sqrt{|z|}} \tag{2.15}$$

holds true with a positive constant M (see [7]).

Note that there is another important class of operators in the mathematical literature, namely, the strongly positive operators which play a significant role in the theory of semi-groups, theory of the strongly elliptic operators, theory of finite-element method, and other fields (see, e.g., [3, 11]). Contrary to the strongly P-positive operators, these operators possess a spectrum Σ placed in a symmetric (with respect to the positive x-axis) angle 2ϕ, $\phi < \frac{\pi}{2}$ with the vertex at the origin, $\Re e\, \Sigma > 0$ and having a resolvent satisfying

$$\|(z - A)^{-1}\| \leqslant \frac{M}{1 + |z|}$$

on the edges and outside of the angle.

The strongly elliptic partial differential operators with $\Re e\, \Sigma > 0$ are important examples of both the strongly P-positive and strongly positive operators. Clearly, good approximations of these partial differential operators have to possess analogous spectral properties.

In this section we show that the solution of (1.1) can be represented by (1.2) with an integration parabola $\Gamma = \{z = (\xi, \eta) : \xi = \tilde{a}\eta^2 + b,\ \tilde{a} > a, b < \gamma_0\}$ containing the spectral parabola $\Gamma_0 = \{z = (\xi, \eta) : \xi = a\eta^2 + \gamma_0\}$ and under assumptions that A is a strongly P-positive operator and $u_0 \in D(A^\epsilon)$ for any $\epsilon > \frac{1}{2}$ (no other information about the operator is necessary).

In fact, after parametrization of the path of Γ and using the strong P-positivity of A we have

$$\begin{aligned}
\|u(t)\| &= \|\frac{1}{2\pi i}\int_\Gamma e^{-zt}(z - A)^{-1}u_0 dz\| \\
&= \|\frac{1}{2\pi i}\int_\infty^0 e^{-(\tilde{a}\eta^2 + b + i\eta)t}(\tilde{a}\eta^2 + b + i\eta - A)^{-1}(2\tilde{a}\eta + i)d\eta\, u_0 \\
&\quad + \frac{1}{2\pi i}\int_0^\infty e^{-(\tilde{a}\eta^2 + b - i\eta)t}(\tilde{a}\eta^2 + b - i\eta - A)^{-1}(2\tilde{a}\eta - i)d\eta\, u_0\| \\
&\leqslant c\int_0^\infty e^{-(\tilde{a}\eta^2 + b)t}\frac{\sqrt{4\tilde{a}^2\eta^2 + 1}}{[1 + ((\tilde{a}\eta^2 + b)^2 + \eta^2)^{1/4}]\,((\tilde{a}\eta^2 + b)^2 + \eta^2)^{\epsilon/2}}d\eta\|A^\epsilon u_0\|.
\end{aligned} \tag{2.16}$$

It is easy to see that this integral converges for all $t > 0$ if $\epsilon = 0$ and for $t = 0$ provided that $\epsilon > \frac{1}{2}$. Analogously we get for the derivative of u

$$\begin{aligned}\|u'(t)\| &= \|\frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0 dz\| \\ &= \| -\frac{1}{2\pi i}\int_\infty^0 (\tilde{a}\eta^2+b+i\eta)e^{-(\tilde{a}\eta^2+b+i\eta)t}(\tilde{a}\eta^2+b+i\eta-A)^{-1}(2\tilde{a}\eta+i)d\eta\, u_0 \\ &\quad -\frac{1}{2\pi i}\int_0^\infty (\tilde{a}\eta^2+b-i\eta)e^{-(\tilde{a}\eta^2+b-i\eta)t}(\tilde{a}\eta^2+b-i\eta-A)^{-1}(2\tilde{a}\eta-i)d\eta\, u_0\| \\ &\leqslant \frac{M}{\pi}\int_0^\infty (\sqrt{(\tilde{a}\eta^2+b)^2+\eta^2})^{1-\epsilon}e^{-(\tilde{a}\eta^2+b)t}\frac{\sqrt{4\tilde{a}^2\eta^2+1}}{1+((\tilde{a}\eta^2+b)^2+\eta^2)^{1/4}}d\eta\|A^\epsilon u_0\|.\end{aligned} \tag{2.17}$$

This integral converges for $t > 0$ and $\epsilon = 0$. The convergence of these integrals implies that $u(0) = u_0$ and moreover for $t > 0$

$$\begin{aligned}u'(t) \;+\; Au(t) &= \frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0 dz + A(\frac{1}{2\pi i}\int_\Gamma e^{-zt}(z-A)^{-1}u_0 dz) \\ &= \frac{1}{2\pi i}\int_\Gamma -ze^{-zt}(z-A)^{-1}u_0 dz + \frac{1}{2\pi i}\int_\Gamma ze^{-zt}(z-A)^{-1}u_0 dz = 0,\end{aligned} \tag{2.18}$$

i.e., (1.2) is the solution of (1.1).
Thus, we have proved the following result.

Theorem 2.1. *Let Γ be an integration parabola containing the spectral parabola Γ_0 of a strongly P-positive operator $A : E \to E$. Then, the solution of problem (1.1) with an initial vector $u_0 \in D(A^\epsilon)$, $\epsilon > 1/2$ can be represented by the integral (1.2).*

The parametrized integral (1.2) can be represented in another form

$$u(t) \;=\; \frac{1}{2\pi i}\int_{-\infty}^{\infty}\tilde{F}(\eta,t)d\eta, \tag{2.19}$$

where

$$\tilde{F}(\eta,t) = e^{-(\tilde{a}\eta^2+b-i\eta)t}\hat{u}(\tilde{a}\eta^2+b-i\eta)(2\tilde{a}\eta-i), \tag{2.20}$$

and $\hat{u}(z)$ with $z = \tilde{a}\eta^2+b-i\eta$ is the solution of the stationary equation

$$(z-A)\hat{u}(z) = u_0. \tag{2.21}$$

A unifying numerical algorithm using representation (2.19) as well as its analysis will be described in the next sections where we will consider the case $u_0 \in E$.

3. Representation of the solution of the first order equations with an initial function from E

Let us show that the solution of problem (1.1) and its derivatives can be represented by the formulas

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{-zt}(z-A)^{-1}u_0 dz, \tag{3.1}$$

$$u'(t) = \frac{1}{2\pi i} \int_\Gamma z e^{-zt}(z-A)^{-1}u_0 dz. \tag{3.2}$$

also in the case $x_0 \in E$ provided that A possesses a discrete spectrum inside of a parabola Γ_0 consisting of the eigenvalues $\lambda_j = \mu_j + i\nu_j, j = 1, 2, \ldots$ with $\Re e\ \lambda_j > \gamma$ which correspond to the eigenfunctions $e_j, j = 1, 2, \ldots$ forming a basis of E. In this case there exists a biorthogonal system $f_j, j = 1, 2, \ldots$ in the dual space X^* such that $< e_k, f_j >= \delta_{k,j}$, where $<,>$ denotes the duality relation. One can represent

$$u_0 = \sum_{j=1}^{\infty} \alpha_j e_j,\ \alpha_j =< u_0, f_j > \tag{3.3}$$

with

$$\|u_0\|^2 = \sum_{j=1}^{\infty} | < u_0, f_j > |^2, \tag{3.4}$$

i.e.,

$$u(t) = \frac{e^{-bt}}{2\pi i} \sum_{j=1}^{\infty} \alpha_j e_j I_{1,j}(t), \tag{3.5}$$

where

$$I_{1,j}(t) = \int_\Gamma e^{-(z-b)t}(z-\lambda_j)^{-1}dz = 2\pi i e^{-(\lambda_j - b)t}$$

as a Cauchy integral. It follows from (3.5) that

$$\|u(t)\|^2 = \frac{e^{-2bt}}{4\pi^2} \sum_j {\alpha_j}^2 |I_{1,j}|^2 \leqslant e^{-2bt} \sum_j {\alpha_j}^2 e^{-2(\mu_j - b)t} \leqslant e^{-2\gamma_0 t}\|u_0\|^2. \tag{3.6}$$

In order to show that the function (3.1) satisfies the differential equation we consider integral (3.2) in the form

$$u'(t) = -\frac{1}{2\pi i} \sum_j \alpha_j e_j I_{1,j}^{(1)}(t) \tag{3.7}$$

with

$$I_{1,j}^{(1)}(t) = \int_\Gamma z e^{-zt}(z-\lambda_j)^{-1}dz = 2\pi i \lambda_j e^{-\lambda_j t} \tag{3.8}$$

$$\begin{aligned} \|u'(t)\|^2 &= \sum_{j=1}^{\infty} \alpha_j^2 |\lambda_j|^2 e^{-2\mu_j t} = \sum_{j=1}^{\infty} \alpha_j^2 (\mu_j^2 + \nu_j^2) e^{-2\mu_j t} \\ &\leqslant \sum_{j=1}^{\infty} \alpha_j^2 (\mu_j^2 + \frac{\mu_j - \gamma_0}{a}) e^{-2\mu_j t} \end{aligned} \tag{3.9}$$

since
$$|\lambda_j|^2 = \mu_j^2 + \nu_j^2 \leqslant \left|\mu_j + i\sqrt{\frac{\mu_j - \gamma_0}{a}}\right|^2.$$
The function
$$f(\mu) = \left(\mu^2 + \frac{\mu - \gamma_0}{a}\right) e^{-2\mu t}$$
possesses a local maximum at the point
$$\mu_{max} = \frac{\frac{1}{a}(1 + 2t\gamma_0)}{\sqrt{\left(\frac{t}{a} - 1\right)^2 + \frac{1}{a}(1 + 2t\gamma_0)2t} + 1 - \frac{t}{a}} \rightarrow_{t\to 0+0} \frac{1}{2a}.$$
For the maximum over the interval $[\gamma_0, \infty)$ we have
$$f_{max}(t) = \begin{cases} f(\mu_{max}) & \text{if} \quad \mu_{max} \geqslant \gamma_0, \\ f(\gamma_0) & \text{if} \quad \mu_{max} < \gamma_0. \end{cases}$$
$$= \begin{cases} \frac{1}{a}\left(a\mu_{max}^2 + \mu_{max} - \gamma_0\right)\exp\left(-2t\mu_{max}\right) & \text{if} \quad \mu_{max} \geqslant \gamma_0, \\ \gamma_0^2 \exp\left(-2t\gamma_0\right) & \text{if} \quad \mu_{max} < \gamma_0. \end{cases}$$
This yields the estimate
$$\|u'(t)\|^2 \leqslant f_{max}(t).$$
Therefore, function (3.1) satisfies the initial condition $u(0) = u_0$ and the differential equation
$$u'(t) + Au(t) = -\frac{1}{2\pi i}\int_\Gamma ze^{-zt}(z - A)^{-1}u_0 dz + \frac{1}{2\pi i}\int_\Gamma ze^{-zt}(z - A)^{-1}u_0 dz = 0, \quad t > 0.$$
Thus, we have proved the following result.

Theorem 3.1. *Let Γ be an integration parabola containing the spectral parabola Γ_0 of a strongly P-positive operator $A : E \to E$ with the discrete spectrum $\{\lambda_j\}_{j=1}^\infty$ and the eigenvectors $\{e_j\}_{j=1}^\infty$ forming a basis of the Banach space E. Then, the solution of problem (1.1) with an initial vector from E can be represented by the integral (1.2).*

4. Computational algorithm and analysis

Following [13], we construct a quadrature rule for the integral in (2.19) by using the Sinc approximation on $(-\infty, \infty)$. For $1 \leqslant p \leqslant \infty$, introduce the family $\mathbf{H}^p(D_d)$ of all vector-valued functions, which are analytic in the infinite strip D_d,
$$D_d = \{z \in \mathbb{C} : -\infty < \Re z < \infty, |\Im z| < d\}, \tag{4.1}$$
such that if $D_d(\epsilon)$ is defined for $0 < \epsilon < 1$ by
$$D_d(\epsilon) = \{z \in \mathbb{C} : |\Re z| < 1/\epsilon, |\Im z| < d(1 - \epsilon)\} \tag{4.2}$$

then for each $\mathcal{F} \in \mathbf{H}^p(D_d)$ there holds $\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} < \infty$ with

$$\|\mathcal{F}\|_{\mathbf{H}^p(D_d)} = \begin{cases} \lim_{\epsilon\to 0}(\int_{\partial D_d(\epsilon)} \|\mathcal{F}(z)\|^p |dz|)^{1/p} & \text{if } 1 \leqslant p < \infty, \\ \lim_{\epsilon\to 0}\sup_{z\in\partial D_d(\epsilon)} \|\mathcal{F}(z)\| & \text{if } p = \infty. \end{cases} \tag{4.3}$$

Let

$$S(k,h)(x) = \frac{\sin[\pi(x-kh)/h]}{\pi(x-kh)/h} \tag{4.4}$$

be the k'th Sinc function with the step size h evaluated in x. Given $\mathcal{F} \in \mathbf{H}^p(D_d), h > 0$ and positive integer N, let us use the notations

$$I(\mathcal{F}) = \int_{\mathbb{R}} \mathcal{F}(x)dx, \qquad T_N(\mathcal{F},h) = h\sum_{k=-N}^{N} \mathcal{F}(kh),$$
$$T(\mathcal{F},h) = h\sum_{k=-\infty}^{\infty} \mathcal{F}(kh),$$
$$C(\mathcal{F},h) = \sum_{k=-\infty}^{\infty} \mathcal{F}(kh)S(k,h),$$
$$\eta_N(\mathcal{F},h) = I(\mathcal{F}) - T_N(\mathcal{F},h), \qquad \eta(\mathcal{F},h) = I(\mathcal{F}) - T(\mathcal{F},h).$$

Applying the quadrature rule T_N with the vector-valued function

$$F(\eta,t) = (2\tilde{a}\eta - i)\varphi(\eta)\hat{u}(\psi(\eta)), \tag{4.5}$$

where

$$\varphi(\eta) = e^{-t\psi(\eta)}, \quad \psi(\eta) = \tilde{a}\eta^2 + b - i\eta, \tag{4.6}$$

we obtain for integral (2.19)

$$u(t) = exp(-tA)u_0 \approx u_N(t) = exp_N(-tA)u_0 = h\sum_{k=-N}^{N} F(kh,t). \tag{4.7}$$

This quadrature rule allows one to introduce the following algorithm for the solution of problem (1.1) at a given time value t.

Algorithm 4.1. The Sinc-algorithm for parallel solution of problem (1.1).

1. Given a, γ_0, choose $k > 1, \tilde{a} = \frac{a}{k}$, $d = (1 - \frac{1}{\sqrt{k}})\frac{k}{2a}$, N and determine z_p, α_p ($p = -N, \dots, N$) by $z_p = \frac{a}{k}(ph)^2 + b - iph$, $\alpha_p = 2ph - i$, where $h = \sqrt[3]{\frac{2\pi dk}{a}}(N+1)^{-2/3}$ and $b = \gamma_0 - \frac{k-1}{4a}$.

2. Solve the equations $(z_p - A)\hat{u}(z) = u_0$, $p = -N, \dots, N$ (note that it can be done in parallel).

3. Find the approximation u_N for the solution of (1.1) in the form

$$u_N(t) = h\sum_{p=-N}^{N} \alpha_p e^{-z_p t}\hat{u}(z_p). \tag{4.8}$$

Remark 4.1. The above algorithm possesses two sequential levels of parallelism: first, one can compute all $\hat{u}(z_p)$ at Step 2 in parallel and, second, each operator exponent at different time values $(t_1, t_2, \dots, t_M)$).

Adapting the ideas of [13], one can prove the following approximation results for functions from $\mathbf{H}^1(D_d)$.

Lemma 4.1. *For any vector-valued function $f \in \mathbf{H}^1(D_d)$, there holds*

$$\eta(f,h) = \frac{i}{2} \int_{\mathbb{R}} \left\{ \frac{f(\xi - id^-) e^{-\pi(d+i\xi)/h}}{\sin\left[\pi(\xi - id)/h\right]} - \frac{f(\xi + id^-) e^{-\pi(d-i\xi)/h}}{\sin\left[\pi(\xi + id)/h\right]} \right\} d\xi \tag{4.9}$$

which yields the estimate

$$\|\eta(f,h)\| \leqslant \frac{e^{-\pi d/h}}{2\sinh(\pi d/h)} \|f\|_{\mathbf{H}^1(D_d)}. \tag{4.10}$$

If, in addition, f satisfies on $\mathbb{R}$ the condition

$$\|f(x)\| < ce^{-\alpha x^2}, \quad \alpha, c > 0, \tag{4.11}$$

then

$$\|\eta_N(f,h)\| \leqslant c\sqrt{\pi} \left[\frac{exp(-2\pi d/h)}{\sqrt{\alpha}(1 - exp(-2\pi d/h))} + \frac{exp[-\alpha N^2 h^2]}{\sqrt{\pi}\alpha h N} \right]. \tag{4.12}$$

Proof. Let $E(f,h)$ be defined as follows:

$$E(f,h)(z) = f(z) - C(f,h)(z).$$

Analogously to [13] (see Theorem 3.1.2) one can get

$$\begin{aligned} E(f,h)(z) &= f(z) - C(f,h)(z) \\ &= \frac{\sin(\pi z/h)}{2\pi i} \int_{\mathbb{R}} \left\{ \frac{f(\xi - id^-)}{(\xi - z - id)\sin\left[\pi(\xi - id)/h\right]} \right. \\ &\quad \left. - \frac{f(\xi + id^-)}{(\xi - z + id)\sin\left[\pi(\xi + id)/h\right]} \right\} d\xi \end{aligned} \tag{4.13}$$

and upon replacing z by x, we have

$$\eta(f,h) = \int_{\mathbb{R}} E(f,h)(x) dx \tag{4.14}$$

(here d^- points to the boundary value of f when the argument tends to the boundary of the strip). After interchanging the order of integration and using the identities

$$\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\sin(\pi x/h)}{\pm(\xi - x) - id} dx = \frac{i}{2} e^{-\pi(d \pm i\xi)/h}, \tag{4.15}$$

we obtain (4.9). Using the estimate (see [13; p. 133]) $\sinh(\pi d/h) \leqslant |\sin[\pi(\xi \pm id)/h]| \leqslant \cosh(\pi d/h)$, the assumption $f \in \mathbf{H}^1(D_d)$, and the identity (4.9), we obtain the desired bound (4.10). Assumption (4.11) now implies

$$\begin{aligned}\|\eta_N(f,h)\| &\leqslant \|\eta(f,h)\| + h \sum_{|k|>N} \|f(kh)\| \\ &\leqslant \frac{exp(-\pi d/h)}{2\sinh(\pi d/h)} \|f\|_{\mathbf{H}^1(D_d)} + c\,h \sum_{|k|>N} exp[-\alpha(kh)^2].\end{aligned} \tag{4.16}$$

For the last sum we use the simple estimate

$$\begin{aligned}\sum_{|k|>N} e^{-\alpha(kh)^2} &= 2\sum_{k=N+1}^{\infty} e^{-\alpha(kh)^2} \leqslant 2\int_N^\infty e^{-\alpha h^2 x^2}dx = \frac{1}{\sqrt{\alpha}h}\int_{\sqrt{\alpha}hN}^{\infty} \frac{1}{x}e^{-x^2}2xdx \\ &\leqslant \frac{2}{\alpha h^2 N}\int_{\sqrt{\alpha}hN}^{\infty} xe^{-x^2}dx = \frac{1}{\alpha h^2 N}e^{-\alpha h^2 N^2}.\end{aligned} \tag{4.17}$$

It follows from (4.11) that

$$\|f\|_{\mathbf{H}^1(D_d)} \leqslant 2c\int_{-\infty}^{\infty} e^{-\alpha x^2}dx = \frac{2c}{\sqrt{\alpha}}\sqrt{\pi} \tag{4.18}$$

which together with (4.16) and (4.17) implies

$$\|\eta_N(f,h)\| \leqslant c\sqrt{\pi}\left[\frac{exp(-\pi d/h)}{\sqrt{\alpha}\sinh(\pi d/h)} + \frac{exp[-\alpha N^2 h^2]}{\sqrt{\pi}\alpha h N}\right],$$

which completes the proof. □

Theorem 4.2. *Under the assumptions of Theorems 2.1 or 3.1, choose* $k > 1$, $\tilde{a} = a/k$, $h = \sqrt[3]{2\pi dk/(N^2 a)}$, $b = b(k) = \gamma_0 - (k-1)/(4a)$ *and the integration parabola* $\Gamma_{b(k)} = \{z = \tilde{a}\eta^2 + b(k) - i\eta : \eta \in (-\infty, \infty)\}$, *then there holds*

$$\begin{aligned}\|u(t) - u_N(t)\| &\equiv \|(exp(-tA) - \exp_N(-tA))u_0\| \\ &\leqslant Mc\sqrt{\pi}\left[\frac{2\sqrt{k}exp[-sN^{2/3}]}{\sqrt{at}(1 - exp(-sN^{2/3}))} + \frac{kexp[-tsN^{2/3}]}{\sqrt{\pi}tN^{1/3}\sqrt[3]{2\pi dka^2}}\right]\|u_0\|,\end{aligned} \tag{4.19}$$

where

$$\begin{aligned}s &= \sqrt[3]{(2\pi d)^2 a/k}, \\ c &= M_1 e^{t[ad^2/k+d-b]}, \quad d = (1 - \frac{1}{\sqrt{k}})\frac{k}{2a}, \\ M_1 &= \max_{z\in\overline{D}_d} \frac{|2\frac{a}{k}z - i|}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}}\end{aligned} \tag{4.20}$$

and M *is the constant from the inequality of the strong P-positiveness.*

Proof. First, we note that one can choose as an integration path any parabola

$$\Gamma_b = \{z = \frac{a}{k}\eta^2 + b + i\eta : \eta \in (-\infty, \infty), k > 1, b < \gamma_0\}, \tag{4.21}$$

which contains the spectral parabola

$$\Gamma_0 = \{z = a\eta^2 + \gamma_0 + i\eta : \eta \in (-\infty, \infty)\}. \tag{4.22}$$

In order to apply Lemma 4.1 for the quadrature rule u_N we have to provide that the integrand $\tilde{F}(\eta, t)$ can be analytically extended in a strip D_d around the real axis η. It is easy to see that it is sufficient that there exists $d > 0$ such that for $|\nu| < d$ the function (transformed resolvent)

$$R(\eta + i\nu, A) = [\psi(\eta + i\nu)I - A]^{-1},\ \eta \in (-\infty, \infty),\ |\nu| < d \tag{4.23}$$

has a bounded norm $\|R\|_{E\to E}$. Due to the strong P-positivity of A, the latter can be easily verified if the parabola set

$$\begin{aligned}\Gamma_b(\nu) &= \{z = \frac{a}{k}(\eta + i\nu)^2 + b + i(\eta + i\nu) :\ \eta \in (-\infty, \infty),\ |\nu| < d\} \\ &= \{z = \frac{a}{k}\eta^2 + b + \frac{k}{4a} - \frac{a}{k}(\nu + \frac{k}{2a})^2 + i\eta(1 + \frac{2a}{k}\nu);\ \eta \in (-\infty, \infty),\ |\nu| < d\}\end{aligned} \tag{4.24}$$

for various ν does not intersect Γ_0. We represent the parabolae from the set $\Gamma_b(\nu)$ in the form $\xi = a'\eta^2 + b'$ with

$$a' = \frac{a}{k}\left(1 + \frac{2a}{k}\nu\right)^{-2}, b' = b + \frac{k}{4a} - \frac{a}{k}\left(\nu + \frac{k}{2a}\right)^2. \tag{4.25}$$

Now, it is easy to see that if we choose

$$\nu = \left(\frac{1}{\sqrt{k}} - 1\right)\frac{k}{2a} \equiv -d,\ b = b(k) = \gamma_0 - \frac{k-1}{4a} \tag{4.26}$$

then

$$\begin{aligned}\Gamma_{b(k)}(-d) &= \{z = \frac{a}{k}\eta^2 + b + \frac{k-1}{4a} + i\frac{\eta}{\sqrt{k}} :\ \eta \in (-\infty, \infty)\} \\ &= \{z = \tilde{a}\eta_*^2 + \gamma_0 + i\eta_* :\ \eta_* \equiv \frac{\eta}{\sqrt{k}} \in (-\infty, \infty)\} \equiv \Gamma_0.\end{aligned} \tag{4.27}$$

From (4.25), one can see that $a' \to 0,\ b' \to -\infty$ monotonically with respect to ν as $\nu \to \infty$, i.e., the parabolae from $\Gamma_b(\nu)$ move away from the spectral parabola Γ_0 monotonically. This means that the parabolae set $\Gamma_b(\nu)$ for $b = b(k),\ |\nu| < d$ lies outside of the spectral parabola Γ_0, i.e., we can extend the integrand into the strip (4.1) with d given by (4.26). Note that the choice $\nu = d = (1 - 1/\sqrt{k})\frac{k}{2a}$ selects from the family $\Gamma_{b(k)}(\nu)$ the particular parabola

$$\begin{aligned}\Gamma_{b(k)}(d) &= \{z = a\eta^2/k + b_+ + i\eta(2 - 1/\sqrt{k}) : \eta \in (-\infty, \infty)\} \\ &= \{z = a_+\eta_*^2 + b_+ + i\eta_* : \eta_* \equiv \eta(2 - 1/\sqrt{k}) \in (-\infty, \infty)\}\end{aligned} \tag{4.28}$$

with

$$a_+ = \frac{a}{k(2 - 1/\sqrt{k})^2},\ b_+ = b - \frac{3k - 4\sqrt{k} + 1}{4a},$$

which for $|\nu| \leqslant d$ is the most remote from the spectral parabola Γ_0. Due to the strong P-positivity of A there holds for $z = \eta + i\nu \in D_d$

$$\begin{aligned} \|F(z,t)\| &\leqslant M \frac{|2\frac{a}{k}z - i|\, exp[-t(\frac{a}{k}z^2 + b - iz)]}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}} \|u_0\| \\ &= M \frac{|2\frac{a}{k}z - i| exp\{-t[\frac{a}{k}(\eta^2 - \nu^2) + b + \nu]\}}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}} \in \mathbf{H}^1(D_d) \quad \forall t > 0. \end{aligned} \tag{4.29}$$

We have also

$$\|F(\eta, t)\| < ce^{-\alpha\eta^2},\ \eta \in \mathbb{R} \tag{4.30}$$

with

$$\alpha = t\frac{a}{k}, \quad c = M_1 e^{t[ad^2/k + d - b]}, \quad M_1 = \max_{z \in \overline{D}_d} \frac{|2\frac{a}{k}z - i|}{1 + \sqrt{|\frac{a}{k}z^2 + b - iz|}}. \tag{4.31}$$

If k is choosen so that

$$1 < \sqrt{k} < 1/2 + \sqrt{1/4 + a\gamma_0}$$

then taking into account that $b = \gamma_0 - (k-1)/(4a)$, we get the estimates

$$-\frac{a}{k}\nu^2 + \nu + b > 0\ \forall \nu \in [-d, d], \quad d < \gamma_0/2.$$

The constant c in (4.31) is then given by

$$c = M_1 e^{-t(-a\nu^2/k + \nu + b)}, \tag{4.32}$$

where $c \to 0$ as $t \to \infty$. Using Lemma 4.1 with $\alpha = t\frac{a}{k}$ in (4.12), we get

$$\|\eta_N(F, h)\| \leqslant Mc\sqrt{\pi} \left[\frac{2\sqrt{k} exp(-2\pi d/h)}{\sqrt{at}(1 - exp(-2\pi d/h))} + \frac{k exp[-N^2 h^2 \frac{a}{k} t]}{\sqrt{\pi} athN} \right]. \tag{4.33}$$

Equalizing the exponents by setting $-2\pi d/h = -N^2 h^2 a/k$, we obtain

$$h = \sqrt[3]{\frac{2\pi dk}{a}} N^{-2/3}. \tag{4.34}$$

Substitution of this value into (4.33) leads to the estimate

$$\|\eta_N(F, h)\| \leqslant Mc\sqrt{\pi} \left[\frac{2\sqrt{k} e^{-sN^{2/3}}}{\sqrt{at}(1 - e^{-sN^{2/3}})} + \frac{k e^{-tsN^{2/3}}}{\sqrt{\pi} t N^{1/3} \sqrt[3]{2\pi dka^2}} \right] \|u_0\|, \tag{4.35}$$

which completes our proof. □

Remark 4.2. Choosing k such that

$$1 < \sqrt{k} < \frac{1}{2} + \sqrt{\frac{1}{4} + a\gamma_0},$$

we get that $d < \gamma_0/2$ and

$$-\frac{a}{k}\nu^2 + \nu + b > 0\ \forall z \in [-d, d],$$

i.e., the constant c in (4.31) tends exponentially to 0 when $t \to \infty$.

Remark 4.3. The theorem 4.2 guarantees the exponential convergence of the Sinc-algorithm provided that $t > 0$. Let us show that the algorithm converges even for $t = 0$ with the convergence rate depending on u_0.

Actually, for $t = 0$ we have to compute the integrals

$$\begin{aligned} I_{1,j}(0) &= \int_\Gamma (z-\lambda_j)^{-1} dz = 2i \int_0^\infty \mathcal{F}(\eta,\lambda_j) d\eta \\ &= -\int_{-\infty}^{\infty} \mathcal{F}_1(\eta,\lambda_j) d\eta, \end{aligned} \tag{4.36}$$

where

$$\begin{aligned} \mathcal{F}(\eta,\lambda_j) &= \frac{\tilde{a}\eta^2 - b + \lambda_j}{(\tilde{a}\eta^2 + b + i\eta - \lambda_j)(\tilde{a}\eta^2 + b - i\eta - \lambda_j)}, \quad \tilde{a} = \frac{a}{k}, \\ \mathcal{F}_1(\eta,\lambda_j) &= \frac{2\tilde{a}\eta - i}{(\tilde{a}\eta^2 + b + i\eta - \lambda_j)}. \end{aligned} \tag{4.37}$$

The Sinc-algorithm 4.1 for the integrand $\mathcal{F}_1(\eta,\lambda_j)$ reads as follows

$$\begin{aligned} I_{1,j}^{(N)}(0) &= -h \sum_{p=-N}^{N} \frac{2\tilde{a}(ph) - i}{(\tilde{a}(ph)^2 + b - iph - \lambda_j)} \\ &= 2ih \left[\frac{-1}{2(b-\lambda_j)} + \sum_{p=1}^{N} \frac{\tilde{a}(ph)^2 - b + \lambda_j}{(\tilde{a}(ph)^2 + b + iph - \lambda_j)(\tilde{a}(ph)^2 + b - iph - \lambda_j)} \right] \end{aligned} \tag{4.38}$$

and the error is given by

$$\begin{aligned} (I_{1,j}(0) - I_{1,j}^{(N)}(0))/(2i) &= \int_0^{(N+1)h} \mathcal{F}(\eta,\lambda_j) d\eta \\ &\quad - h \left\{ \frac{1}{2}\mathcal{F}(0,\lambda_j) + \sum_{p=1}^{N} \mathcal{F}(ph,\lambda_j) + \frac{1}{2}\mathcal{F}((N+1)h,\lambda_j) \right\} \\ &\quad + h\mathcal{F}((N+1)h,\lambda_j)/2 + \int_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j) d\eta \\ &= Tr_N(\lambda_j) + h\mathcal{F}((N+1)h,\lambda_j)/2 + \int_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j) d\eta. \end{aligned} \tag{4.39}$$

where

$$Tr_N(\lambda_j) = \int_0^{(N+1)h} \mathcal{F}(\eta,\lambda_j) d\eta - h \left\{ \frac{1}{2}\mathcal{F}(0,\lambda_j) + \sum_{p=1}^{N} \mathcal{F}(ph,\lambda_j) + \frac{1}{2}\mathcal{F}((N+1)h,\lambda_j) \right\}$$

stays for the error of the trapezoidal rule. Supposing $\Im m\ \lambda_j = 0$, we have

$$\begin{aligned}|\mathcal{F}(\eta,\lambda_j)| = & \left|\frac{2\tilde{a}\eta^2}{(\tilde{a}\eta^2+b-\lambda_j)^2+\eta^2} - \frac{\tilde{a}\eta^2+b-\lambda_j}{(\tilde{a}\eta^2+b-\lambda_j)^2+\eta^2}\right| \\ & \leqslant 2\tilde{a} + \frac{1}{\sqrt{(\tilde{a}\eta^2+b-\lambda_j)^2+\eta^2}} \leqslant \quad 2\tilde{a}+\frac{1}{\eta}.\end{aligned} \tag{4.40}$$

Let us estimate the error of the trapezoidal rule $Tr_N(\lambda_j)$:

$$\begin{aligned}|Tr_N(\lambda_j)| = & \left|\sum_{p=0}^{N}\left[\int\limits_{ph}^{(p+1)h}\mathcal{F}(\eta,\lambda_j) - \frac{h}{2}\mathcal{F}(ph,\lambda_j)+\mathcal{F}((p+1)h,\lambda_j)\right]\right| \\ = & \left|\sum_{p=0}^{N}\int\limits_{ph}^{(p+1)h}\left[\int\limits_{ph}^{\eta}(s-\eta)\frac{\partial^2\mathcal{F}(s,\lambda_j)}{\partial s^2}ds + \int\limits_{(p+1)h}^{\eta}(s-\eta)\frac{\partial^2\mathcal{F}(s,\lambda_j)}{\partial s^2}ds\right]d\eta\right| \\ \leqslant & \frac{1}{3}\max_{\eta\in[0,\infty)}\left|\frac{\partial^2\mathcal{F}(\eta,\lambda_j)}{\partial\eta^2}\right| h^3(N+1),\end{aligned} \tag{4.41}$$

where

$$\begin{aligned}\frac{\partial^2\mathcal{F}(\eta,\lambda_j)}{\partial\eta^2} = & 2(2\tilde{a}\eta - i)\frac{\tilde{a}^2\eta^2-3\tilde{a}(b-\lambda_j)+1-i\tilde{a}\eta}{(\tilde{a}\eta^2+b-i\eta-\lambda_j)^3} \\ & + 2(2\tilde{a}\eta+i)\frac{\tilde{a}^2\eta^2-3\tilde{a}(b-\lambda_j)+1+i\tilde{a}\eta}{(\tilde{a}\eta^2+b+i\eta-\lambda_j)^3}.\end{aligned} \tag{4.42}$$

It is clear that for a bounded operator A there exists an absolute constant $c_{\mathcal{F}}$ such that

$$\max_{\eta\in[0,\infty)}\left|\frac{\partial^2\mathcal{F}(\eta,\lambda_j)}{\partial\eta^2}\right| \leqslant c_{\mathcal{F}}. \tag{4.43}$$

In the case of an unbounded operator we have $\lambda_j - b \to \infty$ as $j \to \infty$ and due to

$$\left|\tilde{a}\eta^2+b-\lambda_j+i\eta\right|^2 = \tilde{a}^2\eta^4 + 2\tilde{a}\frac{1+2\tilde{a}(b-\lambda_j)}{2\tilde{a}}\eta^2 + (b-\lambda_j)^2 \geqslant \beta_j \tag{4.44}$$

with $\beta_j = \frac{4\tilde{a}(\lambda_j-b)-1}{4\tilde{a}^2}$, there exists a number j_* such that $\beta_j > 0\ \forall j > j_*$. For all these j we can see from

$$\begin{aligned}\max_{\eta\in[0,\infty)}\left|\frac{\partial^2\mathcal{F}(\eta,\lambda_j)}{\partial\eta^2}\right| = & 4\max_{\eta\in[0,\infty)}\left\{|2\tilde{a}\eta-i|\frac{|\tilde{a}^2\eta^2-3\tilde{a}(b-\lambda_j)+1-i\tilde{a}\eta|}{|\tilde{a}\eta^2+b-i\eta-\lambda_j|^3}\right\} \\ \leqslant & 4\max_{\eta\in[0,\infty)}\left\{|2\tilde{a}\eta-i|\left[\frac{\tilde{a}}{|\tilde{a}\eta^2+b+i\eta-\lambda_j|^2}+\frac{1+4\tilde{a}(\lambda_j-b)}{|\tilde{a}\eta^2+b+i\eta-\lambda_j|^3}\right]\right\} \\ \leqslant & 4\left(\frac{4\tilde{a}}{\sqrt{\beta_j}}+\frac{2+4\tilde{a}^2}{\beta_j}\right)^{3/2}\Bigg|_{j=j_*}\end{aligned} \tag{4.45}$$

that inequality (4.43) holds also in this case (probably with another constant $c_{\mathcal{F}}$ independent of λ_j). Now, it follows from (4.34),(4.40),(4.41) that

$$\begin{aligned}|Tr_N(\lambda_j)| \leqslant & c(N+1)^{-1}, \\ h|\mathcal{F}(N+1,\lambda_j)| \leqslant & c(N+1)^{-2/3}.\end{aligned} \tag{4.46}$$

Further, we have

$$\int_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta = \int_{(N+1)h}^{\infty} \left[\frac{2\tilde{a}\eta - i}{\tilde{a}\eta^2 + b - i\eta - \lambda_j} - \frac{2\tilde{a}\eta + i}{\tilde{a}\eta^2 + b + i\eta - \lambda_j}\right] d\eta$$
$$= \ln \frac{\tilde{a}\eta^2 + b - i\eta - \lambda_j}{\tilde{a}\eta^2 + b + i\eta - \lambda_j}\Bigg|_{\eta=(N+1)h}^{\infty} = \ln \frac{(\tilde{a}\eta^2 + b - i\eta - \lambda_j)^2}{(\tilde{a}\eta^2 + b - \lambda_j)^2 + \eta^2}\Bigg|_{\eta=(N+1)h}^{\infty}$$
$$= \ln \frac{(\tilde{a}\eta^2 + b - \lambda_j)^2 - \eta^2 - i2\eta(\tilde{a}\eta^2 + b - \lambda_j)}{(\tilde{a}\eta^2 + b - \lambda_j)^2 + \eta^2}\Bigg|_{\eta=(N+1)h}^{\infty}$$
$$= \ln\left[\cos\psi(\eta) - i\sin\psi(\eta)\right]\Big|_{\eta=(N+1)h}^{\infty} = -i\psi(\eta)\Big|_{\eta=(N+1)h}^{\infty}, \tag{4.47}$$

where $-\pi < \psi(\eta) < \pi$. Since

$$\frac{2}{\pi} \leqslant \frac{\sin\alpha}{\alpha} \leqslant 1 \;\forall \alpha \in [0, \pi/2] \tag{4.48}$$

we get from (4.47)

$$2\frac{(N+1)h|\tilde{a}(N+1)^2h^2 + b - \lambda_j|}{[\tilde{a}(N+1)^2h^2 + b - \lambda_j]^2 + (N+1)^2h^2}$$
$$\leqslant \left|\int_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta\right| = |\psi((N+1)h)| \tag{4.49}$$
$$\leqslant \pi\frac{(N+1)h|\tilde{a}(N+1)^2h^2 + b - \lambda_j|}{[\tilde{a}(N+1)^2h^2 + b - \lambda_j]^2 + (N+1)^2h^2}.$$

Now, in order to get an estimate for (4.39), we need to estimate the summand (4.49). Let us show that there exists a strongly P-positive operator and such integration parabola that the uniform bound with respect to j for the right-hand side in (4.49) can be a constant only (in what follows we suppose $N >> 1$).

We consider the operator A from Example 2.1. This operator possesses the eigenvalues $\lambda_j = (j\pi)^2,\ j = 1, 2, \ldots$ and the eigenfunctions $e_j(x) = \sqrt{2}\sin j\pi x,\ j = 1, 2, \ldots$ which form an orthonormal basis in $H^1(0,1) \bigcap \overset{\circ}{H}{}^2(0,1)$. Due to (4.20) and (4.34) we have

$$\tilde{a}(N+1)^2h^2 = \mu(N+1)^{2/3}, \quad \mu = [\frac{\pi^2 k}{a}(1 - \frac{1}{\sqrt{k}})^2]^{1/3}, \tag{4.50}$$

i.e., there exists $k^* \in (1, \infty)$ such that $\mu = \mu(k^*) = \pi^2$. Choosing this k, we get

$$\psi((N+1)h) = \arcsin \frac{2\pi^3\sqrt{k^*/a}(N+1)^{1/3}[(N+1)^{2/3} - j^2 + b/\pi^2]}{\pi^4[(N+1)^{2/3} - j^2 + b/\pi^2]^2 + \pi^2(N+1)^{2/3}k^*/a}. \tag{4.51}$$

It follows from (4.51) that

$$\max_j |\psi((N+1)h)| = \arcsin 1 = \frac{\pi}{2} \tag{4.52}$$

independent of N provided that

$$|(N+1)^{2/3} - j^2 + b/\pi^2| = \frac{\sqrt{k^*}}{\pi\sqrt{a}}(N+1)^{1/3}. \tag{4.53}$$

Further, we have in the space H_0^2

$$\begin{aligned}
&\|u(0) - u_N(0)\|^2 \\
&= \frac{1}{\pi^2}\sum_{j=1}^{\infty}\alpha_j^2 \left[Tr_N(\lambda_j) + h\mathcal{F}((N+1)h,\lambda_j) + \int\limits_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta\right] \\
&\leqslant \frac{3}{\pi^2}\sum_{j=1}^{\infty}\alpha_j^2 \left\{[Tr_N(\lambda_j)]^2 + [h\mathcal{F}((N+1)h,\lambda_j)]^2 + \left[\int\limits_{(N+1)h}^{\infty} \mathcal{F}(\eta,\lambda_j)d\eta\right]^2\right\},
\end{aligned} \tag{4.54}$$

where α_j are the Fourier coefficients of the initial function u_0 and $u_N(0)$ is the initial vector computed by the Sinc-algorithm. Taking into account estimates (4.46) and (4.49), we get

$$\begin{aligned}
\|u(0) - u_N(0)\|^2 &\leqslant c[(N+1)^{-2} + (N+1)^{-4/3}]\|u_0\|^2 \\
&\quad + c\sum_{j=1}^{\infty}\alpha_j^2\left\{\frac{(N+1)^{1/3}[(N+1)^{2/3} - j^2 + b/\pi^2]}{[(N+1)^{2/3} - j^2 + b/\pi^2]^2 + (N+1)^{2/3}}\right\}^2 \\
&\leqslant c(N+1)^{-4/3}\|u_0\|^2 \\
&\quad + c\left\{\frac{(N+1)^{1/3}[(N+1)^{2/3} - 1 + b/\pi^2]}{[3(N+1)^{2/3}/4 + b/\pi^2]^2 + (N+1)^{2/3}}\right\}^2 \sum_{j=1}^{F}\alpha_j^2 + c\sum_{j=F+1}^{F1}\alpha_j^2 \\
&\quad + c\left\{\frac{(N+1)^{1/3}[5(N+1)^{2/3}/4 + b/\pi^2]}{[5(N+1)^{2/3}/4 + b/\pi^2]^2 + (N+1)^{2/3}}\right\}^2 \sum_{j=F1+1}^{\infty}\alpha_j^2 \\
&\leqslant c(N+1)^{-1/3}\|u_0\|^2 + c\sum_{j=F1+1}^{\infty}\alpha_j^2,
\end{aligned} \tag{4.55}$$

where F is the greatest integer which is less than, or equal to, $(N+1)^{1/3}/2$ and $F1$ is the greatest integer which is less than, or equal to, $3(N+1)^{1/3}/2$. The last sum can converge arbitrarily slowly to 0 as $N \to \infty$.

Let us get an estimate from below. We have

$$
\begin{aligned}
\|u(0)-u_N(0)\| \geqslant & \frac{1}{\pi}\left\{\sum_{j=1}^{\infty}\alpha_j^2\left[\int_{(N+1)h}^{\infty}\mathcal{F}(\eta,\lambda_j)d\eta\right]^2\right\}^{1/2} \\
& -\frac{1}{\pi}\left\{\sum_{j=1}^{\infty}\alpha_j^2\left[Tr_N^2(\lambda_j)\right]^2\right\}^{1/2} -\frac{1}{\pi}\left\{\sum_{j=1}^{\infty}\alpha_j^2\left[\mathcal{F}^2((N+1)h,\lambda_j)\right]^2\right\}^{1/2} \\
& \geqslant c(N+1)^{1/3}\left\{\sum_{j=1}^{\infty}\alpha_j^2\frac{[(N+1)^{2/3}-j^2+b/\pi^2]^2}{[((N+1)^{2/3}-j^2+b/\pi^2)^2+(N+1)^{2/3}]^2}\right\}^{1/2} \\
& - c[(N+1)^{-1}+(N+1)^{-2/3}]^{1/2} \\
& \geqslant c(N+1)^{1/3}\alpha_1\left[\frac{[(N+1)^{2/3}-1+b/\pi^2]^2}{[((N+1)^{2/3}-1+b/\pi^2)^2+(N+1)^{2/3}]^2}\right]^{1/2} \\
& - c(N+1)^{-1/3} \geqslant c(N+1)^{-1/3}.
\end{aligned}
\tag{4.56}
$$

Thus, we have proved the following result.

Theorem 4.3. *For any arbitrarily slowly decreasing function $\delta(N)$ there exists at least one strongly P-positive operator and such initial vector u_0 that*

$$c(N+1)^{-1/3} \leqslant \|u(0)-u_N(0)\| \leqslant c\delta(N), \tag{4.57}$$

where c is a constant independent of N.

We now turn to discretization in both space and time. Let $\hat{u}_{h_1}(z_j)$ be the solution of the discrete problem

$$(z_j - A_{h_1})\hat{u}_{h_1}(z_j) = P_{h_1}u_0$$

with a discretization A_{h_1} for A and a projection operator P_{h_1} so that

$$\|\hat{u}_{h_1}(z_j)-\hat{u}(z_j)\| \leqslant ch_1^k\|u_0\|$$

holds with c polynomially dependent on z_j (note that finite difference and finite element approximations of elliptic operators possess this property). The fully discrete approximation for the solution of (1.1) is then defined by

$$u_{N,h_1}(t) = h\sum_{j=-N}^{N}\alpha_j\hat{u}_{h_1}(z_j)e^{-z_jt}. \tag{4.58}$$

The next short statement gives the error estimate for this approximation.

Theorem 4.4. *Let the assumptions of Theorem 4.2 and inequality (4.58) hold, then the error estimate of the fully discrete approximation (4.58) is given by*

$$\|u(t)-u_{N,h_1}(t)\| \leqslant c(e^{-sN^{2/3}}+h_1^k)\|u_0\|,\ t>0,$$

where c, s are constants independent of N.

Proof. Since the errors are additive, $\alpha_je^{-z_jt}$ decays exponentially with respect to z_j and $h \asymp N^{-2/3}$, we get immediately the complete error estimate (4.59). □

5. Treatment of nonhomogeneous equations

Let us consider the following nonhomogeneous problem

$$\frac{du}{dt} + Au = f(t),\ u(0) = u_0. \tag{5.1}$$

Given the solution operator e^{-At} (the operator exponential), one can represent the solution of (5.1) by

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-\xi)f(\xi)d\xi}. \tag{5.2}$$

In order to avoid computational expanses connected with the low convergence of the Sinc-Algorithm for small t when using a quadrature rule, we use a polynomial approximation

$$f^{(n)}(t) = f_0 + tf_1 + ... + t^n f_n \tag{5.3}$$

such that for a given tolerance ϵ it holds $\|f - f^{(n)}\| \leqslant \epsilon$ (for example, if $f^{(n)}$ is the Chebyshev interpolation polynomial then $\|f - f^{(n)}\| \leqslant cn^{-m}$ provided that $f(t) \in C^m[0,T]$) and replace the problem (5.2) by

$$\frac{du_n}{dt} + Au_n = f^{(n)}(t),\ u(0) = u_0. \tag{5.4}$$

Denoting

$$I_k = \int_0^t e^{-A(t-\xi)}\xi^k d\xi \tag{5.5}$$

we get by partial integration

$$I_k = A^{-1}[t^k - kI_{k-1}],\ k = 1, 2, ...,\ I_0 = A^{-1}[I - e^{-At}]. \tag{5.6}$$

It yields

$$I_k = \sum_{p=0}^{k}(-1)^p \frac{k!}{(k-p)!} t^{(k-p)} A^{(-1-p)} + (-1)^{k+1}k!A^{-k-1}e^{-At} \tag{5.7}$$

and, finally, we get the following representation for the solution: of (5.4)

$$\begin{aligned} u_n(t) =& e^{-At}\left[u_0 + \sum_{k=0}^{n}(-1)^{k+1}k!A^{-k-1}f_k\right] \\ &+ \sum_{k=0}^{n}\sum_{p=0}^{k}(-1)^p\frac{k!}{(k-p)!}t^{k-1}A^{-1-p}f_p. \end{aligned} \tag{5.8}$$

6. Numerical examples

Example 6.1. Let us consider the following problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t) = u(1,t) = 0, u(x,0) = \sin \pi x$$

Table 1. The error of Algorithm 4.1.

t	ε_{16}	ε_{32}	ε_{64}	ε_{128}
0	6.1458e-01	3.2313e-01	1.7426e-01	1.1123e-01
0.2	4.7224e-02	1.3944e-02	1.8814e-03	1.1108e-04
0.4	8.4978e-04	2.3909e-04	3.9416e-06	3.4460e-08
0.6	5.2588e-04	2.0196e-05	2.1643e-07	1.5619e-10
0.8	1.1102e-04	1.9911e-06	2.6417e-09	6.3806e-14
1.0	1.1094e-05	5.8952e-08	2.0010e-11	3.6594e-16

t	ε_{256}	ε_{512}	ε_{1024}
0	7.8104e-02	5.7705e-02	4.3870e-02
0.2	1.6160e-06	2.0480e-09	3.4646e-14
0.4	4.8676e-11	3.9544e-16	3.2618e-25
0.6	3.6177e-16	5.4619e-23	3.4073e-36
0.8	8.8512e-20	4.9334e-30	2.4506e-47
1.0	2.3290e-24	5.1397e-38	3.0872e-58

with the exact solution $u(x,t) = e^{-\pi^2 t}\sin\pi x$. The numerical solution was computed in accordance with the Sinc- algorithm 4.1 ($a = 1, b = 1, k = 2$) where step 2 was performed using explicit formulas. The error $\varepsilon_N = \varepsilon_N(x,t) = u(x,t) - u_N(x,t)$ for $x = 0.5$ as a function of N is given by Table 1 and is in a good agreement with the estimate (4.19).

The next table presents the error of Algorithm 4.1 at $x = 0.5,\ t = 0$ and the experimental convergence rate with respect to $N^{-\rho}$ indicating the order $\rho \approx 1/3$.

Table 2. The convergence rate at $t = 0$.

N	ε_N	ρ_N
2048	0.03391	0.369
4096	0.02648	0.356

References

[1] *Schwarz methods for obstacle problems with convection-diffusion operators*, in: *Handbook of Numerical Analysis* (P. Ciarlet and J. Lions, eds.), vol. 1, North-Holland, Amsterdam, 1990, pp. 251–256.

[2] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Commuications on Pure and Applied Mathematics, **XV** (1962), pp. 119–147.

[3] A. Ashyralyev and P. Sobolevskii, *Well-Posedness of Parabolic Difference Equations*, Birkhäuser Verlag, 1994.

[4] H. Bateman and A. Erdelyi, *Higher Transcendental Functions*, Mc Graw-Hill Book Company, Inc., 1953.

[5] C. Bernardi and Y. Maday, *Approximations Spectrales de Problemes aux Limites Elliptiques*, Springer-Verlag, 1992.

[6] T. Carleman, *Uber die asymptotische verteilung der eigenwerte partieller differentialgleichungen*, Ber. der Sächs. Akad Wiss. Leipzig, Math.-Nat.Kl., **88** (1936), pp. 119–132.

[7] I. Gavrilyuk, *Strongly p-positive operators and explicit representation of the solutions of initial value problems for second order differential equations in banach space*, Journ. of Math. Analysis and Appl., **236** (1999), pp. 327–349.

[8] A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, Clarendon Press, New York, Oxford, 1985.

[9] S. Krein, *Linear Differential Operators in Banach Spaces*, TAMS, New York, 29, 1971.

[10] R. Kress, *Linear Integral Equations*, Springer-Verlag, Berlin et al, 1989.

[11] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Difference Equations*, Springer-Verlag, Berlin et al, 1983.

[12] D. Sheen, I. Sloan, and V. Thomeé, *A parallel method for time-discretization of parabolic problems based on contour integral representation and quadrature*, Mathematics of Computation, **69**, pp. 177–195.

[13] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, Beijing, 1993.

[14] G. Szegö, *Orthogonal Polynomials*, AMS, New York, 1959.

[15] V. Thomeé, *Finite difference methods for linear parabolic equations*, in: *Handbook of Numerical Analysis*, vol. 1, North-Holland, Amsterdam, 1990, pp. 251–256.

Received 22 Oct. 2001
Revised 23 Jan. 2002