COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 1 (2001), No. 4, pp. 356–366

CONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR THE THIRD BOUNDARY-VALUE PROBLEM FOR AN ELLIPTIC EQUATION WITH VARIABLE COEFFICIENTS

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Abstract — In this paper we study the convergence of a finite difference scheme that approximates the third boundary-value problem for an elliptic equation with variable coefficients on a unit square. We assume that the generalized solution of the problem belongs to the Sobolev space W_2^3. An "almost" second-order convergence rate estimate (with additional logarithmic multiplier) in the discrete W_2^1 norm is obtained.

2000 Mathematics Subject Classification: 65N15.

Keywords: third boundary-value problem, finite differences, Sobolev spaces, convergence rate estimates.

1. Introduction

For a class of finite difference schemes (FDSs) approximating elliptic boundary-value problems (BVPs) with generalized solutions, convergence rate estimates consistent with the smoothness of the data

$$\|u-v\|_{W_p^k(\omega)} \leqslant Ch^{s-k}\|u\|_{W_p^s(\Omega)}, \qquad s \geqslant k, \tag{1}$$

are of great interest (see [13], [7]). Here $u = u(x)$ denotes the solution of the BVP, v denotes the solution of corresponding FDS, h is the discretization parameter, $W_p^k(\omega)$ is the Sobolev space of mesh functions, and C is a positive generic constant, independent of h and u.

A standard technique for the derivation of such a type of estimates (see [17], [7]) is based on the Bramble—Hilbert lemma [3], [5]. In particular, for the third BVP $O(h^{3/2})$ convergence rate estimates of type (1) were derived in [15] and [12]. Improved results are obtained for the Poisson equation [10], and for the problems with mixed boundary conditions [2], [4]. In the present paper, for the FDS that approximates the third BVP for a selfadjoint elliptic equation with variable coefficients in the square an $O(h^2 \ln^{3/2} h^{-1})$ convergence rate estimate in the discrete W_2^1 norm is obtained.

2. Boundary-Value Problem and its Approximation

As a model problem we consider the third-boundary value problem for an elliptic equation with variable coefficients in the unit square $\Omega = (0,1)^2$:

$$
\begin{aligned}
L(u) &\equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\,\frac{\partial u}{\partial x_j}\Big) = f \quad \text{in } \Omega,\\
l(u) &\equiv \sum_{i,j=1}^{2} a_{ij}\,\frac{\partial u}{\partial x_j}\,\cos(\nu, x_i) + \sigma u = g \quad \text{on } \Gamma = \partial\Omega\,,
\end{aligned}
\tag{2}
$$

where ν is the unit outward normal to Γ. Let us denote $\Gamma_{1k} = \{k\}\times[0,1]$, $\Gamma_{2k} = [0,1]\times\{k\}$, $k = 0, 1$.

We assume that the generalized solution of problem (2) belongs to the Sobolev space $W_2^3(\Omega)$, $f \in W_2^1(\Omega)$, $a_{ij} \in W_2^2(\Omega)$, $g \in W_2^{3/2}(\Gamma)$ and $\sigma \in W_2^{3/2}(\Gamma)$. Also, let the conditions of strong ellipticity be satisfied:

$$
\begin{gathered}
\sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geqslant c_0 \sum_{i=1}^{2}\xi_i^2, \quad \forall x \in \bar{\Omega},\ \forall \xi \in R^2, \quad c_0 = \text{const} > 0,\\
\sigma(x) \geqslant \sigma_0 = \text{const} > 0.
\end{gathered}
\tag{3}
$$

Let $\bar\omega$ be an uniform mesh in Ω with the step size $h = 1/n$, $\omega = \bar\omega\cap\Omega$, $\gamma = \bar\omega \cap \Gamma$, $\bar\gamma_{i\alpha} = \{x \in \bar\omega \,:\, x_i = \alpha = \text{const},\ 0 \leqslant x_{3-1} \leqslant 1\}$, $\gamma_{i\alpha} = \{x \in \bar\gamma_{i\alpha} \,:\, 0 < x_{3-i} < 1\}$, $\gamma_{i\alpha}^- = \{x \in \bar\gamma_{i\alpha} \,:\, 0 \leqslant x_{3-1} < 1\}$, $\gamma_{i\alpha}^+ = \{x \in \bar\gamma_{i\alpha} \,:\, 0 < x_{3-1} \leqslant 1\}$, $\gamma_{i\alpha}^\star = \bar\gamma_{i\alpha}\cap\gamma_{i\alpha}$, $\gamma^\star = \gamma \setminus \{\cup_{i,k}\gamma_{ik}\}$, $i = 1,2$, $k = 0,1$. The finite difference operators are defined in the usual manner [16]: $v_{x_i} = (v^{+i} - v)/h$, $v_{\bar x_i} = (v - v^{-i})/h$, where $v^{\pm i}(x) = v(x \pm h r_i)$, and r_i is the unit vector of the axis x_i.

We define the following inner products and discrete norms:

$$[u,v] = h^2 \sum_{x\in\omega} u(x)v(x) + \frac{h^2}{2}\sum_{x\in\gamma\setminus\gamma^\star} u(x)v(x) + \frac{h^2}{4}\sum_{x\in\gamma^\star} u(x)v(x), \qquad |[v]|^2 = [v,v],$$

$$[u,v)_i = h^2 \sum_{x\in\omega\cup\gamma_{i0}} u(x)v(x) + \frac{h^2}{2}\sum_{x\in\gamma_{3-i,0}^-\cup\gamma_{3-i,1}^-} u(x)v(x), \qquad |[v\|_i^2 = [v,v)_i,$$

$$(u,v]_i = h^2 \sum_{x\in\omega\cup\gamma_{i1}} u(x)v(x) + \frac{h^2}{2}\sum_{x\in\gamma_{3-i,0}^+\cup\gamma_{3-i,1}^+} u(x)v(x), \qquad \|v]|_i^2 = (v,v]_i,$$

$$[u,v) = h^2 \sum_{x\in\omega\cup\gamma_{10}^-\cup\gamma_{20}^-} u(x)v(x), \qquad |[v\|^2 = [v,v), \qquad (u,v] = h^2 \sum_{x\in\omega\cup\gamma_{11}^+\cup\gamma_{21}^+} u(x)v(x),$$

$$\|v]|^2 = (v,v], \qquad |[v]|^2_{W_2^1(\bar\omega)} = |[v]|^2 + |[v_{x_1}\|_1^2 + |[v_{x_2}\|_2^2, \qquad |[v]|_{C(\bar\omega)} = \max_{x\in\bar\omega}|v(x)|,$$

$$[u,v]_{\bar\gamma_{i\alpha}} = h\sum_{x\in\gamma_{i\alpha}} u(x)v(x) + \frac{h}{2}\sum_{x\in\gamma_{i\alpha}^\star} u(x)v(x), \qquad |[v]|^2_{\bar\gamma_{i\alpha}} = [v,v]_{\bar\gamma_{i\alpha}},$$

$$[u,v)_{\gamma_{i\alpha}^-} = h\sum_{x\in\gamma_{i\alpha}^-} u(x)v(x), \qquad |[v\|^2_{\gamma_{i\alpha}^-} = [v,v)_{\gamma_{i\alpha}^-}.$$

We also define the Steklov smoothing operators with the step size h [17]:

$$T_i^+ f(x) = \int_0^1 f(x + htr_i)\, dt = T_i^- f(x + hr_i) = T_i f(x + \frac{h}{2} r_i),$$

$$T_{i\pm}^2 f(x) = 2 \int_0^1 (1-t) f(x \pm htr_i)\, dt,$$

where $i = 1, 2$. These operators commute and transform derivatives into differences:

$$T_i^+ \left(\frac{\partial u}{\partial x_i}\right) = u_{x_i}\,, \quad T_i^- \left(\frac{\partial u}{\partial x_i}\right) = u_{\bar{x}_i}\,, \quad T_i^2 \left(\frac{\partial^2 u}{\partial x_i^2}\right) = u_{\bar{x}_i x_i}\,, \quad i = 1, 2\,.$$

We approximate BVP (2) with the following FDS:

$$L_h v = \tilde{f}, \qquad x \in \bar{\omega}, \tag{4}$$

where

$$L_h v = \begin{cases} -\frac{1}{2} \sum_{i,j=1}^2 \Big[\big(a_{ij}\, v_{x_j}\big)_{\bar{x}_i} + \big(a_{ij}\, v_{\bar{x}_j}\big)_{x_i} \Big], & x \in \omega \\ -\frac{2}{h} \Big(\frac{a_{11} + a_{11}^{+1}}{2}\, v_{x_1} + \frac{a_{12} v_{x_2} + a_{12}^{+1} v_{\bar{x}_2}^{+1}}{2} - \tilde{\sigma} v \Big) - \big(a_{21}\, v_{x_1}\big)_{\bar{x}_2} - \Big(\frac{a_{22} + a_{22}^{+2}}{2}\, v_{x_2} \Big)_{\bar{x}_2}, & x \in \gamma_{10} \\ -\frac{2}{h} \Big[\frac{a_{11} + a_{11}^{+1}}{2}\, v_{x_1} + a_{12} v_{x_2} + a_{21} v_{x_1} + \frac{a_{22} + a_{22}^{+2}}{2}\, v_{x_2} - (\tilde{\sigma}_1 + \tilde{\sigma}_2) v \Big], & x = (0,0) \\ \text{and analogously at the other boundary nodes,} & x \in \gamma \setminus \gamma_{10}^- \end{cases}$$

$$\tilde{f} = \begin{cases} T_1^2 T_2^2 f, & x \in \omega \\ T_{i\pm}^2 T_{3-i}^2 f + \frac{2}{h}\, T_{3-i}^2 g, & x \in \gamma_{i,\, 0.5 \mp 0.5} \\ T_{1\pm}^2 T_{2\pm}^2 f + \frac{2}{h} \big(T_{1\pm}^2 g + T_{2\pm}^2 g \big), & x = (0.5 \mp 0.5,\, 0.5 \mp 0.5) \in \gamma^\star \end{cases}$$

$$\tilde{\sigma} = T_{3-i}^2 \sigma\,, \qquad x \in \gamma_{i0} \cup \gamma_{i1}\,, \quad i = 1,\, 2$$

and

$$\tilde{\sigma}_i = T_{3-i\pm}^2 \sigma\,, \quad x \in \gamma^\star, \quad x_{3-i} = 0.5 \mp 0.5\,, \quad i = 1,\, 2\,.$$

Let u be the solution of the BVP (2), and v the solution of the FDS (4). The error $z = u - v$ is defined on $\bar{\omega}$ and satisfies the condition

$$L_h z = \psi, \qquad x \in \bar{\omega}, \tag{5}$$

where

$$\psi = \begin{cases} \displaystyle\sum_{i,j=1}^2 \eta_{ij,\, \bar{x}_i}\,, & x \in \omega \\ \dfrac{2}{h}\, \eta_{11} + \dfrac{2}{h}\, \eta_{12} + \tilde{\eta}_{21,\, \bar{x}_2} + \tilde{\eta}_{22,\, \bar{x}_2} + \dfrac{2}{h}\, \zeta, & x \in \gamma_{10} \\ \dfrac{2}{h}\, \tilde{\eta}_{11} + \dfrac{2}{h}\, \tilde{\eta}_{12} + \dfrac{2}{h}\, \tilde{\eta}_{21} + \dfrac{2}{h}\, \tilde{\eta}_{22} + \dfrac{2}{h} \left(\zeta_1 + \zeta_2 \right), & x = (0,0) \\ \text{and analogously at the other boundary nodes,} & x \in \gamma \setminus \gamma_{10}^- \end{cases}$$

$$\eta_{ij} = T_i^+ T_{3-i}^2 \Big(a_{ij} \, \frac{\partial u}{\partial x_j} \Big) - \frac{1}{2} \left(a_{ij} u_{x_j} + a_{ij}^{+i} u_{\bar{x}_j}^{+i} \right), \qquad x \in \omega$$

$$\tilde{\eta}_{ii} = T_i^+ T_{3-i\pm}^2 \Big(a_{ii} \, \frac{\partial u}{\partial x_i} \Big) - \frac{a_{ii} + a_{ii}^{+i}}{2} \, u_{x_i}, \qquad x \in \gamma_{3-i,\,0.5\mp 0.5}^-$$

$$\tilde{\eta}_{i,3-i} = \begin{cases} T_i^+ T_{3-i+}^2 \Big(a_{i,3-i} \, \frac{\partial u}{\partial x_{3-i}} \Big) - a_{i,3-i} \, u_{x_{3-i}}, & x \in \gamma_{3-i,0}^- \\ T_i^+ T_{3-i-}^2 \Big(a_{i,3-i} \, \frac{\partial u}{\partial x_{3-i}} \Big) - a_{i,3-i}^{+i} \, u_{\bar{x}_{3-i}}^{+i}, & x \in \gamma_{3-i,1}^- \end{cases}$$

$$\zeta = (T_i^2 \sigma) u - T_i^2 (\sigma u), \qquad x \in \gamma_{3-i,0} \cup \gamma_{3-i,1}$$

$$\zeta_i = (T_{i\pm}^2 \sigma) u - T_{i\pm}^2 (\sigma u), \qquad x \in \gamma^\star, \quad x_{3-i} = 0.5 \mp 0.5.$$

3. Stability of FDS

We will prove an a priori estimate for the FDS (5).

Lemma 1. *Let $a_{ij} \in W_2^{3/2}(\Omega)$ and $\sigma \in C(\Gamma)$. Then, for sufficiently small $h \leqslant h_0$, there exist the positive constants C_1 and C_2 such that*

$$C_1 |[v]|_{W_2^1(\bar{\omega})}^2 \leqslant [L_h v, v] \leqslant C_2 |[v]|_{W_2^1(\bar{\omega})}^2.$$

Proof. The proof follows immediately from

$$[L_h v,\, v] = \frac{1}{2} \sum_{i,j=1}^{2} \Big\{ [a_{ij} v_{x_j},\, v_{x_i}) + (a_{ij} v_{\bar{x}_j},\, v_{\bar{x}_i}] \Big\} + \frac{h}{4} \sum_{i=1}^{2} [a_{ii}^{+i} - a_{ii},\, v_{x_i}^2)_{\gamma_{3-i,\,0}^-}$$
$$+ h \sum_{x \in \gamma \setminus \gamma^\star} \tilde{\sigma} \, v^2 + \frac{h}{2} \sum_{x \in \gamma^\star} (\tilde{\sigma}_1 + \tilde{\sigma}_2) \, v^2.$$

□

Lemma 2. *The following inequality holds true:*

$$\Big| \, [v,\, w_{x_{3-i}})_{\gamma_{i\alpha}^-} \Big| \leqslant C \Bigg\{ h^2 \sum_{x,\, x' \in \gamma_{i\alpha}^-,\; x' \neq x} \left[\frac{v(x) - v(x')}{x_{3-i} - x'_{3-i}} \right]^2$$
$$+ h \sum_{x \in \gamma_{i\alpha}^-} \left(\frac{1}{x + h/2} + \frac{1}{1 - x - h/2} \right) v^2(x) \Bigg\}^{1/2} |[w]|_{W_2^1(\bar{\omega})}.$$

Proof. Let us set $i = 1$ (the proof is analogous for $i = 2$). The mesh function w can be presented in the form

$$w(x_1, x_2) = \sum_{l=0}^{n}{}' \sum_{k=0}^{n}{}' a_{kl} \cos k\pi x_1 \, \cos l\pi x_2 = \sum_{l=0}^{n}{}' C_l(x_1) \cos l\pi x_2, \tag{6}$$

where $\sum_{l=0}^{n}{}' b_l = b_0/2 + \sum_{l=1}^{n-1} b_l + b_n/2$. From here follows

$$w_{x_2} = \sum_{l=0}^{n}{}' C_l(\alpha)\Big(-\frac{2}{h}\sin\frac{l\pi h}{2}\Big)\sin l\pi(x_2+h/2), \quad x=(\alpha,x_2)\in\gamma_{1\alpha}^-. \tag{7}$$

Using the orthogonality of sines, one immediately obtains

$$|[w_{x_2}||^2_{\bar\gamma_{1\alpha}} = \frac{1}{2}\sum_{l=0}^{n}{}' \lambda_l\, C_l^2(\alpha), \quad \lambda_l = \Big(\frac{2}{h}\sin\frac{l\pi h}{2}\Big)^2. \tag{8}$$

Let us consider an analogous sine-expansion of the mesh function v:

$$v = \sum_{l=0}^{n}{}' D_l(\alpha)\,\sin l\pi(x_2+h/2), \quad x=(\alpha,x_2)\in\gamma_{1\alpha}^-. \tag{9}$$

From (7) and (9) one easily obtains

$$\Big|[v,\, w_{x_{3-i}})_{\gamma_{i\alpha}^-}\Big| \leqslant \bigg(\frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, C_l^2(\alpha)\bigg)^{1/2}\bigg(\frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, D_l^2(\alpha)\bigg)^{1/2}. \tag{10}$$

Using the inequality (see [1])

$$\max_{x_1}|C(x_1)|^2 \leqslant \varepsilon\, h\sum_{m=0}^{n-1} C_{x_1}^2(mh) + \Big(\frac{1}{\varepsilon}+1\Big)\, h\sum_{m=0}^{n}{}' C^2(mh), \quad \varepsilon>0,$$

where $C(x_1)$ is the mesh function defined for $x_1\in\{0,\, h,\, 2h,\, \dots,\, nh\}$, we obtain

$$\frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, C_l^2(\alpha) \leqslant \frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\bigg\{\varepsilon_l h\sum_{m=0}^{n-1} C_{l,x_1}^2(mh) + \Big(\frac{1}{\varepsilon_l}+1\Big)\, h\sum_{m=0}^{n}{}' C_l^2(mh)\bigg\}.$$

Choosing $\varepsilon_0 = 1,\ \varepsilon_l = (1+\sqrt{1+4\lambda_l})/2\lambda_l,\ l\geqslant 1$, we obtain

$$\frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, C_l^2(\alpha) \leqslant \max_{1\leqslant l\leqslant n}\frac{1+\sqrt{1+4\lambda_l}}{2\sqrt{\lambda_l}}\cdot\frac{1}{2}\sum_{l=0}^{n}{}' \bigg\{h\sum_{m=0}^{n-1} C_{l,x_1}^2(mh) + \lambda_l h\sum_{m=0}^{n}{}' C_l^2(mh)\bigg\}.$$

Using the well-known inequality $\lambda_l\geqslant\lambda_1\geqslant 8,\ l\geqslant 1$, from here we obtain

$$\begin{aligned}\frac{1}{2}\sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, C_l^2(\alpha) &\leqslant \frac{5}{4}\, h\sum_{m=0}^{n-1}\bigg(\frac{1}{2}\sum_{l=0}^{n}{}' C_{l,x_1}^2(mh)\bigg) + \frac{5}{4}\, h\sum_{m=0}^{n}{}'\bigg(\frac{1}{2}\sum_{l=0}^{n}{}' \lambda_l\, C_l^2(mh)\bigg)\\ &\leqslant \frac{5}{4}\, h\sum_{m=0}^{n-1}|[w_{x_1}]|^2_{\bar\gamma_{1,mh}} + \frac{5}{4}\, h\sum_{m=0}^{n}{}' |[w_{x_2}\|^2_{\gamma_{1,mh}^-} = \frac{5}{4}\big(|[w_{x_1}\|_1^2 + |[w_{x_2}\|_2^2\big) \leqslant \frac{5}{4}\,|[w]|^2_{W_2^1(\bar\omega)}.\end{aligned} \tag{11}$$

Let us consider the sum

$$N^2(v) = h^2\sum_{x_2,\, t_2=-1,\ t_2\neq 0}^{1-h}\bigg[\frac{v(\alpha,x_2)-v(\alpha,\, x_2-t_2)}{t_2}\bigg]^2, \tag{12}$$

where the mesh function v is extended outside $\bar{\omega}$ by (9). Using the periodicity and orthogonality of sines, from (9) follows

$$N^2(v) = 4 \sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, D_l^2(\alpha)\, I_l,$$

where $I_0 = 0$, and

$$I_l = \frac{\frac{l\pi h}{2}}{\sin\frac{l\pi h}{2}} J_l, \qquad J_l = \frac{l\pi h}{2} \left\{ \sum_{t=h}^{1-h} \left(\frac{\sin\frac{l\pi t}{2}}{\frac{l\pi t}{2}} \right)^2 + \frac{1}{2} \left(\frac{\sin\frac{l\pi}{2}}{\frac{l\pi}{2}} \right)^2 \right\}, \qquad l \geqslant 1.$$

Further,

$$1 \leqslant \frac{\frac{l\pi h}{2}}{\sin\frac{l\pi h}{2}} \leqslant \frac{\pi}{2}, \qquad l \geqslant 1,$$

while J_l is the integral sum for $\int\limits_0^{l\pi/2} \Big(\frac{\sin x}{x}\Big)^2 dx$ and can be estimated from both sides:

$$\frac{1}{4\pi} \leqslant J_l \leqslant 2\pi + \frac{2}{\pi}\,.$$

Hence we conclude that $\left(\frac{1}{2} \sum\limits_{l=0}^{n}{}' \sqrt{\lambda_l}\, D_l^2(\alpha) \right)^{1/2}$ and $N(v)$ are equivalent, i.e.,

$$\frac{1}{4\pi}\,\frac{1}{2} \sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, D_l^2(\alpha) \leqslant N^2(v) \leqslant (\pi^2+1)\,\frac{1}{2} \sum_{l=0}^{n}{}' \sqrt{\lambda_l}\, D_l^2(\alpha). \tag{13}$$

From (12) and the periodicity of sines follows

$$\begin{aligned} N^2(v) &= h^2 \sum_{x_2=-1}^{1-h} \; \sum_{x_2'=x_2-1+h,\; x_2'\neq x_2}^{x_2+1} \left[\frac{v(\alpha, x_2) - v(\alpha,\, x_2')}{x_2 - x_2'} \right]^2 \\ &\leqslant 2h^2 \sum_{x,\, x' \in \gamma_{i\alpha}^-,\; x'\neq x} \left[\frac{v(x) - v(x')}{x_{3-i} - x_{3-i}'} \right]^2 + 8h \sum_{x\in\gamma_{i\alpha}^-} \left(\frac{1}{x+h/2} + \frac{1}{1-x-h/2} \right) v^2(x). \end{aligned} \tag{14}$$

Finally, the assertion follows from (10), (11), (13) and (14). □

Lemma 3. *Let v be a mesh function on $\bar{\omega}$, then*

$$||[v]||_{C(\bar{\omega})} \leqslant C \sqrt{ln\frac{1}{h}}\; ||[v]||_{W_2^1(\bar{\omega})}.$$

Proof. The proof follows immediately from cosine expansion (6) (see also [6]). □

From (5), using partial summation, one obtains

$$\begin{aligned} [L_h z,\, z] = &- \sum_{i=1}^{2} h^2 \sum_{x\in\omega\cup\gamma_{i0}} (\eta_{i1} + \eta_{i2})\, z_{x_i} - \sum_{i=1}^{2} \frac{h^2}{2} \sum_{x\in\gamma_{3-i,0}^-} (\tilde{\eta}_{i1} + \tilde{\eta}_{i2})\, z_{x_i} \\ &+ h \sum_{x\in\gamma\setminus\gamma^\star} \zeta\, z + \frac{h}{2} \sum_{x\in\gamma^\star} (\zeta_1 + \zeta_2)\, z. \end{aligned} \tag{15}$$

Let us set $\tilde{\eta}_{ij} = \overline{\eta}_{ij} + \hat{\eta}_{ij}$, where $\overline{\eta}_{ij}$ and $\hat{\eta}_{ij}$ will be defined later. From (15), using Lemmas 1—3, we obtain the following assertion.

Lemma 4. *The finite difference scheme* (5) *is stable in the sense of the a priori estimate*

$$
\begin{aligned}
\|[z]\|^2_{W_2^1(\bar{\omega})} \leqslant C \Bigg\{ \sum_{i=1}^{2} \Bigg[h^2 \sum_{x\in\omega\cup\gamma_{i0}} (\eta_{i1}^2 + \eta_{i2}^2) + h^2 \sum_{x\in\gamma^-_{3-i,0}\cup\gamma^-_{3-i,1}} (\overline{\eta}_{i1}^2 + \overline{\eta}_{i2}^2) \Bigg] + h \sum_{x\in\gamma\setminus\gamma^\star} \zeta^2 \\
+ h^2 \, ln \frac{1}{h} \sum_{x\in\gamma^\star} (\zeta_1^2 + \zeta_2^2) + \sum_{k=0}^{1} \sum_{i,j=1}^{2} \Bigg[h^4 \sum_{x,\, x'\in\gamma^-_{3-i,k},\; x'\neq x} \left(\frac{\hat{\eta}_{ij}(x) - \hat{\eta}_{ij}(x')}{x_{3-i} - x'_{3-i}} \right)^2 \\
+ h^3 \sum_{x\in\gamma^-_{3-i,k}} \left(\frac{1}{x+h/2} + \frac{1}{1-x-h/2} \right) \hat{\eta}_{ij}^2(x) \Bigg] \Bigg\}.
\end{aligned}
\tag{16}
$$

4. Convergence of FDS

We will prove the following result.

Theorem 1. *Let $a_{ij} \in W_2^2(\Omega)$, $\sigma \in W_2^{3/2}(\Gamma)$, the conditions* (3) *of strong ellipticity be satisfied, and the solution of BVP* (2) *belongs to $W_2^3(\Omega)$. Then FDS* (4) *converges in the norm $W_2^1(\bar{\omega})$ and the convergence rate estimate*

$$
\|[z]\|_{W_2^1(\bar{\omega})} \leqslant Ch^2 \, ln^{3/2} \frac{1}{h} \left(\max_{ij} \|a_{ij}\|_{W_2^2(\Omega)} + \|\sigma\|^2_{W_2^{3/2}(\Gamma)} \right) \|u\|_{W_2^3(\Omega)}. \tag{17}
$$

holds true.

Proof. Under the previous assumptions, if $u \in W_2^3(\Omega)$, then $f \in W_2^1(\Omega)$ and $g \in W_2^{3/2}(\Gamma)$. As is known, the converse is true if f and g satisfy some additional compatibility conditions (see [11], [19]).

In accordance with Lemma 4, the problem of deriving the convergence rate estimate for the FDS (4) is reduced to estimating the right-hand side of inequality (16).

The term η_{ij} can be estimated in the same manner as in the case of the Dirichlet boundary-value problem (see [9], [7]):

$$
h^2 \sum_{x\in\omega\cup\gamma_{i0}} \eta_{i,x_i}^2 \leqslant Ch^{2s-2} \, \|a_{ij}\|^2_{W_2^{s-1}(\Omega)} \|u\|^2_{W_2^s(\Omega)}, \quad 2 < s \leqslant 3. \tag{18}
$$

On γ^-_{10} the term $\tilde{\eta}_{22}$ can be decomposed in the following way:

$$
\tilde{\eta}_{22} = \overline{\eta}_{22} + \hat{\eta}_{22},
$$

where

$$
\begin{aligned}
\overline{\eta}_{22} = {} & \frac{1}{2h^3} \int\limits_0^h \int\limits_{x_2}^{x_2+h} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2''}^{x_2'} \int\limits_{x_2''}^{x_2'} \frac{\partial a_{22}}{\partial x_2}(x_1', x_2''') \, \frac{\partial^2 u}{\partial x_2^2}(x_1', x_2'''') \, dx_2'''' dx_3''' dx_2'' dx_2' dx_1' \\
& + \frac{1}{h^3} \int\limits_0^h \int\limits_{x_2}^{x_2+h} \int\limits_{x_2+h/2}^{x_2'} \int\limits_{x_2}^{x_2+h} \big(x_2 + h/2 - x_2'\big) \, \frac{\partial^2 a_{22}}{\partial x_2^2}(x_1', x_2'') \, \frac{\partial u}{\partial x_2}(x_1', x_2''') \, dx_2''' dx_2'' dx_2'' dx_1' \\
& + \frac{1}{h^3} \int\limits_0^h \int\limits_0^{x_1'} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2'}^{x_2''} \bigg[\frac{\partial a_{22}}{\partial x_1}(x_1'', x_2') \, \frac{\partial^2 u}{\partial x_2^2}(x_1'', x_2''') \\
& \qquad + a_{22}(x_1'', x_2') \, \frac{\partial^3 u}{\partial x_1 \, \partial x_2^2}(x_1'', x_2''') \bigg] \, dx_2''' dx_2'' dx_2' dx_1'' dx_1' \\
& + \frac{1}{2h^3} \int\limits_0^h \int\limits_0^{x_1'} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2}^{x_2'} \bigg[\frac{\partial^2 a_{22}}{\partial x_1 \partial x_2}(x_1'', x_2''') \, \frac{\partial u}{\partial x_2}(0, x_2'') \\
& \qquad + \frac{\partial a_{22}}{\partial x_2}(x_1', x_2''') \, \frac{\partial^2 u}{\partial x_1 \, \partial x_2}(x_1'', x_2'') \bigg] \, dx_2''' dx_2'' dx_2' dx_1'' dx_1' \\
& + \frac{1}{2h^3} \int\limits_0^h \int\limits_0^{x_1'} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2}^{x_2+h} \int\limits_{x_2+h}^{x_2'} \bigg[\frac{\partial^2 a_{22}}{\partial x_1 \partial x_2}(x_1'', x_2''') \, \frac{\partial u}{\partial x_2}(0, x_2'') \\
& \qquad + \frac{\partial a_{22}}{\partial x_2}(x_1', x_2''') \, \frac{\partial^2 u}{\partial x_1 \, \partial x_2}(x_1'', x_2'') \bigg] \, dx_2''' dx_2'' dx_2' dx_1'' dx_1' \\
& + \frac{2}{h^3} \int\limits_0^h \int\limits_0^h \int\limits_{x_1''}^{x_1'} \int\limits_0^{x_1'''} \int\limits_{x_2}^{x_2+h} \left(1 - \frac{x_1'}{h}\right) \frac{\partial^2}{\partial x_1^2}\Big(a_{22} \, \frac{\partial u}{\partial x_2}\Big)(x_1'''', x_2') \, dx_2' dx_1'''' dx_1''' dx_1'' dx_1' \\
& + \frac{1}{h^2} \int\limits_0^h \int\limits_0^{x_1'} \int\limits_0^{x_1''} \int\limits_{x_2}^{x_2+h} \frac{\partial^2}{\partial x_1^2}\Big(a_{22} \, \frac{\partial u}{\partial x_2}\Big)(x_1''', x_2') \, dx_2' dx_1''' dx_1'' dx_1'
\end{aligned}
$$

and

$$
\hat{\eta}_{22} = \frac{1}{3} \int\limits_{x_2}^{x_2+h} \frac{\partial}{\partial x_1}\Big(a_{22} \, \frac{\partial u}{\partial x_2}\Big)(0, x_2') \, dx_2'.
$$

From here we easily obtain

$$
h^2 \sum_{x \in \gamma_{10}^-} \overline{\eta}_{22}^2 \leqslant C h^4 \, \|a_{22}\|^2_{W_2^2(\Omega)} \|u\|^2_{W_2^3(\Omega)} \tag{19}
$$

and

$$
\begin{aligned}
h^4 \sum_{x,\,x'\in\gamma_{10}^-,\ x'\neq x} \left(\frac{\hat\eta_{22}(x)-\hat\eta_{22}(x')}{x_2-x_2'}\right)^2 &\leqslant Ch^4 \left\|\frac{\partial}{\partial x_1}\left(a_{22}\,\frac{\partial u}{\partial x_2}\right)\right\|^2_{W_2^{1/2}(\Gamma_{10})} \\
\leqslant Ch^4 \left\|\frac{\partial}{\partial x_1}\left(a_{22}\,\frac{\partial u}{\partial x_2}\right)\right\|^2_{W_2^1(\Omega)} &\leqslant Ch^4\,\|a_{22}\|^2_{W_2^2(\Omega)}\|u\|^2_{W_2^3(\Omega)}.
\end{aligned} \tag{20}
$$

Using the inequality [14]

$$
\|F\|_{L_2(0,\varepsilon)} \leqslant C\varepsilon^{1/2}\ln\frac{1}{\varepsilon}\,\|F\|_{W_2^{1/2}(0,1)}, \qquad 0<\varepsilon<1,
$$

we obtain

$$
\begin{aligned}
&h^3\sum_{x\in\gamma_{10}^-}\left(\frac{1}{x+h/2}+\frac{1}{1-x-h/2}\right)\hat\eta_{22}^2(x) \leqslant Ch^4\ln^3\frac{1}{h}\,\left\|\frac{\partial}{\partial x_1}\left(a_{22}\,\frac{\partial u}{\partial x_2}\right)\right\|^2_{W_2^{1/2}(\Gamma_{10})} \\
&\leqslant Ch^4\ln^3\frac{1}{h}\,\left\|\frac{\partial}{\partial x_1}\left(a_{22}\,\frac{\partial u}{\partial x_2}\right)\right\|^2_{W_2^1(\Omega)} \leqslant Ch^4\ln^3\frac{1}{h}\,\|a_{22}\|^2_{W_2^2(\Omega)}\|u\|^2_{W_2^3(\Omega)}.
\end{aligned} \tag{21}
$$

Analogously, on γ_{10}^- the term $\tilde\eta_{21}$ can be decomposed in the following way:

$$
\tilde\eta_{21} = \overline{\eta}_{21} + \hat\eta_{21},
$$

where

$$
\begin{aligned}
\overline{\eta}_{21} &= \frac{1}{h^2}\int_0^h\int_0^{x_1'}\int_0^{x_1''}\int_{x_2}^{x_2+h} \frac{\partial^2 a_{21}}{\partial x_1^2}(x_1''',x_2')\,\frac{\partial u}{\partial x_1}(x_1',x_2')\,dx_2'dx_1'''dx_1''dx_1' \\
&+\frac{1}{h^2}\int_0^h\int_0^{x_1'}\int_{x_2}^{x_2+h} x_1'\,\frac{\partial a_{21}}{\partial x_1}(0,x_2')\,\frac{\partial^2 u}{\partial x_1^2}(x_1'',x_2')\,dx_2'dx_1''dx_1' \\
&-\frac{1}{h^2}\int_0^h\int_0^{x_1'}\int_{x_2}^{x_2+h}\int_{x_2}^{x_2'}\left[\frac{\partial^2 a_{21}}{\partial x_1\,\partial x_2}(x_1'',x_2'')\,\frac{\partial u}{\partial x_1}(x_1',x_2'')\right. \\
&\qquad\left.+\frac{\partial a_{21}}{\partial x_1}(x_1'',x_2'')\,\frac{\partial^2 u}{\partial x_1\,\partial x_2}(x_1',x_2'')\right]dx_2''dx_2'dx_1''dx_1' \\
&+\frac{2}{h^3}\int_0^h\int_0^h\int_{x_1''}^{x_1'}\int_0^{x_1'''}\int_{x_2}^{x_2+h}\left(1-\frac{x_1'}{h}\right)\frac{\partial^2}{\partial x_1^2}\left(a_{21}\,\frac{\partial u}{\partial x_1}\right)(x_1'''',x_2')\,dx_2'dx_1''''dx_1'''dx_1''dx_1' \\
&+\frac{1}{h^2}\int_0^h\int_{x_2}^{x_2+h}\int_{x_2+h/2}^{x_2'}\left(x_2+h/2-x_2'\right)\frac{\partial^2}{\partial x_2^2}\left(a_{21}\,\frac{\partial u}{\partial x_1}\right)(x_1',x_2'')\,dx_2''dx_2'dx_1' \\
&+\frac{1}{2h}\int_0^h\int_0^{x_1'}\int_{x_2}^{x_2+h}\frac{\partial^2}{\partial x_1\partial x_2}\left(a_{21}\,\frac{\partial u}{\partial x_1}\right)(x_1'',x_2')\,dx_2'dx_1''dx_1'
\end{aligned}
$$

and

$$\hat{\eta}_{21} = \frac{1}{2}\int\limits_{x_2}^{x_2+h}\left(\frac{\partial a_{21}}{\partial x_1}\,\frac{\partial u}{\partial x_1}\right)(0,x_2')\,dx_2' - \frac{1}{6}\int\limits_{x_2}^{x_2+h}\frac{\partial}{\partial x_1}\left(a_{21}\,\frac{\partial u}{\partial x_1}\right)(0,x_2')\,dx_2'$$
$$+\frac{1}{2}\int\limits_{x_2}^{x_2+h}\frac{\partial}{\partial x_2}\left(a_{21}\,\frac{\partial u}{\partial x_1}\right)(0,x_2')\,dx_2'.$$

From here one obtains estimates of the types (19)—(21). Analogous estimates hold for other terms $\tilde{\eta}_{ij}$.

The term ζ on γ_{10} can be represented in the following manner:

$$\zeta = \frac{1}{h}\int\limits_{x_2-h}^{x_2+h}\int\limits_{x_2}^{x_2'}\int\limits_{x_2'}^{x_2}\left(1-\frac{|x_2'-x_2|}{h}\right)\frac{\partial\sigma}{\partial x_2}(0,x_2'')\,\frac{\partial u}{\partial x_2}(0,x_2''')\,dx_2'''dx_2''dx_2'$$
$$+\frac{\sigma(0,x_2)}{h}\int\limits_{x_2-h}^{x_2+h}\int\limits_{x_2'}^{x_2}\int\limits_{x_2}^{x_2''}\left(1-\frac{|x_2'-x_2|}{h}\right)\frac{\partial^2 u}{\partial x_2^2}(0,x_2''')\,dx_2'''dx_2''dx_2'\ ,$$

wherefrom follows

$$h\sum_{x\in\gamma_{10}}\zeta^2 \leqslant Ch^4\,\|\sigma\|^2_{W_2^{3/2}(\Gamma_{10})}\|u\|^2_{W_2^3(\Omega)}. \tag{22}$$

Analogous estimates hold on other γ_{ik}.

The term $\zeta_1(0,0)$ can be represented in the form

$$\zeta_1(0,0) = \frac{2}{h}\int\limits_0^h\int\limits_0^{x_2'}\left(1-\frac{x_2'}{h}\right)\sigma(0,x_2')\,\frac{\partial u}{\partial x_2}(0,x_2'')\,dx_2''dx_2'\ .$$

From this representation we simply obtain

$$h^2\ln\frac{1}{h}\,\zeta_1^2(0,0) \leqslant Ch^4\ln\frac{1}{h}\,\|\sigma\|^2_{W_2^{3/2}(\Gamma_{10})}\|u\|^2_{W_2^3(\Omega)}. \tag{23}$$

Analogous estimates hold for other similar terms.

Finally, the assertion follows from Lemma 4 and estimates (18)—(23). □

Note that estimate (17) is "almost" consistent with the smoothness of data (with an additional logarithmic multiplier).

Using interpolation theory of Banach spaces (see [18], [8]), from (17) and the known estimate [12]

$$\|[z]\|_{W_2^1(\bar{\omega})} \leqslant Ch^{s-1}\left(\max_{ij}\|a_{ij}\|_{W_2^{s-1}(\Omega)} + \|\sigma\|^2_{W_2^{s-3/2}(\Gamma)}\right)\|u\|_{W_2^s(\Omega)},\quad 2<s<2.5$$

we can obtain the corresponding convergence rate estimate for $2 < s \leqslant 3$.

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Received 6 Aug. 2001
Revised 8 Oct. 2001