COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 1 (2001), No. 4, pp. 383–397

THE FAST SOLUTION OF PERIODIC INTEGRAL AND PSEUDODIFFERENTIAL EQUATIONS BY GMRES

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Abstract — In this paper we consider GMRES to solve finite-dimensional approximations of a class of well-posed linear operator equations in Hilbert spaces. It is shown that the speed of convergence is superlinear. As a consequence we have that GMRES can be used as a fast solver of a fully discrete variant of the trigonometric Galerkin equations associated with periodic integral equations.

2000 Mathematics Subject Classification: 31A30; 35J05; 42A05; 42A15; 45E05; 45E10; 65F10; 65F25; 65R20; 65N38; 65T40; 65Y20.

Keywords: periodic integral equations, pseudodifferential equations, Symm's integral equation, biharmonic equation, Cauchy integral equation, Hilbert integral equation, the hypersingular integral equation, conjugate gradient type methods, GMRES, trigonometric Galerkin method, fast solvers, complexity.

1. Introduction

In this paper we consider GMRES in order to solve an equation $Tu = f$ approximately, where $T : \mathcal{H} \to \mathcal{H}$ is a bijective bounded linear operator in a complex Hilbert space $\mathcal{H}$, and moreover $f \in \mathcal{H}$ holds. For the subsequent numerical analysis we suppose that T and f are replaced by some bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ and $g \in \mathcal{H}$ with $S \approx T$ and $g \approx f$, respectively. Using these approximations, GMRES by definition generates a sequence $x_n \in \mathcal{H},\ n = 1, 2, \ldots$, that has the following properties:

$$x_n \in \mathcal{K}_n(S, g), \tag{1}$$

$$\|Sx_n - g\| = \min_{x \in \mathcal{K}_n(S,g)} \|Sx - g\|, \tag{2}$$

with Krylov subspaces

$$\mathcal{K}_n(S, g) = \operatorname{span}\{\, g, Sg, \ldots, S^{n-1}g \,\}, \qquad n = 1, 2, \ldots .$$

The sequence formally terminates when the residual $Sx_n - g \in \mathcal{H}$ vanishes for some n. If the right-hand side g belongs to a finite-dimensional subspace of $\mathcal{H}$ which is invariant with respect to the operator S, then (1)–(2) can be formulated in a matrix-vector-setting. In that situation there exist some schemes for the computation of the approximations x_n. The most well-known scheme is based on Arnoldi's method where the Krylov subspaces $\mathcal{K}_n(S, g)$ are successively orthonormalized for $n = 1, 2, \ldots$; see, e. g., Greenbaum [5], Trefethen and Bau [18] or [13] for details. Another scheme is based on an orthonormalization of $S\mathcal{K}_n(S, g)$, see [17] for details.

We return to the general situation considered in (1)–(2). From the basic property (2) it follows immediately that

$$\|Sx_n - g\| \leqslant \inf_{p_n \in \Pi_n, p_n(0)=1} \|p_n(S)\| \, \|g\|, \tag{3}$$

where Π_n denotes the set of polynomials of degree $\leqslant n$. The outline of this paper is as follows: first a condition on the spectrum of the original operator T is imposed that allows us to provide an estimate of the right-hand side of inequality (3) showing that the speed of convergence of the residuals (2) is r-superlinear (for that notation see Ortega and Rheinboldt [12]). The corresponding result is applied to a class of linear equations that arise if the boundary integral method is applied to boundary value problems on two-dimensional bounded and simply connected domains with smooth boundaries. Finally, the results of some numerical experiments are presented.

2. Convergence speed of GMRES for perturbations of a class of well-posed equations

In the sequel we specify the conditions on the bounded linear operator $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a complex Hilbert space: we suppose that the spectrum $\sigma(T) \subset \mathbb{C}$ of the operator T satisfies the following conditions:

$$\left.\begin{array}{l} 0 \notin \sigma(T), \\ \sigma(T) \text{ is a countable set, i.e., } \sigma(T) = \{\, \lambda_1, \lambda_2, \ldots \}, \\ \lim\limits_{k\to\infty} \lambda_k \text{ exists, if } \sigma(T) \text{ is infinite.} \end{array}\right\} \tag{4}$$

The most prominent examples of the operators $T : \mathcal{H} \to \mathcal{H}$ satisfying the conditions in (4) are of the form $T = I + K$ where the operator $K : \mathcal{H} \to \mathcal{H}$ is compact and T is supposed to have a trivial null space.

As a preparation we recall the formula $r_A = \lim_{k\to\infty} \|A^k\|^{1/k}$ where $r_A = \max\{\, |\lambda| : \lambda \in \sigma(A)\,\}$ denotes the spectral radius of a bounded linear operator $A : \mathcal{H} \to \mathcal{H}$. We are now in a position to state our main result on the superlinear convergence of the GMRES residuals (2).

Theorem 2.1. *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator in a Hilbert space $\mathcal{H}$, with a spectrum $\sigma(T)$ that satisfies the conditions (4). Then, for each real number $0 < \theta < 1$ there exists a constant c_θ and a real number $\eta = \eta_\theta > 0$ such that for each bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ with $\|S - T\| \leqslant \eta$ the following holds:*

$$\inf_{p_n \in \Pi_n, p_n(0)=1} \|p_n(S)\| \leqslant c_\theta \theta^n \qquad \textit{for} \quad n = 1, 2, \ldots \,. \tag{5}$$

Proof. For notational convenience we restrict the considerations to the case that the spectrum $\sigma(T)$ is infinite, and we use the notation $\lambda_* = \lim_{k\to\infty} \lambda_k$, where $\sigma(T) = \{ \lambda_1, \lambda_2, \dots \}$. Note that $\lambda_* \neq 0$ is satisfied due to the first condition in (4). Without loss of generality we suppose moreover that $|\lambda_1 - \lambda_*| \geqslant |\lambda_2 - \lambda_*| \geqslant \dots$ holds. We define

$$s_n(\lambda) \;=\; \prod_{j=1}^{n} \Big(1 - \frac{\lambda}{\lambda_j}\Big) \qquad \text{for} \quad n \;=\; 1, 2, \dots \;.$$

Then, $s_n \in \Pi_n$ and $s_n(0) = 1$. Clearly $\sigma(s_n(T))$ consists of 0 and $\prod_{j=1}^{n} \frac{\lambda_j - \lambda_k}{\lambda_j},\ k > n$. Since $|\lambda_j - \lambda_k| \leqslant 2|\lambda_j - \lambda_*|$ for $k > n$, we have

$$r_{s_n(T)} \;\leqslant\; \prod_{j=1}^{n} \frac{2|\lambda_j - \lambda_*|}{|\lambda_j|} \qquad \text{for} \quad n \;=\; 1, 2, \dots \;.$$

Hence $r_{s_n(T)}^{1/n} \to 0$ as $n \to \infty$ and thus for each real number $0 < \theta < 1$ there exists an integer $n_\theta \geqslant 1$ with $r_{s_{n_\theta}(T)} \leqslant \frac{1}{2}\theta^{n_\theta}$. Since $\|s_{n_\theta}^k(T)\|^{1/k} \to r_{s_{n_\theta}(T)}$ holds as $k \to \infty$, there exists an integer $k_\theta \geqslant 1$ with

$$\|s_{n_\theta}^{k_\theta}(T)\| \;\leqslant\; \tfrac{1}{2}\theta^{k_\theta n_\theta}.$$

Since $s_{n_\theta}^{k_\theta}$ is a polynomial, there exists a real number $\eta = \eta_\theta > 0$ such that

$$\|s_{n_\theta}^{k_\theta}(S) - s_{n_\theta}^{k_\theta}(T)\| \;\leqslant\; \tfrac{1}{2}\theta^{k_\theta n_\theta} \qquad \text{for} \quad \|S - T\| \leqslant \eta$$

is satisfied. It follows that $\|s_{n_\theta}^{k_\theta}(S)\| \;\leqslant\; \theta^{k_\theta n_\theta}$ holds for each bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ with $\|S - T\| \leqslant \eta$, and thus

$$\|s_{n_\theta}^{k_\theta m}(S)\| \;\leqslant\; \theta^{k_\theta n_\theta m} \qquad \text{for} \quad \|S - T\| \leqslant \eta, \quad m \;=\; 1, 2, \dots \;.$$

We are now in a position to prove the statement of the theorem. Let n be an arbitrary integer $\geqslant 1$. In the situation $n < k_\theta n_\theta$ the polynomial $p_n = 1$ obviously satisfies $p_n \in \Pi_n,\ p_n(0) = 1$ and $\|p_n(S)\| = 1 \leqslant \theta^{-k_\theta n_\theta}\theta^n$ for each bounded linear operator $S : \mathcal{H} \to \mathcal{H}$. If the opposite, i.e., $n \geqslant k_\theta n_\theta$, holds, then for some integer $m \geqslant 1$ we have $k_\theta n_\theta m \leqslant n \leqslant k_\theta n_\theta(m+1)$, and then the polynomial $p_n = s_{n_\theta}^{k_\theta m}$ satisfies $p_n \in \Pi_{k_\theta n_\theta m} \subset \Pi_n$ and $p_n(0) = 1$, and moreover $\|p_n(S)\| \leqslant \theta^{k_\theta n_\theta m} \leqslant (\theta^{-k_\theta n_\theta})\theta^n$ holds for each bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ with $\|S - T\| \leqslant \eta$. Thus, the statement of the theorem follows, with the constant $c_\theta = \theta^{-k_\theta n_\theta}$. □

Remark 2.1. 1. The basic purpose of Theorem 2.1 (see also [17] for a similar result) is to show for GMRES superlinear convergence which is uniform with respect to operator perturbations. Note that only assumptions on the spectrum of the underlying operator are required, normality or diagonalizability is not needed. Moreover note that no resolvent integration is involved in the proof.

2. Superlinear convergence of GMRES for solving a class of finite-dimensional approximations of well-posed problems is obtained in Campbell, Ipsen, Kelley, Meyer and Xue [2]. The general setting considered there is motivated by applying Nyström's method to integral equations of the second kind, and the obtained convergence results for GMRES are also uniform with respect to the considered operator perturbations, which is known also as mesh independence. Other results on the superlinear convergence of GMRES in an infinite-dimensional setting can be found in Campbell, Ipsen, Kelley and Meyer [1],

Kelley and Xue [8] and Moret [10]. For further results on GMRES under more general conditions on the spectrum of the underlying operator we refer to Nevanlinna [11; Chapter 3.3].

3. Additionally we mention here some papers in which GMRES is considered in a finite-dimensional setting and for different purposes than ours, e. g., Elman [3], Freund, Golub and Nachtigal [4], Greenbaum, Pták and Strakŏs [6], Liesen [9], van der Vorst and Vuik [20] and Saad and Schulz [15]; in the latter paper GMRES is introduced.

The result in Theorem 2.1 on the convergence speed of the residuals associated with GMRES can be applied to provide upper bounds for the number of iterations that is needed until a specific stopping criterion applies. In fact, we consider a posteriori stopping criterions of the following form: for some appropriate value of $\delta > 0$, compute the GMRES iterations $x_1, x_2, \ldots$ (see (1)-(2)) until the condition

$$\|Sx_n - g\| \leqslant \delta\|g\| \tag{6}$$

is satisfied for the first time. The stopping index is denoted by $n_\delta := n \geqslant 0$.

As an immediate consequence of Theorem 2.1 we obtain the following asymptotical estimate of the stopping index.

Theorem 2.2. *Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator in a Hilbert space $\mathcal{H}$, with a spectrum $\sigma(T)$ that satisfies the conditions in (4). Then, for each real number $\varepsilon > 0$ there exist real numbers $\delta_\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that the following holds: for each bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ with $\|S - T\| \leqslant \eta_\varepsilon$ and each vector $g \in \mathcal{H}$ we have*

$$n_\delta \leqslant \varepsilon \log(1/\delta) \qquad \text{for} \quad 0 < \delta \leqslant \delta_\varepsilon,$$

where n_δ is the stopping index considered in (6).

Proof. For any real number $0 < \theta < 1$ and any bounded linear operator $S : \mathcal{H} \to \mathcal{H}$ satisfying $\|S - T\| \leqslant \eta$, with $\eta > 0$ chosen according to Theorem 2.1, we obtain $\delta\|g\| \leqslant \|Sx_{n_\delta - 1} - g\| \leqslant c_\theta \theta^{n_\delta - 1}\|g\|$, where without loss of generality we may assume that $n_\delta \geqslant 1$ holds. Thus, we have

$$n_\delta \leqslant \underbrace{1 + \frac{\log c_\theta}{\log(1/\theta)}}_{=:\,K_\theta} + \frac{\log(1/\delta)}{\log(1/\theta)}. \tag{7}$$

Now for an arbitrary real number $\varepsilon > 0$ we choose a real number θ with $0 < \theta = \theta_\varepsilon < 1$ so small such that $2/\varepsilon \leqslant \log(1/\theta)$ is satisfied. Then, we obtain the statement of the theorem from estimate (7) by choosing $\eta = \eta_\varepsilon$ according to Theorem 2.1, and by choosing δ_ε sufficiently small such that $2K_\theta/\varepsilon \leqslant \log(1/\delta_\varepsilon)$ is satisfied. □

Remark 2.2. Thus, $n_\delta = o(\log(1/\delta))$ as $\delta \to 0$ holds uniformly with respect to operator perturbations as considered in Theorem 2.2.

3. An application to periodic integral operators

In the sequel we consider a class of periodic integral equations that arise, e. g., from the boundary integral equation formulation of interior or exterior boundary value problems in a two-dimensional domain with a smooth boundary. In what follows we have several quotations of the monograph [17] but most of the statements are covered also by the papers [19] and [16]; see also [14] where it is shown that the CGNR–method, that is, the conjugate gradient method of Hestenes and Stiefel applied to the normal equations, can be used also as a fast solver for the class of periodic integral equations which is considered in what follows.

3.1. The class of operators

Below we consider equations of the following form:

$$\mathcal{A}u \;=\; f, \tag{8}$$

where $f:\mathbb{R}\to\mathbb{C}$ is a 1-periodic function, and the operator $\mathcal{A}$ has the form

$$\mathcal{A}=\sum_{p=0}^{q} A_p, \qquad (A_0u)(t)=\int_0^1 \left[\kappa_0^+(t-s)\,a_0^+(t,s) \;+\; \kappa_0^-(t-s)\,a_0^-(t,s)\right]u(s)\;ds, \tag{9}$$

$$(A_pu)(t)=\int_0^1 \kappa_p(t-s)\,a_p(t,s)\,u(s)\,ds, \quad t\in[\,0,1\,], \quad p=1,2,\dots,q. \tag{10}$$

Here $q\in\mathbb{N}=\{\,1,2,\dots\}$, and $a_0^\pm$ and a_p are complex-valued 1-biperiodic C^∞-smooth functions defined on $\mathbb{R}^2$. It is supposed that

$$b^+(t) \;:=\; a_0^+(t,t) \;+\; a_0^-(t,t)\neq 0, \tag{11}$$

$$b^-(t) \;:=\; a_0^+(t,t) \;-\; a_0^-(t,t)\neq 0 \quad (t\in\mathbb{R}), \qquad W(b^+) \;=\; W(b^-), \tag{12}$$

where $W(b)$ denotes the winding number of a continuous 1-periodic function b. In particular, $W(b^+)=0=W(b^-)$ if $a_0^\pm(t,s)$ are real functions. Often A_0 has the form (10), i.e., $a_0^-(t,s)=0$, $a_0^+(t,s) =: a_0(t,s)$ and $\kappa_0^+(t) =: \kappa_0(t)$. In that case conditions (11), (12) reduce to $a_0(t,t)\neq 0$ $(t\in\mathbb{R})$.

Further, $\kappa_0^\pm$ and κ_p, $p=1,\dots,q$, are 1-periodic functions or distributions with the known Fourier coefficients

$$\hat\kappa_0^\pm(m) := \int_0^1 \kappa_0^\pm(t)\,e^{-im2\pi t}\;dt, \qquad \hat\kappa_p(m) := \int_0^1 \kappa_p(t)\,e^{-im2\pi t}\;dt, \qquad m\in\mathbb{Z}.$$

We suppose that the following conditions are satisfied:

$$\hat\kappa_0^-(m) \;=\; \operatorname{sign}(m)\hat\kappa_0^+(m) \qquad (0\neq m\in\mathbb{Z}) \tag{13}$$

$$c_{00}|m|^\alpha \leqslant |\hat\kappa_0^+(m)| \;\leqslant\; c_{01}|m|^\alpha \qquad (0\neq m\in\mathbb{Z}) \tag{14}$$

$$\left|\hat\kappa_0^+(m) \;-\; \hat\kappa_0^+(m-1)\right| \;\leqslant\; c_1\underline{m}^{\alpha-1} \qquad (m\in\mathbb{Z}) \tag{15}$$

$$|\hat\kappa_p(m)| \;\leqslant\; c_0\underline{m}^{\alpha-\beta_p} \qquad (m\in\mathbb{Z}, \quad p=1,2,\dots,q), \tag{16}$$

with a certain parameter $\alpha \in \mathbb{R}$ and positive integers $\beta_1, \beta_2, \ldots, \beta_q$, and $c_0,\ c_1,\ c_{00}$ and c_{01} are some positive constants. Moreover, we use the notation

$$\llcorner m \lrcorner \;=\; \begin{cases} |m|, & \text{if } \; m \neq 0 \\ 1\ , & \text{if } \; m = 0 \end{cases} \qquad\qquad (m \in \mathbb{Z}).$$

Later, conditions (15), (16) will be strengthened (see (30), (31)). Equations of the form (8) with the operators $\mathcal{A}$ as in (9)–(10) satisfying conditions (11)–(16) arise, if the boundary integral method is applied to a boundary value problem on a two-dimensional bounded and simply connected domain with a smooth boundary. An associated example will be presented in Section 3.2 but first the basic mapping properties of the operator $\mathcal{A}$ considered in (9)–(10) are stated and some transformation of the equation $\mathcal{A}u = f$ is considered. As a preparation for any $\lambda \in \mathbb{R}$ we consider the Sobolev space H^λ of those functions or distributions u which satisfy

$$\|u\|_\lambda := \Big(\sum_{m \in \mathbb{Z}} \llcorner m \lrcorner^{2\lambda} |\hat{u}(m)|^2 \Big)^{1/2} < \infty,$$

$$\text{where} \qquad \hat{u}(m) := \int_0^1 u(t)\, e^{-\mathrm{i}m2\pi t}\, dt, \qquad m \in \mathbb{Z},$$

and $\mathcal{L}(H^{\lambda_1}, H^{\lambda_2})$ denotes the space of bounded linear operators from H^{λ_1} into H^{λ_2} ($\lambda_1,\ \lambda_2 \in \mathbb{R}$). As a consequence of the conditions (13), (14) and (16) we have for any $\lambda \in \mathbb{R}$

$$A_0 \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha}), \qquad\qquad A_p \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta_p}) \qquad \text{for} \quad p = 1, 2, \ldots, q,$$

see [17] for details. We thus have $\mathcal{A} = \sum_{p=0}^q A_p \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$. Under the given conditions, this operator $\mathcal{A}$ moreover can be transformed into an operator that differs from the identity operator only by some compact operator $K : H^\lambda \to H^\lambda$. For this purpose we consider the operator

$$\mathcal{B} \;=\; \big[(1/b^+)P^+ \;+\; (1/b^-)P^-\big]\mathcal{G}_0^{-1}, \tag{17}$$

where

$$P^+u \;=\; \sum_{m \geqslant 0} \hat{u}(m) e^{\mathrm{i}m2\pi t}, \qquad\qquad P^-u \;=\; \sum_{m<0} \hat{u}(m) e^{\mathrm{i}m2\pi t}, \tag{18}$$

$$(\mathcal{G}_0 u)(t) \;=\; \hat{u}(0) \;+\; \sum_{0 \neq m \in \mathbb{Z}} \hat{\kappa}_0^+(m) \hat{u}(m) e^{\mathrm{i}m2\pi t}. \tag{19}$$

The operators $\mathcal{G}_0 \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ and $[(1/b^+)P^+ \;+\; (1/b^-)P^-] \in \mathcal{L}(H^\lambda, H^\lambda)$ are isomorphisms for each $\lambda \in \mathbb{R}$, and thus $\mathcal{B} \in \mathcal{L}(H^{\lambda-\alpha}, H^\lambda)$ is also an isomorphism for each $\lambda \in \mathbb{R}$. Thus multiplying both sides of the equation $\mathcal{A}u = f$ in (8) by the operator $\mathcal{B}$ yields the equivalent equation

$$\underbrace{\mathcal{B} \sum_{p=0}^q A_p}_{=:\,T} u \;=\; \mathcal{B}f. \tag{20}$$

It is easy to see that $\mathcal{B}A_0 = I + R$ holds with some operator $R \in \mathcal{L}(H^\lambda, H^{\lambda+1})$, and therefore the operator T introduced in (20) can be written as follows:

$$Tu = u + Ku \quad \text{with} \quad K = \mathcal{B}\sum_{p=0}^{q} A_p - I : H^\lambda \to H^{\lambda+\beta}, \quad \beta = \min\{1, \beta_1, \beta_2, \ldots, \beta_q\}. \tag{21}$$

This in particular means that $K : H^\lambda \to H^\lambda$ is a compact operator for each $\lambda \in \mathbb{R}$. We finally note that from the property (21) it follows that the null space $N(T)$ of the operator T satisfies $N(T) \subset C^\infty$. Thus, if the condition

$$v \text{ 1–periodic } C^\infty\text{–function}, \quad Tv = 0 \quad \Longrightarrow \quad v = 0 \tag{22}$$

is satisfied, then for each $\lambda \in \mathbb{R}$ the operator $T \in \mathcal{L}(H^\lambda, H^\lambda)$ is an isomorphism with a spectrum that satisfies condition (4).

3.2. An example

In the sequel we consider a prominent example, cf. [17].

Example 3.1. Symm's integral equation for closed C^∞-smooth boundaries in parametrized form looks as follows:

$$(\mathcal{A}u)(t) \quad = \quad -\int_0^1 \log|x(t) - x(s)|\, u(s)\, ds \quad = \quad f(t), \qquad t \in [\,0, 1\,],$$

where $x : \mathbb{R} \to \mathbb{R}^2$ is a C^∞–smooth 1-periodic parametrization of the corresponding boundary with $x'(t) \neq 0$ for $t \in \mathbb{R}$. We consider the following decomposition:

$$(\mathcal{A}u)(t) \quad = \quad \int_0^1 \kappa_0(t-s)\, u(s)\, ds \;+\; \int_0^1 a_1(t,s)\, u(s)\, ds, \qquad t \in [\,0, 1\,],$$

with $\kappa_0(t) = -\log|\sin \pi t|$ and

$$a_1(t,s) \quad = \quad \begin{cases} -\log \dfrac{|x(t) - x(s)|}{|\sin \pi(t-s)|}, & \text{if} \quad t \neq s, \\[2ex] -\log \dfrac{|x'(t)|}{\pi}, & \text{if} \quad t = s. \end{cases}$$

Here $|\cdot|$ also denotes the Euclidean norm in $\mathbb{R}^2$. Note that a_1 is a 1-biperiodic C^∞-function, and the Fourier coefficients of κ_0 have the following form:

$$\hat{\kappa}_0(m) \quad = \quad \begin{cases} \dfrac{1}{2|m|}, & \text{if} \quad 0 \neq m \in \mathbb{Z}, \\ \log 2, & \text{if} \quad m = 0. \end{cases}$$

Thus, the conditions (11)–(16) are satisfied (with $\kappa_1 \equiv 1$) for $\alpha = -1$ and any $\beta_1 > 0$.

Further examples are, e.g., some integral equation formulations of the biharmonic problem, the Cauchy integral equation, the Hilbert integral equation, and the hypersingular integral equation.

3.3. A specific approximation of T

3.3.1. Some preparations. We again suppose that $\mathcal{A}$ is an operator of the form (9)–(10) that satisfies the conditions (11)–(16). For the subsequent considerations on the full discretization of the considered equation (20) we need spaces of trigonometric trial polynomials $\mathcal{T}_N$. They are defined as follows:

$$\mathcal{T}_N \;:=\; \Big\{ \sum_{m\in\mathbb{Z}_N} b_m e^{\mathrm{i}m2\pi t} \;:\; b_m \in \mathbb{C} \text{ for } \; m \in \mathbb{Z}_N \Big\},$$
$$\text{where } \; \mathbb{Z}_N := \Big\{ \; m \in \mathbb{Z} : \; -\frac{N}{2} < m \leqslant \frac{N}{2} \Big\}, \quad N \in \mathbb{N}.$$

In the sequel for a given integer N we construct an operator $S_N : \mathcal{T}_N \to \mathcal{T}_N$ that approximates T, and moreover for each function $v_N \in \mathcal{T}_N$ the function $S_N v_N \in \mathcal{T}_N$ can be computed fully discretely by $\mathcal{O}(N \log N)$ arithmetical operations. We continue with several preparations. The Fourier projectors associated with $\mathcal{T}_N$ are given by

$$P_N u \;=\; \sum_{m\in\mathbb{Z}_N} \hat{u}(m) e^{\mathrm{i}m2\pi t} \qquad \big(u \in H^\lambda \;\text{ for some }\; \lambda \in \mathbb{R}\big).$$

We shall need also the interpolation projector Q_N onto the space $\mathcal{T}_N$ which is defined as follows:

$$Q_N u \in \mathcal{T}_N, \qquad (Q_N u)\big(\tfrac{j}{N}\big) \;=\; u\big(\tfrac{j}{N}\big), \qquad j = 1, 2, \ldots, N \qquad \big(u \in H^\lambda \;\text{ for some }\; \lambda > \tfrac{1}{2}\big).$$

The following estimates will be needed later:

$$\|(I - P_N)u\|_\lambda \;\leqslant\; (\tfrac{N}{2})^{\lambda-\mu} \|u\|_\mu \qquad \text{for} \quad u \in H^\mu \qquad (-\infty < \lambda \leqslant \mu < \infty), \tag{23}$$
$$\|u_N\|_\mu \;\leqslant\; (\tfrac{N}{2})^{\mu-\lambda} \|u_N\|_\lambda \qquad \text{for} \quad u_N \in \mathcal{T}_N \qquad (-\infty < \lambda \leqslant \mu < \infty), \tag{24}$$
$$\|(I - Q_N)u\|_\lambda \;\leqslant\; \gamma_\mu (\tfrac{N}{2})^{\lambda-\mu} \|u\|_\mu \qquad \text{for} \quad u \in H^\mu \qquad (0 \leqslant \lambda \leqslant \mu < \infty, \quad \mu > \tfrac{1}{2}), \tag{25}$$

with $\gamma_\mu = (1 + \sum_{j=1}^\infty \frac{1}{j^{2\mu}})^{1/2}$, cf. [17] for details. Further, let $L \in \mathbb{N}, L < N$. By $Q_{L,L}$ we denote the following two-dimensional interpolation operator:

$$Q_{L,L}\psi \;\in\; \mathcal{T}_{L,L}, \qquad (Q_{L,L}\psi)\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big) \;=\; \psi\big(\tfrac{j_1}{L}, \tfrac{j_2}{L}\big) \qquad \text{for } j_1, \; j_2 \;=\; 1, 2, \ldots, L,$$
$$\mathcal{T}_{L,L} := \Big\{ \sum_{m_1,m_2\in\mathbb{Z}_L} b_{m_1m_2} e^{\mathrm{i}m_1 2\pi t} e^{\mathrm{i}m_2 2\pi s} \;:\; b_{m_1m_2} \in \mathbb{C} \text{ for } \; m_1, \; m_2 \in \mathbb{Z}_L \Big\},$$

where $\psi : \mathbb{R}^2 \to \mathbb{C}$ is a 1-biperiodic C^∞-smooth function. Finally, we consider the following set $D_L \subset \mathbb{Z}^2$:

$$D_L = \big\{ \, (m_1, m_2) \in \mathbb{Z}^2 \;:\; |m_1| + |m_2| \;\leqslant\; L/2 \, \big\}, \tag{26}$$

and P_{D_L} denotes the corresponding Fourier projection operator:

$$P_{D_L} v \;=\; \sum_{(m_1,m_2)\in D_L} \hat{v}(m_1, m_2) e^{\mathrm{i}m_1 2\pi t} e^{\mathrm{i}m_2 2\pi s},$$
$$\text{where} \qquad \hat{v}(m_1, m_2) = \int_0^1 \int_0^1 v(t, s)\, e^{-\mathrm{i}m_1 \pi t}\, e^{-\mathrm{i}m_2 2\pi s}\, ds\, dt, \qquad m_1, \; m_2 \in \mathbb{Z}.$$

3.3.2. A specific approximation of a_p ***for*** $p \geqslant 0$***.*** We again suppose that $\mathcal{A}$ is an operator of the form (9)–(10) that satisfies the conditions (11)–(16). In the sequel we recall the basic features of [17] on a specific approximation to $\mathcal{A}$ and will present its elementary consequences for the transformed operator $T = \mathcal{B}\mathcal{A}$. In the sequel for $L \in \mathbb{N}, L < N$, we consider the functions

$$a_{0,L}^{\pm} \;=\; P_{D_L} Q_{L,L} a_0^{\pm}, \qquad a_{p,L} \;=\; P_{D_L} Q_{L,L} a_p, \quad p \;=\; 1,2,\ldots,q, \tag{27}$$

where the set $D_L \subset \mathbb{Z}^2$ is as in (26). Note that for the computation of the function $a_{p,L}$, only the values of the function a_p at the grid points $\left(\frac{j_1}{L}, \frac{j_2}{L}\right)$, j_1, $j_2 = 1,2,\ldots,L$, are needed.

3.3.3. Approximation of A_p ***on*** $\mathcal{T}_L$***.*** An approximation of the operator A_p is obtained if the kernels $a_0^{\pm}$ and a_p are replaced by $a_{0,L}^{\pm}$ and $a_{p,L}$, as defined in (27), respectively:

$$\begin{aligned} (A_{0,L}u)(t) \;&=\; \int_0^1 \left[\kappa_0^+(t-s)\, a_{0,L}^+(t,s) \;+\; \kappa_0^-(t-s)\, a_{0,L}^-(t,s)\right] u(s)\, ds, \\ (A_{p,L}u)(t) \;&=\; \int_0^1 \kappa_p(t-s)\, a_{p,L}(t,s)\, u(s)\, ds, \qquad p \;=\; 1,2,\ldots,q. \end{aligned} \tag{28}$$

We have $A_{p,L} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta_p})$ for each $\lambda \in \mathbb{R}$, with $\beta_0 := 0$, and moreover the following estimate holds:

$$\|A_{p,L} - A_p\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \leqslant c_{\lambda,r} L^{-r} \qquad (\lambda \in \mathbb{R},\; p \;=\; 0,1,\ldots,q) \qquad \forall\, r \geqslant 0, \tag{29}$$

cf. [17] for details. This approximation $A_{p,L}$ to A_p in fact is used on $\mathcal{T}_L$.

The following can be stated on the computational costs: if $L \sim N^\sigma$ holds for some $0 < \sigma \leqslant 1/2$, then the system matrix associated with $A_{p,L} : \mathcal{T}_L \to \mathcal{T}_{2L}$ has fully discrete entries that can be computed by $\mathcal{O}(N \log N)$ arithmetical operations. Moreover, for each $v_L \in \mathcal{T}_L$ with the known Fourier coefficients, the vector $w_L = A_{p,L} v_L \in \mathcal{T}_{2L}$ can be computed by $\mathcal{O}(L^2) = \mathcal{O}(N)$ arithmetical operations.

3.3.4. Approximation of A_p ***on the subspace*** $\mathcal{T}_N \ominus \mathcal{T}_L$***.*** In order to obtain an approximation of the operator A_p that allows one to keep the number of arithmetical operations sufficiently small, on the subspace $\mathcal{T}_N \ominus \mathcal{T}_L = \operatorname{span}\{\, e^{im2\pi t} : m \in \mathbb{Z}_N \backslash \mathbb{Z}_L \,\}$, an asymptotic approximation to A_p by operators of simpler structure is considered. For this we need the following additional conditions on the operator $\mathcal{A}$:

$$|\triangle^j \hat{\kappa}_0^{\pm}(m)| \;\leqslant\; c_j \llcorner m \lrcorner^{\alpha-j} \qquad (m \in \mathbb{Z}, \quad j \;=\; 0,1,\ldots), \tag{30}$$

$$|\triangle^j \hat{\kappa}_p(m)| \;\leqslant\; c_j \llcorner m \lrcorner^{\alpha-\beta_p-j} \qquad (m \in \mathbb{Z}, \quad j = 0,1,\ldots, \quad p = 1,2,\ldots,q). \tag{31}$$

Here c_j denote some positive constants, and $\triangle$ denotes the forward difference operator, i.e.,

$$\triangle \hat{v}(m) \;=\; \hat{v}(m+1) \;-\; \hat{v}(m), \qquad m \in \mathbb{Z}.$$

For Symm's integral operator considered in Example 3.1, for instance, the conditions (30)–(31) are satisfied. The mentioned asymptotic approximation $A_{p,L,d}$ of $A_{p,L}$ with an integer $d \geqslant 0$ has the following form for $p = 1, 2, \ldots, q$:

$$A_{p,L,d} \;=\; \sum_{j=0}^{d-\lfloor\beta_p\rfloor-1} B_{p,L,j}, \qquad (B_{p,L,j}u)(t) \;=\; b_{p,L,j}(t) \sum_{m\in\mathbb{Z}} \big[\triangle^j \hat{\kappa}_p(m)\big]\, \hat{u}(m)\, e^{\mathrm{i}m2\pi t},$$

if $d \geqslant \lfloor\beta_p\rfloor + 1$, and $A_{p,L,d} = 0$ if $d \leqslant \lfloor\beta_p\rfloor$. The asymptotic approximation $A_{0,L,d}$ of $A_{0,L}$ is similarly constructed. The function $b_{p,L,j} \in \mathcal{T}_{2L}$ has the following specific form:

$$b_{p,L,j}(t) \;=\; \frac{1}{j!}\,\partial_s^{[j]} a_{p,L}(t,s)\Big|_{s=t}, \qquad \begin{array}{l} t \in [\,0,1\,], \\ j \;=\; 0,1,\ldots,d-\lfloor\beta_p\rfloor-1, \end{array}$$

where $\lfloor x \rfloor$ denotes the biggest integer smaller or equal to a real number $x \in \mathbb{R}$. Moreover,

$$\begin{aligned} \partial_s^{[0]} &= 1, \qquad \partial_s^{[1]} \;=\; \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s}, \\ \partial_s^{[j]} &= \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s} - j+1\Big)\cdots\Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s} - 1\Big)\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial s}, \qquad j \;=\; 2,3,\ldots\,. \end{aligned}$$

The Fourier coefficients of the functions $b_{p,L,j}$ can be obtained recursively for $j = 0, 1, \ldots, d-\lfloor\beta_p\rfloor - 1$, cf. [17] for details. It follows from the conditions (30)–(31) that $A_{p,L,d} \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\beta_p})$ holds, and – as a basic purpose of this construction – we moreover have that the difference $A_{p,L,d} - A_p$ is an operator of lower order than A_p: $A_{p,L,d} - A_p \in \mathcal{L}(H^\lambda, H^{\lambda-\alpha+\max\{d,\beta_p\}})$ and

$$\big\| \big(A_{p,L,d} - A_p\big)(I - P_L) \big\|_{\mathcal{L}(H^\lambda, H^{\lambda-\alpha})} \;\leqslant\; c_\lambda L^{-\max\{d,\beta_p\}} \qquad (\lambda \in \mathbb{R},\; p = 0,\ldots,q), \tag{32}$$

with a constant c_λ which in fact is independent of the parameter L; cf. again [17] for details.

The following can be stated on the computational costs: for each $v_N \in \mathcal{T}_N$ with known Fourier coefficients, the function $w_N = \mathcal{A}_{L,d} v_N \in \mathcal{T}_{N+2L}$ can be computed by a fully discrete scheme that requires $\mathcal{O}(N \log N)$ arithmetical operations, if the FFT is applied.

3.3.5. Approximation of A_p on the subspace $\mathcal{T}_N$. The basic approaches considered in [17] – which we recalled in the Sections 3.3.3 and 3.3.4 – in our situation finally yield the following approximation to the operator $\mathcal{A} = \sum_{p=0}^q A_p$:

$$\mathcal{A}_{L,d} \;=\; \Big(\sum_{p=0}^{q} A_{p,L}\Big) P_L \;+\; \Big(\sum_{p=0}^{q} A_{p,L,d}\Big)(I - P_L) \tag{33}$$

with the following properties $\mathcal{A}_{L,d} \in \mathcal{L}(H^\mu, H^{\lambda-\alpha})$ for each $\mu, \lambda \in \mathbb{R}$ with $\lambda \leqslant \mu$ and

$$\|\mathcal{A}_{L,d} - \mathcal{A}\|_{\mathcal{L}(H^\mu, H^{\lambda-\alpha})} \;\leqslant\; c_{\lambda,\mu} L^{-(d+\mu-\lambda)} \qquad (\mu, \lambda \in \mathbb{R} \;\text{ with } \lambda \leqslant \mu), \tag{34}$$

with some constant $c_{\lambda,\mu} \geqslant 0$.

3.3.6. Approximation of $\mathcal{B}$. An approximation of the operator $\mathcal{B}$ is obtained by replacing, in the definition of $\mathcal{B}$, the functions $1/b^+$ and $1/b^-$ by their trigonometric interpolants $\in \mathcal{T}_L$, respectively:

$$\mathcal{B}_L \;=\; \big[Q_L(1/b^+)P^+ \;+\; Q_L(1/b^-)P^-\big]\mathcal{G}_0^{-1}. \tag{35}$$

The functions $1/b^+$ and $1/b^-$ are 1-periodic C^∞-functions, and we thus have

$$\|\mathcal{B}_L - \mathcal{B}\|_{\mathcal{L}(H^{\lambda-\alpha},H^\lambda)} \;\leqslant\; c_{\lambda,r}L^{-r} \qquad (\lambda \in \mathbb{R}) \qquad \forall\, r \geqslant 0, \tag{36}$$

cf. [17] for details. The following can be stated on the computational costs: for each $v_N \in \mathcal{T}_{N+2L}$ with the known Fourier coefficients, the function $w_N = P_N\mathcal{B}_Lv_N \in \mathcal{T}_N$ can be computed by a fully discrete scheme that requires $\mathcal{O}(N\log N)$ arithmetical operations, if the FFT is applied.

4. GMRES for the specific application

In the sequel we suppose that the right-hand side f in equation (8) satisfies

$$f \;\in\; H^{\mu-\alpha} \qquad \text{for some} \;\; \mu > \alpha + 1/2, \tag{37}$$

and consider the following approximations:

$$S_N := P_N\mathcal{B}_L\mathcal{A}_{L,d} : \mathcal{T}_N \to \mathcal{T}_N, \qquad g_N := P_N\mathcal{B}_LQ_Nf \in \mathcal{T}_N. \tag{38}$$

In the sequel we consider GMRES applied with the operator S_N and the right-hand side g_N in (38), and with respect to the Sobolev space H^α: let the sequence $x_n \in \mathcal{T}_N,\ n = 0, 1, \ldots,$ be given by

$$x_n \in \mathcal{K}_n(S_N, g_N), \qquad \|S_Nx_n - g_N\|_\alpha = \min_{x\in\mathcal{K}_n(S_N,g_N)} \|S_Nx - g_N\|_\alpha. \tag{39}$$

We note that at each step of GMRES, one application of the operator S_N to an element of $\mathcal{T}_N$ has to be employed. The following a posteriori stopping criterion is considered: terminate the iteration at the step $n =: n_N$ when

$$\|S_Nx_n - g_N\|_\alpha \;\leqslant\; cN^{\alpha-\mu}\|g_N\|_\alpha \tag{40}$$

is satisfied for the first time, where c denotes some positive constant. As a preparation for the formulation of the following basic theorem we specify the conditions on the parameters L and d:

$$L \,\sim\, N^\sigma, \quad 0 < \sigma < 1, \qquad d \geqslant \tfrac{1-\sigma}{\sigma}(\mu - \alpha). \tag{41}$$

We recall that for any function $u \in H^\mu$, the best approximation to u by functions from $\mathcal{T}_N$ in the norm of H^λ can be estimated by $(N/2)^{\lambda-\mu}\|u\|_\mu$, cf. (23). This means that the basic estimate (42) in the following theorem is of optimal order.

Theorem 4.1. *Suppose that $\mathcal{A}$ is an operator of the form (9)–(10) which satisfies the conditions (11)–(16) and (22) as well as (30)–(31), and moreover let (37) and (41) be satisfied. Then, there exists an N_0 such that for each integer N with $N \geqslant N_0$ we have*

$$\|x_{n_N} - u\|_\lambda \;\leqslant\; c_{\lambda,\mu}N^{\lambda-\mu}\|u\|_\mu \qquad (\alpha \leqslant \lambda \leqslant \mu), \tag{42}$$

$$n_N \;=\; \mathcal{O}(\log N). \tag{43}$$

Proof. As a preparation for the proof of the two statements of the theorem we consider the operator $\widetilde{S}_N = I - P_N + P_N \mathcal{B}_L \mathcal{A}_{L,d} \in \mathcal{L}(H^\lambda, H^\lambda)$. On $\mathcal{T}_N$ the operator $\widetilde{S}_N$ coincides with S_N, so that the corresponding GMRES sequences in fact coincide. Moreover, we have $\widetilde{S}_N - \mathcal{B}\mathcal{A} = (I - P_N)(I - \mathcal{B}\mathcal{A}) + P_N(\mathcal{B}_L \mathcal{A}_{L,d} - \mathcal{B}A)$, and from the two estimates (36) and (34) with the specific choice $\mu = \lambda$ as well as from the mapping properties of the operator $I - \mathcal{B}\mathcal{A}$ stated in (21) it then follows that

$$\|\widetilde{S}_N - \mathcal{B}\mathcal{A}\|_{\mathcal{L}(H^\lambda, H^\lambda)} \leqslant c_\lambda (N^{-\beta} + N^{-\sigma d}) \qquad (\lambda \in \mathbb{R}) \tag{44}$$

holds with some constant $c_\lambda \geqslant 0$. From this estimate it follows, for sufficiently large $N \geqslant N_0$ and for each $\lambda \in \mathbb{R}$, that the operators $\widetilde{S}_N \in \mathcal{L}(H^\lambda, H^\lambda)$ are invertible with $\|\widetilde{S}_N^{-1}\|_{\mathcal{L}(H^\lambda, H^\lambda)} \leqslant c'_\lambda$ for some constant $c'_\lambda \geqslant 0$.

We now are in a position to present a proof of the estimate (42). In fact, we have

$$\|x_{n_N} - u\|_\lambda \leqslant c'_\lambda \|\widetilde{S}_N x_{n_N} - \widetilde{S}_N u\|_\lambda \leqslant c'_\lambda \big(\|\widetilde{S}_N x_{n_N} - g_N\|_\lambda + \|\widetilde{S}_N u - g_N\|_\lambda \big). \tag{45}$$

The first term on the right-hand side of the above estimate can be estimated with the help of the inverse estimate (24):

$$\|\widetilde{S}_N x_{n_N} - g_N\|_\lambda \leqslant \big(\tfrac{N}{2}\big)^{\lambda - \alpha} \|\widetilde{S}_N x_{n_N} - g_N\|_\alpha \leqslant c_\lambda N^{-(\mu - \lambda)} \|u\|_\mu \qquad (\alpha \leqslant \lambda \leqslant \mu)$$

with some constant $c_\lambda \geqslant 0$. The above estimate in fact follows from the stopping criterion (40) and the mapping properties of the operators that are used in the definition of the function $g_N \in \mathcal{T}_N$ in (38). For the estimation of the last term in (45) we observe that $\widetilde{S}_N u - g_N = (I - P_N)u + P_N \mathcal{B}_L(\mathcal{A}_{L,d} u - Q_N \mathcal{A} u)$, and the last term of the right-hand side of the latter identity can be written as follows: $\mathcal{A}_{L,d} u - Q_N \mathcal{A} u = (\mathcal{A}_{L,d} - \mathcal{A})u + (I - Q_N)\mathcal{A} u$. We thus obtain

$$\|\widetilde{S}_N u - g_N\|_\lambda \leqslant \big(\tfrac{N}{2}\big)^{-(\mu - \lambda)} \|u\|_\mu + c_\lambda \|\mathcal{A}_{L,d} u - Q_N \mathcal{A} u\|_{\lambda - \alpha} \leqslant c'_{\lambda,\mu} N^{-(\mu - \lambda)} \|u\|_\mu,$$

where the mapping properties of the operator $\mathcal{A}$ and the condition (41) have been used. Moreover, the error estimate (34) as well as the approximation properties (23), (25) of the Fourier projection and the interpolation projection have been used. This completes the proof of estimate (42). The estimate (43) follows from Theorem 2.2 applied with the operator $S = \widetilde{S}_N$ and the function $g = g_N$, and with $\delta = cN^{-(\mu - \alpha)}$. Note that it follows from estimate (44) that Theorem 2.2 is applicable in fact. □

Remark 4.1. From estimate (43) and the considerations in Section 3.3 on the complexity of one application of the involved operators, respectively, it follows that $\mathcal{O}(N(\log N)^2)$ arithmetical operations are needed to compute the approximation $x_{n_N} \in \mathcal{T}_N$ if the condition $0 < \sigma \leqslant 1/2$ is satisfied.

5. Numerical experiments

5.1. Introductory remarks

In each of the following two Sections 5.2 and 5.3, a specific equation of the form $\mathcal{A}u = f$ is considered where the operator $\mathcal{A}$ fulfils the conditions (11)–(16) and (30)–(31) for $\alpha = -1$, respectively. In each case the right-hand side f is chosen so that the solution $u : \mathbb{R} \to \mathbb{R}$ of $\mathcal{A}u = f$ is given by the 1-periodic extension of the following function:

$$u(t) \quad = \quad \begin{cases} 1, & \text{if } 0.25 \leqslant t \leqslant 0.75, \\ 0, & \text{if } 0 \leqslant t < 0.25 \quad \text{or} \quad 0.75 < t \leqslant 1, \end{cases} \tag{46}$$

and then

$$u \in H^{1/2-\varepsilon}, \ f \in H^{3/2-\varepsilon} \ \text{ for each } \varepsilon > 0, \tag{47}$$
$$u \notin H^{1/2}, \ f \notin H^{3/2}. \tag{48}$$

Numerical experiments with equations having smoother solutions are also of interest but will not be considered here. For each specific equation, different choices of N are considered, and for each choice of N, the values of the function $f = \mathcal{A}u$ at the grid points are in fact computed numerically with a high precision.

We consider the following specific choices of N and L:

$$N \ = \ 2^k, \qquad L \ = \ 2^{\lceil k/2 \rceil}. \tag{49}$$

Relation (49) means that $L \sim N^{1/2}$ as $k \to \infty$, and in the numerical experiments we consider the specific choices $k = 5, 6, 7, 8$. In the present situation, we may choose $d = 2$ for the asymptotical approximation.

GMRES is applied with $S_N = P_N \mathcal{B}_L \mathcal{A}_{L,d} : \mathcal{T}_N \to \mathcal{T}_N$ and $g_N \in \mathcal{T}_N$ as in Section 4. According to the general analysis presented in Section 4, it is reasonable to terminate the iteration at the step $n =: n_N$ when

$$\| S_N x_{n_N} - g_N \|_{-1} \ \leqslant \ N^{-3/2} \| g_N \|_{-1}$$

is satisfied for the first time, where x_n denotes the n-th iterate of GMRES. The error estimate (42) yields $\| x_{n_N} - u \|_{-1} = \mathcal{O}(N^{-3/2+\varepsilon})$ for any $\varepsilon > 0$. Note that due to the property (48) one cannot conclude from the error estimate (42) that the quotient $\| x_{n_N} - u \|_{-1} / N^{-3/2}$ stays bounded for experiments with different and increasingly ordered values of N. On the other hand, however, due to (47) it is not surprising that these quotients stay bounded in our experiments; notice also that $\| u - P_N u \|_{-1} \sim \big(\sum_{|m| \geqslant N/2} m^{-4} \big)^{1/2} \sim N^{-3/2}$. All computations are performed in MATLAB.

5.2. Symm's integral equation for an ellipse

In the sequel we present the numerical results for Symm's integral equation, cf. Example 3.1 which is considered here for $x(t) = (\frac{1}{2} \cos 2\pi t, \ \frac{1}{4} \sin 2\pi t)^\top$, $t \in \mathbb{R}$, parametrizing a special ellipse Γ. Table 1 contains the results obtained by GMRES.

5.3. A model problem

In the sequel we consider the following model problem, cf. [7] for a similar example:

$$\int_0^1 \kappa_0(t - s) a_0(t, s) \, u(s) \, ds = f(t), \qquad t \in [\,0, 1\,],$$

Table 1. Numerical results with GMRES for Symm's integral equation for an ellipse

N	L	$\|x_{n_N} - u\|_{-1}$	$\|x_{n_N} - u\|_{-1}/N^{-3/2}$	n_N
64	8	2.03e–02	10.41	3
128	8	1.01e–02	14.58	3
256	16	4.59e–03	18.79	4
512	16	1.97e–03	22.86	4
1024	32	6.59e–04	21.61	4

with

$$\hat{\kappa}_0(m) = \begin{cases} \dfrac{4}{\pi}\dfrac{1}{4m-1}, & \text{if } 0 \neq m \in \mathbb{Z}, \\ \dfrac{4}{3\pi}, & \text{if } m = 0, \end{cases}$$

$$a_0(t,s) = b(t)\,b(s), \qquad b(t) = 3 + \sum_{0 \neq m \in \mathbb{Z}} 2^{-4|m|} e^{\mathrm{i}m2\pi t}.$$

Here, we have $q = 0$, with an asymptotical approximation corresponding to A_0 which is nontrivial. Table 2 contains the results obtained by GMRES.

Table 2. Numerical results with GMRES for the model problem

N	L	$\|x_{n_N} - u\|_{-1}$	$\|x_{n_N} - u\|_{-1}/N^{-3/2}$	n_N
64	8	2.02e–02	10.33	4
128	8	9.81e–03	14.20	4
256	16	4.59e–03	18.80	4
512	16	1.97e–03	22.84	5
1024	32	6.59e–04	21.60	5

Acknowledgements

The authors wish to thank an anonymous referee for useful suggestions.

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Received 25 Jul. 2001
Revised 23 Jan. 2002