COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol.1(2001), No.4, pp.367–382

INTERIOR PENALTY DISCONTINUOUS APPROXIMATIONS OF ELLIPTIC PROBLEMS[1]

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Abstract — This paper studies an interior penalty discontinuous approximation of elliptic problems on nonmatching grids. Error analysis, interface domain decomposition type preconditioners, as well as numerical results illustrating both discretization errors and condition number estimates of the problem and reduced forms of it are presented.

2000 Mathematics Subject Classification: 65F10; 65N20; 65N30.

Keywords: nonmatching grids, interior penalty discretization, discontinuous elements, error estimates, preconditioning.

1. Introduction

In this paper we propose and analyze a simple strategy to construct composite discretizations of self-adjoint second-order elliptic equations on nonmatching grids. The need for discretizations on nonmatching grids is motivated partially by parallel adaptive solution methods for PDEs, which is a much easier task if nonmatching grids are allowed across the subdomain boundaries.

Our method can be described as an interior penalty approximation based on partially discontinuous elements. The mortar method is a general technique of handling discretizations

[1]The work of the first and the second authors has been partially supported by the National Science Foundation under Grant DMS-9973328. The work of the last author was performed under the auspices of the U. S. Department of Energy by University of California Lawrence Livermore National Laboratory under contract W-7405-Eng-48.

on nonmatching grids. However, our motivation for using the penalty approach is that it eliminates the need for additional (Lagrange multiplier or mortar) spaces. There is a vast number of publications devoted to the mortar finite element method as a general strategy for deriving discretization methods on nonmatching grids. We refer the interested reader to the series of Proceedings of the International Conferences on Domain Decomposition Methods (for more information see, http://www.ddm.org).

In the present paper, we assume a model situation where the domain is split into a fixed number of nonoverlapping subdomains, and each subdomain is meshed independently. Therefore in general, since the global mesh is not aligned along the subdomain interfaces, the employed finite element spaces consist of the functions which are discontinuous along these interfaces. The jump in the values of the functions along these interfaces is penalized in the variational formulation, which is a standard approach in the interior penalty method. Such methods have been studied in the past by Rivkind [17], Babuška [3], Arnold [2], and Douglas and Dupont [9] and more recently in [18] and [15, 16] in a different context. We study the approach of Rivkind [17] in the context of nonmatching grids and weak solutions and derive error estimates in both "energy" and L^2-norms. An important feature of this approach is that it omits the term in the weak formulation that involves the conormal derivative of the solution to the interface boundaries, since the latter leads to nonsymmetric discretization (cf. [16]). Thus, for smooth solutions we lose the optimal accuracy due to the poor approximation at the interface, but on the other hand we produce symmetric and positive definite discrete problem which has optimal condition number, provided that condition (A.2) holds. More involved analysis provided in [11] allows one to obtain an almost optimal error of the scheme without any additional assumptions.

One can improve the accuracy by increasing the weight in the penalty term at the expense of increased condition number and regularity of the solution, cf., e.g., [8]. Another approach that requires H^2-regularity which has optimal order error estimates is based on a negative norm for the penalty term (see Remark 3.1). This approach is quite feasible but also increases the condition number and in general requires a more involved implementation. Both approaches (increased penalty weight or negative norm penalty terms) will increase the computational complexity of the method. Here, we play down this issue, since we believe that the adaptive grid refinement approach discussed in the present paper is a good alternative. To compensate for the low accuracy near the subdomain interfaces, we use local grid refinement based on suitable a posteriori error estimators and indicators. Adaptive methods have been extensively used for problems with local singular behavior. Our experience shows that the proposed interior penalty method embedded in a multilevel adaptive grid refinement environment leads to reasonably accurate and fast computations.

The Galerkin finite element method with penalty for this class of problems on matching grids has been proposed and studied in [3, 17]. Similarly, in [8], the interface problem has been addressed by recasting the problem as a system of first order (by introducing the gradient of the solution as a new vector variable) and applying the least-squares method for the system. Integrals of the squared jumps in the scalar and the normal component of the vector functions on the interface are added as penalty terms in the least-squares functional. In both cases an optimal with respect to the error method leads to a nonoptimal condition number of the discrete problem.

Other approaches for handling discretizations on nonmatching grids can involve different discretizations in the different subdomains, for example, mixed finite element method in one subdomain and standard Galerkin on the other as proposed in [20] and studied further

in [12], or mixed finite element discretizations in both subdomains, cf. e.g., [1, 13].

In this paper we also address the issue of constructing preconditioners for solving the system on composite nonmatching grids. We propose and investigate an interface domain decomposition type preconditioner (for two subdomains), that is spectrally equivalent to a reduced problem on the interface.

Finally, the accuracy of the proposed method and the optimal condition number of the preconditioned problem are demonstrated on a series of numerical experiments on model problems.

The structure of the present paper is as follows. In Section 2 we formulate the problem and its discretization. Section 3 contains the error analysis. The construction and analysis of the interface domain decomposition preconditioners are given in Section 4. The numerical results can found in the final section, Section 5.

2. Notations and problem formulation

In this paper we use the standard notation for Sobolev spaces of functions defined in a bounded domain $\Omega \subset \mathcal{R}^d$, $d = 2, 3$. For example, $H^s(\Omega)$, for s being an integer, denotes the Hilbert space of the functions u defined on Ω and having generalized derivatives up to the order s that are square integrable in Ω. For noninteger $s > 0$ the spaces are obtained by the real method of interpolation (cf. [14]). $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ which vanish on $\partial\Omega$. The norm of $u \in H^s(\Omega)$ is denoted by $||u||_{s,\Omega}$. We also use the notation $|u|_{s,\Omega}$ for the seminorm in $H^s(\Omega)$. For the traces of functions in $H_0^1(\Omega)$ on a manifold Γ of dimension $d-1$ (curves and surfaces) with $\partial\Gamma \subset \partial\Omega$, we use the Sobolev fractional-order spaces commonly denoted by $H_{00}^{1/2}(\Gamma)$. The corresponding norm in this space can be characterized, for example, as the infimum of the $H^1(\Omega)$-norm of all possible H^1-extensions vanishing on $\partial\Omega$.

For a given bounded simply-connected polygon (polytope) Ω, a source term $f \in L^2(\Omega)$, and coefficient matrix $a(x)$ that is symmetric, uniformly positive definite and bounded in Ω we consider the following boundary value problem for $u(x)$:

$$\left\{\begin{array}{rcll} -\nabla \cdot a\nabla u & = & f(x), & x \in \Omega, \\ u(x) & = & g_D(x), & x \in \partial\Omega_D, \\ a\nabla u \cdot \mathbf{n} & = & g_N(x), & x \in \partial\Omega_N, \end{array}\right. \tag{2.1}$$

where $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$, $\partial\Omega_D \cap \partial\Omega_N = \emptyset$, $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ (pointing outward Ω), $\partial\Omega_D$ has a positive measure, and g_D and g_N are given functions.

To simplify our notation and the overall exposition, we reduce our considerations to the case of homogeneous Dirichlet data given on the whole boundary $\partial\Omega$, i.e., $\Gamma_N = \emptyset$ ($\Gamma_D = \partial\Omega$) and $g_D = u|_{\Gamma_D} = 0$. However, most of our numerical experiments were done for the general case (2.1).

We shall study a discretization of this problem by the finite element method while using meshes that may not align along certain interfaces. This situation may arise when the domain Ω is split initially into p nonoverlapping subdomains $\Omega_i, i = 1, \ldots, p$ and each subdomain is meshed (triangulated) independently of the others. We assume that for any $i = 1, \ldots, p$, the mesh $\mathcal{T}_i$ is a quasiuniform triangulation of Ω_i, i.e., if $h_T := \mathrm{diam}(T)$, $h_i := \max_{T\in\mathcal{T}_i} h_T$ and $|T| := \mathrm{meas}(T)$ then $|T| \geqslant Ch_i^d$ for any $T \in \mathcal{T}_i$, $d = 2, 3$ with a constant C independent of the triangulation. We also denote $h := \max h_i$ and $\mathcal{T} := \cup_i \mathcal{T}_i$. Let Γ_{ij} be the interface

between two subdomains Ω_i and Ω_j and Γ be the union of all interfaces Γ_{ij}. We note that there is no assumption that along each interface Γ_{ij} the triangulations $\mathcal{T}_i$ and $\mathcal{T}_j$ produce the same mesh.

Let V_i be the conforming finite element space of piecewise linear functions associated with the triangulation $\mathcal{T}_i$ and vanishing on $\partial\Omega_D \cap \partial\Omega_i$. Further, the finite element space V on $\mathcal{T}$ will be the direct sum of $V_i,\ i = 1, \ldots, p$, that is $V = \{v :\ v|_{\Omega_i} := v_i \in V_i,\ i = 1, \ldots, p\}$. Since the meshes $\mathcal{T}_i$ in $\Omega_i,\ i = 1, \ldots, p$, are generally not aligned along the subdomain interfaces Γ_{ij}, the functions $v \in V$ are, in general, discontinuous across Γ_{ij}. However, their traces on Γ_{ij} from Ω_i and Ω_j are welldefined.

We now introduce the second order elliptic bilinear forms

$$a_i(v,\ \psi) := \int_{\Omega_i} a\nabla v \cdot \nabla\psi \, dx, \quad \text{for all } v,\ \psi \in H_0^1(\Omega).$$

Note that the form $a_i(\cdot, \cdot)$ is well defined in $V_i \times V_i$, as well.

The weak form of the boundary-value problem (2.1) is: Find $u \in H_0^1(\Omega)$ such that

$$a(u,\ \varphi) := \sum_i a_i(u,\ \varphi) = (f,\ \varphi), \quad \text{for all } \varphi \in H_0^1(\Omega). \tag{2.2}$$

For the interior penalty finite element method we shall need some additional constructions. To simplify our notation and exposition, we take $d = 2$ (for $d = 3$ in the construction below we have to take $|e|^{1/2}$ instead of $|e|$). We specify a "master" side of each interface Γ_{ij}, i.e., the mesh from $\mathcal{T}_k$ for k fixed, either equal to i or to j, will generate partition $\mathcal{E}_{ij}$ of Γ_{ij}. Then $e \in \mathcal{E}_{ij}$ defined as $e = \Gamma_{ij} \cap \partial T$, for all $T \in \mathcal{T}_k$. Finally, we define the set of interface "master" elements $\mathcal{E} = \cup_{i<j}\mathcal{E}_{ij}$.

We introduce the following bilinear form on $V \times V$:

$$b_h(v, \varphi) := \sum_{e\in\mathcal{E}} \frac{1}{|e|} \int_e [v]_e[\varphi]_e \, d\varrho. \tag{2.3}$$

Here, $[\varphi]_e$ is the jump of φ across $e \in \mathcal{E}_{ij} \subset \mathcal{E}$, i.e., $[\varphi]_e = \varphi_i|_e - \varphi_j|_e$ where $\mathcal{E}_{ij}$ is a partition of Γ_{ij} obtained from the master triangulation $\mathcal{T}_k$ ($k = i$ or $k = j$) and $|e|$ is the measure of e. Since the triangulations $\mathcal{T}_i$ are assumed quasiuniform, we have that $|e| \simeq h_k$.

Remark 2.1. In the case of coefficients with large jumps across the interface Γ or anisotropy, more appropriate is to replace $\frac{1}{|e|}$ by $\frac{\sigma(e)}{|e|}$ in (2.3), where the factor $\sigma(e)$ uses harmonic averages of the coefficient matrix $a(x)$:

$$\sigma(e) = \frac{2}{\kappa_i + \kappa_j},\ \kappa_l = \frac{1}{|e|} \int_e a_l^{-1}(x)\mathbf{n}_e(x) \cdot \mathbf{n}_e(x) \, d\varrho,\ l = i, j.$$

Here $\mathbf{n}_e(x)$ denotes the unit normal vector to e at the point $x \in e$.

We study the following discretization method which is subsequently referred to as the interior penalty discretization:

Find $u_h \in V$ such that

$$A(u_h,\ \varphi) := a(u_h,\ \varphi) + b_h(u_h,\ \varphi) = (f,\ \varphi), \quad \text{for all } \varphi \in V. \tag{2.4}$$

Since the finite element space contains functions that are discontinuous across Γ, the penalty form $b_h(\cdot,\cdot)$ imposes a weak compatibility of the solution across Γ, i.e., it controls the size of the jump $[u_h]_e$.

The bilinear form $A(\cdot,\cdot)$, defined by (2.4), is symmetric and positive definite. It is related to, but much simpler than, the one from the corresponding discontinuous Galerkin method used in [2, 16]. The simplification comes from the fact that we have disregarded the term involving the conormal derivative $a\nabla u \cdot \mathbf{n}$ along the interface Γ. This simplification comes with a cost: the proposed approximation will not have optimal order of convergence, in contrast to the nonsymmetric interior penalty Galerkin method studied in [16]. However, our formulation leads to a symmetric and positive definite problem and combined with local grid refinement generated by an a posteriori analysis produces efficient and accurate computational method as demonstrated by our numerical experiments.

3. Error estimates

In this section we derive the basic error estimates for the proposed interior penalty method (2.4). We assume the following two conditions:

(A.1) The solution $u(x)$ of (2.2) is $H^{\frac{3}{2}+\alpha}(\Omega)$-regular, with $\alpha > 0$, and the following estimate $\|u\|_{\frac{3}{2}+\alpha,\Omega} \leqslant C\|f\|_{0,\Omega}$ holds;

(A.2) The maximum mesh-size h_i of $\mathcal{T}_i$ satisfies $h_i \simeq h$, for $i = 1, \ldots, p$, i.e. the mesh $\mathcal{T}$ is a global quasiuniform partition of Ω.

For $u_h \in V$ we define the "energy" norm $||u_h||^2_{1,h} := A(u_h, u_h)$. Obviously this norm is well defined for $u \in H^1_0(\Omega)$ and $||u||^2_{1,h} = a(u,u)$.

The following theorem is the main result in this section:

Theorem 3.1. *Assume that the conditions (A.1), (A.2) hold. Then*

$$||u - u_h||^2_{0,\Omega} + h||u - u_h||^2_{1,h} \leqslant Ch^2||u||^2_{\frac{3}{2}+\alpha,\Omega} \tag{3.1}$$

with a constant C independent of h and $\alpha > 0$.

Proof. We first estimate $||u - u_h||^2_{1,h}$. Note that the exact solution u satisfies the identity

$$A(u,\ \varphi) = (f,\ \varphi) + \int_\Gamma a\nabla u \cdot \mathbf{n}[\varphi]\, d\varrho, \quad \text{for all } \varphi \in V.$$

Here, we used that the exact solution has continuous (in a weak sense) normal flux.

Recall that the discrete solution u_h satisfies $A(u_h,\ \varphi) = (f,\ \varphi)$ for all $\varphi \in V$, so that the error $e_h := u - u_h$ satisfies the identity

$$A(u - u_h,\ \varphi) = \int_\Gamma a\nabla u \cdot \mathbf{n}\,[\varphi]\, d\varrho, \quad \text{for all } \varphi \in V. \tag{3.2}$$

Let ψ_h be the nodal interpolant of the exact solution in V. Note that ψ_h is discontinuous on Γ, but its jump $[\psi_h]$ is small, since u is a continuous function in Ω.

Now we split the error e_h in the following way: $e_h = u - \psi_h - (u_h - \psi_h)$. Using the above identities, we get the following basic equality for e_h:

$$||e_h||^2_{1,h} \equiv A(e_h,\ e_h) = A(e_h,\ u - \psi_h) + A(e_h,\ \psi_h - u_h). \tag{3.3}$$

We apply Schwarz' inequality to the term $A(e_h,\ u-\psi_h)$:

$$A(e_h,\ u-\psi_h) \leqslant A(e_h,\ e_h)^{1/2} A(u-\psi_h,\ u-\psi_h)^{1/2} = ||e_h||_{1,h}\, A(u-\psi_h,\ u-\psi_h)^{1/2}$$

and use identity (3.2) with $\varphi = \psi_h - u_h$ for the second term to get

$$A(e_h,\ \psi_h - u_h) = \int_\Gamma a\nabla u\cdot\mathbf{n}\,[\psi_h - u_h]\,d\varrho.$$

Now we transform this term by adding and subtracting u and use again Schwarz' inequality to arrive at

$$\begin{aligned}\int_\Gamma a\nabla u\cdot\mathbf{n}\,[\psi_h-u_h]\,d\varrho &= \int_\Gamma a\nabla u\cdot\mathbf{n}\,[\psi_h-u]\,d\varrho + \int_\Gamma a\nabla u\cdot\mathbf{n}\,[u-u_h]\,d\varrho\\ &\leqslant \left(\sum_{e\in\mathcal{E}}|e|\int_e (a\nabla u\cdot\mathbf{n})^2\,d\varrho\right)^{1/2} A(u-\psi_h,\ u-\psi_h)^{1/2}\\ &\quad + \left(\sum_{e\in\mathcal{E}}|e|\int_e (a\nabla u\cdot\mathbf{n})^2\,d\varrho\right)^{1/2} A(e_h,\ e_h)^{1/2}.\end{aligned}$$

Since the solution $u(x)$ is $H^{\frac{3}{2}+\alpha}(\Omega)$-regular, $\alpha>0$, the integral term is easily bounded by the trace theorem

$$\sum_{e\in\mathcal{E}}|e|\int_e (a\nabla u\cdot\mathbf{n})^2\,d\varrho \leqslant Ch\|\nabla u\cdot\mathbf{n}\|^2_{0,\Gamma} \leqslant Ch\|u\|^2_{\frac{3}{2}+\alpha,\Omega}. \tag{3.4}$$

From (3.3) and (3.4) it follows that

$$\|e_h\|_{1,\,h} \leqslant Ch^{\frac{1}{2}}\|u\|_{\frac{3}{2}+\alpha,\,\Omega} + CA(u-\psi_h,\ u-\psi_h)^{\frac{1}{2}}. \tag{3.5}$$

It is clear that the first term (coming from the boundary integral involving the normal derivative of the unknown solution) gives the largest contribution to the error. We show below that the remaining terms on the right-hand side of (3.5) are asymptotically smaller.

We first estimate the term $a(u-\psi_h,\ u-\psi_h)$ in $A(\cdot,\cdot)=a(\cdot,\cdot)+b_h(\cdot,\cdot)$. For this we use the standard error estimate for the interpolant ψ_h of u on V:

$$a(u-\psi_h,\ u-\psi_h) = \sum_i a_i(u-\psi_h,\ u-\psi_h) \leqslant Ch^{1+2\alpha}|u|^2_{\frac{3}{2}+\alpha,\Omega}. \tag{3.6}$$

For the second term $b_h(\cdot,\cdot)$, we proceed as follows. First, we note that

$$\begin{aligned}\sum_{e\in\mathcal{E}}\frac{1}{|e|}\int_e [u-\psi_h]_e^2\,d\varrho &\leqslant Ch^{-1}\sum_{\mathcal{E}_{ij}\subset\mathcal{E}_{\Gamma_{ij}}}\int [u-\psi_h]^2\,d\varrho\\ &\leqslant Ch^{-1}\sum_{\mathcal{E}_{ij}\subset\mathcal{E}_{\Gamma_{ij}}}\int \Big((u-\psi_h)|_{\Gamma_{ij}\cap\partial\Omega_i} - (u-\psi_h)|_{\Gamma_{ij}\cap\partial\Omega_j}\Big)^2\,d\varrho\\ &\leqslant Ch^{-1}\sum_{l=1}^{p}\int_{\partial\Omega_l}(u-\psi_h)^2 d\varrho.\end{aligned} \tag{3.7}$$

Fix l, $1 \leqslant l \leqslant p$. Let T be an element of $\mathcal{T}_l$ such that $e_l = \partial T \cap \partial\Omega_l$ is an edge of T. Since the triangulation $\mathcal{T}_l$ is quasiuniform, we have $|e_l| \simeq Ch_T$ and $|T| \simeq Ch_T^2$. We next recall the following (trace type) inequality:

$$\frac{1}{|e_l|}\int_{e_l} \varphi^2 \, d\varrho \leqslant C\left\{|h|^{-2}\|\varphi\|^2_{0,\,T} + |\varphi|^2_{1,T}\right\}, \text{ for } \varphi \in H^1(T), \tag{3.8}$$

which is verified first for a domain of unit size and then by transforming T to a domain of unit size to get the appropriate scaling.

Summing (3.8) over all edges of $T \in \mathcal{T}_l$ along $\partial\Omega_l$ and using the approximation properties of the space V_l (ψ_h is the nodal interpolant of u in V) we get

$$\begin{aligned} h^{-1}\textstyle\int_{\partial\Omega_l}(u-\psi_h)^2 d\varrho &\leqslant C\left(h^{-2}\|u-\psi_h\|^2_{0,\,\Omega_l} + \|u-\psi_h\|^2_{1,\,\Omega_l}\right) \\ &\leqslant Ch^{1+2\alpha}||u||^2_{\frac{3}{2}+\alpha,\Omega_l} \end{aligned} \tag{3.9}$$

and the result follows by summation over l. That is, (3.7), the last estimate, (3.9) and the earlier one, (3.6), lead to the estimate

$$A(u-\psi_h,\ u-\psi_h) \leqslant Ch^{1+2\alpha}|u|^2_{\frac{3}{2}+\alpha,\,\Omega},$$

which, based on (3.5), completes the error analysis in energy ($\|.\|_{1,h}$) norm.

We now continue with bounding the error in L^2-norm, which is obtained by a standard duality argument. Consider the dual problem: Find $z \in H_0^1(\Omega)$ such that $a(z,\varphi) = (e_h,\varphi)$ for every $\varphi \in H_0^1(\Omega)$. Let $z_h \in V$ be the discrete solution of the dual problem obtained by the above described penalty method, i.e., $A(z_h,\varphi) = (e_h,\varphi)$ for all $\varphi \in V$.

Using the fact that z_h and u_h are the solutions of the discrete problems approximating z and u, respectively, one gets

$$\begin{aligned} \|e_h\|_0^2 &= a(z,\ e_h) + \int_\Gamma a\nabla z\cdot\mathbf{n}\,[e_h]d\varrho \\ &= a(z-z_h,\ e_h) + a(z_h,\ e_h) + \int_\Gamma a\nabla z\cdot\mathbf{n}\,[e_h]d\varrho \\ &\leqslant \|z-z_h\|_{1,h}\|e_h\|_{1,h} - b_h(z_h,\ u-u_h) + \int_\Gamma a\nabla u\cdot\mathbf{n}\,[z_h]d\varrho + \int_\Gamma a\nabla z\cdot\mathbf{n}\,[e_h]d\varrho. \end{aligned}$$

Thus, from the Cauchy—Schwarz inequality and the energy error estimate (3.1), we have

$$\begin{aligned} -b_h(z_h,\ u-u_h) &= b_h(z-z_h,\ u-u_h) \\ &\leqslant [b_h(z-z_h,\ z-z_h)b_h(u-u_h,\ u-u_h)]^{\frac{1}{2}} \\ &\leqslant \|z-z_h\|_{1,h}\|u-u_h\|_{1,h} \\ &\leqslant Ch^{\frac{1}{2}}\|z\|_{\frac{3}{2}+\alpha,\Omega}\|e_h\|_{1,h}. \end{aligned}$$

One also has, using the Cauchy—Schwarz, the trace inequality, the energy error estimate (3.1) for $z-z_h$, and assumption (A.1)

$$\begin{aligned} \int_\Gamma a\nabla u\cdot\mathbf{n}\,[z_h]d\varrho &\leqslant Ch^{\frac{1}{2}}\left[\int_\Gamma (a\nabla u\cdot\mathbf{n})^2\,d\varrho\right]^{\frac{1}{2}}[b_h(z-z_h,\ z-z_h)]^{\frac{1}{2}} \\ &\leqslant Ch\|u\|_{\frac{3}{2}+\alpha,\Omega}\|z\|_{\frac{3}{2}+\alpha,\Omega} \leqslant Ch\|u\|_{\frac{3}{2}+\alpha,\Omega}\|e_h\|_0. \end{aligned}$$

Finally, apply again the trace inequality and assumption (A.1) to get

$$\int_\Gamma (a\nabla z \cdot \mathbf{n})^2 \, d\varrho \leqslant C\|z\|^2_{\frac{3}{2}+\alpha,\Omega} \leqslant C||e_h||^2_0.$$

Using these estimates we get

$$||e_h||^2_0 \leqslant C\left(h^{\frac{1}{2}}||z||_{\frac{3}{2}+\alpha,\Omega}||e_h||_{1,h} + h||z||_{\frac{3}{2}+\alpha,\Omega}\,||u||_{\frac{3}{2}+\alpha,\Omega}\right) \leqslant Ch||u||_{\frac{3}{2}+\alpha,\Omega}\,||e_h||_0,$$

which yields (3.1) for the L^2-norm. □

Remark 3.1. One can achieve optimal order error estimates if the penalty term is taken in a negative norm which will require the use of a different penalty weight. More precisely, one may use the following negative norm interior penalty boundary form:

$$\frac{1}{h^{1+2\alpha}}\ \left\langle \Lambda_h^{-\alpha}[v],\ [w]\right\rangle_\Gamma,$$

where $\Lambda_h : \ V_1 \mapsto V_1$ defines an $H^1_0(\Gamma)$-equivalent norm on $V_1|_\Gamma$, that is $< \Lambda_h v, v >_\Gamma \simeq \|v\|^2_{H^1_0(\Gamma)}$ for $v \in V_1|_\Gamma$. Here for simplicity we have assumed two subdomains, $p = 2$ and taken $k = 1$ as the master side of the boundary $\Gamma = \Gamma_{12}$. For $\alpha \in [0, \frac{1}{2}]$ one gets a scale of interior penalty forms. By a straightforward modification of the above error analysis one can get a $\mathcal{O}(h^{\frac{1}{2}+\alpha})$ error estimates in energy norm for $u \in H^{\frac{3}{2}+\alpha}(\Omega)$. Unfortunately, the condition number of the resulting matrices increases to $\mathcal{O}(h^{-2-2\alpha})$, instead of $\mathcal{O}(h^{-2})$, the condition number of the unpenalized problem. Also, use of the negative norm penalty forms raises the question of computing the actions of the corresponding boundary operator, which in general gives rise to dense matrices. If one assumes a multilevel structure of the mesh in Ω_1, then one potential candidate for $\Lambda_h^{-\alpha}$ which is inexpensively computable can come from the (boundary) Sobolev norms of negative fractional order studied in [7]. It is clear that the use of negative norm operators leads to more involved implementation. In the present paper we have taken the somewhat simpler approach of utilizing local refinement near the boundary in order to improve the accuracy.

4. Iterative solution of the resulting linear system

In this section we study a preconditioning technique for solving the system of algebraic equations produced by the interior penalty method described above. Here we shall study the case $p = 2$, so that $\Gamma_{12} = \Gamma$. Further we shall assume that Γ splits Ω into two simply-connected subdomains and Γ has both ends on $\partial\Omega$.

We introduce the reduced problem on the interface Γ. It is obtained by eliminating the interior to Ω_1 and all Ω_2 degrees of freedom, thus leading to the Schur interface complement system for the unknowns on Γ.

First, we introduce the Schur complement operators $S_i : \ V_i|_\Gamma \mapsto \ V_i|_\Gamma$, $i = 1, 2$:

$$(S_i\psi_i,\ \psi_i) := \inf_{v_i \in V_i:\ v_i|_\Gamma = \psi_i} a_i(v_i,\ v_i),$$

where the pairing $(\cdot, \cdot)$ represents the L_2-inner product on Γ.

Further, for $v, w \in V$ we denote by $\varphi_i = v_i|_\Gamma$ and $\psi_i \in w_i|_\Gamma$ the traces of v_i and w_i, $i = 1, 2$ on Γ. One can eliminate the unknowns in the subdomain Ω_2 which is equivalent to introducing the Schur complement operator σ_1 (see Remark 4.1 for more details):

$$(\sigma_1\varphi_1,\ \varphi_1) := \inf_{v_2 \in V_2} \left[a_2(v_2,\ v_2) + b_h(\varphi,\ \varphi)\right]. \tag{4.1}$$

So we get the following reduced problem on Γ:

$$(S_1\varphi_1,\ \psi_1) + (\sigma_1\varphi_1,\ \psi_1) = (g_1,\ \psi_1), \quad \text{for all } \psi_1 \in V_1|_\Gamma. \tag{4.2}$$

Obviously, $S_1 + \sigma_1 :\ V_1|_\Gamma \mapsto\ V_1|_\Gamma$ is a symmetric and positive definite operator.

The following main result holds:

Theorem 4.1. *The reduced boundary operator $S_1 + \sigma_1$ is spectrally equivalent to S_1, that is*

$$(S_1\psi_1,\ \psi_1) \leqslant ((S_1 + \sigma_1)\psi_1,\ \psi_1) \leqslant C(S_1\psi_1,\ \psi_1), \quad \textit{for all } \psi_1 \in V_1|_\Gamma$$

with a constant independent of h and $\frac{h_1}{h_2}$.

Proof. We only have to prove that σ_1 is bounded in terms of S_1. Note that $(S_1\cdot, \cdot)$ defines a norm equivalent to $H^{\frac{1}{2}}_{00}(\Gamma)$-norm restricted to the traces of the finite element space V_1. The definition of σ_1 can be rewritten as

$$(\sigma_1\psi_1,\ \psi_1) = \inf_{\psi_2 \in V_2|_\Gamma} \left[\inf_{v_2 \in V_2,\, v_2|_\Gamma = \psi_2} a_2(v_2,\ v_2) + b_h(\psi,\ \psi)\right] \quad \text{for all } \psi_2 \in V_2|_\Gamma,$$

which will lead to the inequality

$$(\sigma_1\psi_1,\ \psi_1) \leqslant (S_2\psi_2,\ \psi_2) + b_h(\psi,\ \psi), \quad \text{for all } \psi_2 \in V_2|_\Gamma.$$

Choose now $\psi_2 = Q_2^h\psi_1$ where Q_2^h is the L^2-projection onto V_2. In order to define $Q_2^h\psi_1$, we assume that ψ_1 has been harmonically extended in Ω_2, which means that the $H^1(\Omega_2)$-norm of the extension is bounded by the $H^{\frac{1}{2}}_{00}(\Gamma)$-norm of ψ_1. Then, using inequality (3.8), the L^2-approximation and H^1-boundedness properties of Q_2^h, and the fact that $|e| \simeq h_2$, the boundary term is estimated as follows:

$$\begin{aligned} b_h(\psi,\ \psi) &= \sum_{e \in \mathcal{E}} |e|^{-1} \int_e \left[(I - Q_2^h)\psi_1\right]^2 \, d\varrho \\ &\leqslant C\left[|e|^{-2}\|(I - Q_2^h)\psi_1\|^2_{0,\,\Omega_2} + \|(I - Q_2^h)\psi_1\|^2_{1,\Omega_2}\right] \\ &\leqslant C\|\psi_1\|^2_{1,\,\Omega_2} \leqslant C\|\psi_1\|^2_{\frac{1}{2},\,\Gamma} \leqslant C(S_1\psi_1,\ \psi_1). \end{aligned}$$

The rest is also straightforward. One has

$$\begin{aligned} (S_2\psi_2,\ \psi_2) &\leqslant a_2(Q_2^h\psi_1,\ Q_2^h\psi_1) \leqslant C\|Q_2^h\psi_1\|^2_{1,\,\Omega_2} \\ &\leqslant C\|\psi_1\|^2_{1,\,\Omega_2} \leqslant C\|\psi_1\|^2_{\frac{1}{2},\,\Gamma} \leqslant C(S_1\psi_1,\ \psi_1). \end{aligned}$$

□

Remark 4.1. In order to compute the action of σ_1, one has to eliminate the unknowns in Ω_2 (or to evaluate (4.1)). This is equivalent to the following subdomain problem: given a ψ_1 on Γ, find a $\psi_2 \in V_2$ which solves the variational problem

$$a_2(\psi_2,\ \varphi_2) + \sum_{e\in\mathcal{E}} \frac{1}{|e|} \int_e \psi_2\varphi_2\, d\varrho = \sum_{e\in\mathcal{E}} \frac{1}{|e|} \int_e \psi_1\varphi_2\, d\varrho, \quad \text{for all} \quad \varphi_2 \in V_2;$$

then $\psi_2|_\Gamma = \sigma_1\psi_1$. Obviously, the condition number of the resulting matrix depends on the size of $|e|$, $e \in \mathcal{E}$. It is clear that if $\mathcal{E}$ was based on $\mathcal{T}_2$, then the condition number of the above system would have been independent of h_1. The actions of σ_1 can be computed using the preconditioned conjugate gradient method exploiting (a variant of) the preconditioner for interior penalty bilinear form studied in [18].

5. Numerical results

The performance of the proposed penalty method is described in the following four subsections. In Subsection 5.1 we give results for nonmatching and matching grids. In Subsection 5.2, we have incorporated a weight $\delta > 0$ in the penalty term, and studied its effect on the accuracy and on the condition number of the resulting matrices. Subsection 5.3 deals with locally refined meshes obtained as a result of a posteriori error analysis in order to improve the overall accuracy of the method. Finally, in Subsection 5.4, condition number estimates for the original problem (2.4), as well as for the reduced problem (4.2), are given. The same is done for the preconditioned reduced problem (4.2), using the interface domain decomposition type preconditioner S_1 described in Section 4.

Our finite element implementation handles arbitrary triangulations of the domain and linear finite elements. The code includes a refinement technique, which yields a sequence of nested triangulations that are further used to define multilevel preconditioners for the subdomain problems.

In all tables we present the computational results for various test problems with smooth solutions. The domain is split into two subdomains that are triangulated independently so that the meshes do not necessarily match along the interface Γ. In Tables 1, 2, and 5, we present the number of nodes for each level of grid refinement, the error in maximum (L^∞), L^2 and H^1-norm, and the condition number of the algebraic system for the penalty approximation. The results are given for each subdomain separately, in each box the first line is for the "bigger" domain and the second line is for the "smaller" one (see Figure 1, left).

5.1. Uniform refinement results on nonmatching grids

In the first test we use the mesh shown in Figure 1. The grids are nonmatching along the interface between the subdomains Ω_1 (upper left part of Figure 1) and Ω_2 (lower right corner). The exact solution is $u(x, y) = x^2 - y^2$, and the coefficients are $a_1 = a_2 = I$. Dirichlet boundary conditions are imposed on the lines $x = 0$, $y = 0$, and Neumann boundary conditions on the lines $x = 1$, $y = 1$. The symmetric and positive definite discrete problem is solved using the CG method.

In Figure 1, in addition to the mesh, we have given the approximate solution and the error on refinement level 2. Table 1 summarizes the numerical results. The last column gives the condition number of the discrete problem. The error on the finest (5th) level is 0.22%,

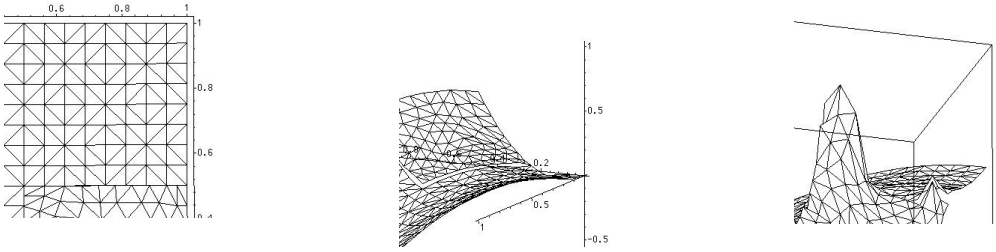

Figure 1. Mesh on level 2 (left), the approximate solution on level 2 (middle) and the error on level 2 (right)

Table 1. Numerical results for uniform refinement; nonmatching grids.

Uniformly refined grid					
level	# nodes	L^∞-error	L^2-error	H^1-error	condition #
1	65	0.027682	0.009983	0.095231	149
	31	0.038660	0.009136	0.069610	
2	225	0.015073	0.005105	0.049991	519
	102	0.022372	0.004702	0.038111	
3	833	0.008204	0.002527	0.025563	2278
	367	0.011481	0.002291	0.019864	
4	3201	0.004255	0.001263	0.013130	9781
	1389	0.005830	0.001131	0.010353	
5	12545	0.002176	0.000632	0.006796	40083
	5401	0.002940	0.000566	0.005512	
order		$\approx$ 1	$\approx$ 1	$\approx$ 1	

0.20% and 0.47% correspondingly in the discrete maximum, L^2 and H^1-norms.

To make an assessment of the performance of the penalty method, we solved the same problem on matching grids uniformly refined to level 5. The results of these computations are presented in Table 2. Comparing Table 1 and Table 2, one can see that the condition numbers and the accuracy are very close. The error on the finest (5th) level is 0.35%, 0.34% and 0.57% correspondingly in the discrete maximum-norm, L^2-norm and H^1-norm.

5.2. Variable penalty weight

Here we test how the penalty weight δ, incorporated in front of the interior penalty boundary form, affects the accuracy and condition number of the resulting discrete problems:

$$a(u_h,\ \varphi) + \sum_{e\in\mathcal{E}} \frac{\delta}{|e|} \int_e [u_h]_e[\varphi]_e \, d\varrho = (f,\ \varphi) + \int_{\Gamma_N} g_N \varphi d\varrho, \quad \text{for all } \varphi \in V.$$

On Γ_D we take u_h to be equal to the piecewise linear interpolant of the boundary data g_D.

The domain is as in the previous subsections and Dirichlet boundary conditions are applied on the whole boundary $\partial\Omega$. In Table 3 we give the results for varying δ on nonmatching

Table 2. Numerical results for uniform refinement; matching grids.

Uniformly refined grid					
level	# nodes	L^∞-error	L^2-error	H^1-error	condition #
1	65	0.039405	0.016707	0.106070	143
	25	0.054990	0.016297	0.084409	
2	225	0.022533	0.008476	0.056189	498
	81	0.027814	0.008517	0.046363	
3	833	0.012380	0.004281	0.029705	1829
	289	0.014366	0.004374	0.025188	
4	3201	0.006617	0.002154	0.015650	6982
	1089	0.007399	0.002221	0.013546	
5	12545	0.003456	0.001081	0.008212	27247
	4225	0.003770	0.001120	0.007223	
order		$\approx$ 1	$\approx$ 1	$\approx$ 1	

grids, and in Table 4 are the results for the case of matching grids. The meshes are kept fixed. The nonmatching grid has 833 nodes in subdomain Ω_1 and 231 nodes in Ω_2, corresponding to $h = 0.04$ ($h^{-\frac{1}{2}} = 5$). The matching grid has 833 nodes in Ω_1 and 289 nodes in Ω_2. The

Table 3. Numerical results for varying δ; nonmatching fixed grids.

δ	L^∞-error	L^2-error	H^1-error	condition #
0.1	0.255030	0.141213	0.497448	420
	0.605293	0.227123	0.642739	
1	0.049788	0.019099	0.135558	420
	0.088299	0.029629	0.132201	
$h^{-1/2}$=5	0.013678	0.004131	0.092995	1190
	0.020614	0.006253	0.074047	
10	0.007464	0.002485	0.089832	2273
	0.011123	0.003379	0.069903	
1000	0.013250	0.002558	0.093995	88288
	0.011979	0.002308	0.075255	

computations for both matching and nonmatching grids show that increasing δ puts more weight on the penalty term and leads to a decrease in the errors in all norms. However, for δ larger than $h^{-1/2}$ there is no significant improvement in the accuracy. Moreover, for nonmatching grids very large δ causes deterioration of the error in maximum and H^1-norm and, as expected, increases the condition number.

5.3. Local refinement results

We consider the problem from Subsection 5.1. There is a wide range of well-established a posteriori local error estimators that are used to generate locally refined meshes that guarantee accurate discretizations, e.g., "*residual based refinement*" [4, 6, 19] "*Zienkiewicz-*

Table 4. Numerical results for varying δ; matching fixed grids.

δ	L^∞-error	L^2-error	H^1-error	condition #
0.1	0.080767	0.024755	0.159351	440
	0.184311	0.039591	0.203872	
1	0.012907	0.003594	0.036052	440
	0.022984	0.005555	0.035535	
$h^{-1/2} = 5$	0.003310	0.000759	0.023133	615
	0.004671	0.001158	0.014892	
10	0.001778	0.000417	0.022384	1162
	0.002340	0.000590	0.013363	
1000	0.000019	0.000173	0.022098	42261
	0.000024	0.000100	0.012758	

Zhu error estimator" [21], "*hierarchical refinement*" [5], and "*second derivative refinement*" [10].

In our context we had to adapt the estimators near Γ due to the interior penalty form. The grids obtained as a result of applying the above four error estimators differ slightly, but in all cases of smooth solutions the estimators lead to meshes that are refined in the areas around the interior boundary. Here we present the results from the "*residual based refinement*" estimator only.

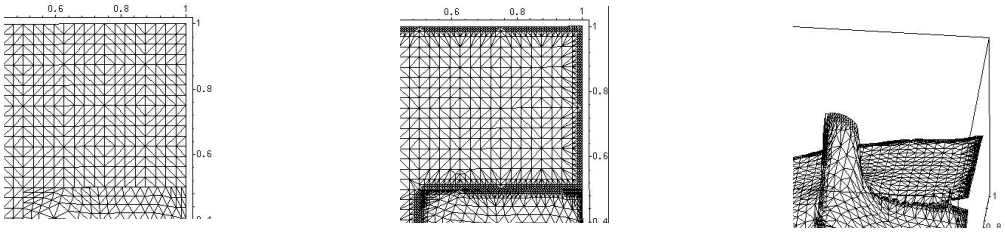

Figure 2. Mesh on level 2 (left), mesh on level 4 (middle), and the error on level 4 (right).

The method is based on equilibrating certain residuals over the elements. The residual over one element is decomposed into two parts. The first part contains the contribution from the interior of the finite element and the second part consists of the contribution from jumps of the normal flux across the finite element boundary.

More specifically, in our computations for every element $T \in \mathcal{T}_i$ we compute and equilibrate the following quantities:

$$\rho_T \equiv h_T ||f + \nabla \cdot a \nabla u_h||_T + h_T^{1/2} \sum_{e \in \partial T} R_e \ ,$$

where the residuals R_e are defined as

$$R_e = \begin{cases} 0 & \text{if } e \in \partial\Omega_D \ , \\ ||g_N - a\nabla u_h \cdot \mathbf{n}||_e & \text{if } e \in \partial\Omega_N \ , \\ \frac{1}{2}|||[a\nabla u_h] \cdot \mathbf{n}||_e & \text{if } e \in \Omega \setminus \{\partial\Omega \cup \Gamma\}, \\ \frac{1}{2}|||[a\nabla u_h] \cdot \mathbf{n}||_e + ||a\nabla u_h \cdot \mathbf{n}||_e & \text{if } e \in \Gamma. \end{cases} \tag{5.1}$$

Removing the jump term $0.5|||[a\nabla u_h] \cdot \mathbf{n}||_e$ for $e \in \Gamma$ from R_e (the last line of (5.1)) will not change substantially the quality of the error estimator, but the resulting refinement procedure will be fully parallel.

Asymptotically, this error indicator, as the a priori estimate, is of order 1/2 because of the term $||a\nabla u_h \cdot \mathbf{n}||_e$. Equilibration of the local error indicators ρ_T ensures that the term $h_1 \int_\Gamma (a\nabla u \cdot \mathbf{n})^2 \, d\varrho$ (see, the error estimates in Theorem 3.1) will be of size Ch^2, where h is the quasiuniform size of the mesh away from Γ, i.e., due to the local refinement we should get close to a first order scheme.

Table 5. Numerical results for local refinement; nonmatching grids.

level	# nodes	L^∞-error	L^2-error	H^1-error	condition #
1	225	0.022438	0.007905	0.052656	521
	102	0.032274	0.007905	0.052656	
2	833	0.012507	0.003926	0.027194	2328
	307	0.016949	0.003248	0.024047	
3	1139	0.006597	0.002000	0.022904	5036
	589	0.008623	0.001591	0.015972	
4	1759	0.003334	0.001017	0.021487	9814
	1109	0.004390	0.000802	0.013027	

The computational results are summarized in Table 5. One notices that for our interior penalty approximation the numerical experiments show that the local refinement reduces the L^2-error according to the theory, the error in the maximum norm is slightly better than one may expect and the error in H^1-norm is slightly worse than expected.

Figure 3. Mesh on level 3 (left), the error (middle), and the approximate solution on the same level (right).

Further, on Figure 3 we show the mesh and the error for two independently meshed subdomains. The solution in this example is chosen in such a way that $a\nabla u \cdot \mathbf{n} = 0$ accross the interface. The plot shows that the error is uniformly the same in the whole domain which in turn indicates that the term $a\nabla u \cdot \mathbf{n}$ is the main contributor to the error of the method.

5.4. Estimates for the condition numbers

The results given in this section are for the problem solved in Subsection 5.1. The meshes are nonmatching and uniform refinement is used. Table 6 compares the condition numbers for matrices corresponding to the original problem (2.4), the reduced problem (4.2), and the reduced problem (4.2) preconditioned with S_1^{-1} (see Section 4). As one can see in

Table 6. In columns 1, 2, and 3 we give the condition numbers for the problems (2.4), (4.2), and (4.2) preconditioned with S_1^{-1}, respectively.

Condition Numbers				
Level	# nodes	1	2	3
1	65 31	149	4	1.90
2	225 102	519	8	1.93
3	833 367	2278	16	2.15
4	3201 1389	9781	35	2.46
5	12545 5401	40083	70	2.57
order	2	2	1	0

Table 6, the condition numbers behave like: $\mathcal{O}(h^{-2})$ for the original problem, $\mathcal{O}(h^{-1})$ for (nonpreconditioned) reduced problems, and $\mathcal{O}(1)$ when the preconditioner from Section 4 is applied, all in good agreement with the theory.

Acknowledgment

Part of this research has been done during the summer visits of the first and the second authors to Lawrence Livermore National Laboratory. The authors thank the Institute for Scientific Computing Research and the Center for Applied Scientific Computing for their hospitality, and for the technical and financial support.

The authors are grateful to Professor J. Pasciak for his careful reading of the manuscript and his comment and remarks that led to an improved presentation of the results.

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Received 17 Aug. 2001
Revised 10 Dec. 2001