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A HIGH-ORDER DIFFERENCE SCHEME FOR A NONLOCAL BOUNDARY-VALUE PROBLEM FOR THE HEAT EQUATION[1]

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Abstract — This paper is concerned with a high order difference scheme for a nonlocal boundary-value problem of parabolic equation. The integrals in the boundary equations are approximated by the composite Simpson rule. The unconditional solvability and L_∞ convergence of the difference scheme is proved by the energy method. The convergence rate of the difference scheme is second order in time and fourth order in space. Some numerical examples are provided to illustrate the convergence.

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1. Introduction

In the field of quasistatic thermoelasticity, one needs to solve the following nonlocal boundary value problem:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \qquad 0 < x < 1, \quad 0 < t \leqslant T, \tag{1.1.1}$$

$$u(0,t) = \int_0^1 \alpha(s)u(s,t)ds + g_0(t), \qquad 0 \leqslant t \leqslant T, \tag{1.1.2}$$

$$u(1,t) = \int_0^1 \beta(s)u(s,t)ds + g_1(t), \qquad 0 \leqslant t \leqslant T, \tag{1.1.3}$$

$$u(x,0) = \phi(x), \qquad 0 \leqslant x \leqslant 1, \tag{1.1.4}$$

where $f, \alpha, \beta, g_0, g_1$, and ϕ are known functions and the initial-value and boundary-value conditions satisfy the compatibility conditions:

$$\phi(0) = \int_0^1 \alpha(s)\phi(s)ds + g_0(0), \quad \phi(1) = \int_0^1 \beta(s)\phi(s)ds + g_1(0). \tag{1.1.5}$$

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Day [1, 2] showed that the maximum modulus, $\max_{x\in[0,1]}|u(x,t)|$, is a decreasing function in t. In [3], Friedman extended this result to a general parabolic equation (in n dimensions) using a method based on the maximum principle. He also showed the existence and uniqueness of the solution to his extended version of the problem. Ekolin [4] presented the forward Euler, the backward Euler, and the Crank-Nicolson difference schemes for (1.1), respectively. The integrals in the boundary conditions for $x = 0, 1$ are approximated by the composite trapezoidal rule. By maximum principle, Ekolin showed that, if the mesh ratio $\lambda \equiv \tau/h^2 \leqslant 0.5$ (τ temporal step size, h spatial step size), then the error in the forward Euler method is of second order in space, and that the error in the backward Euler method is of first order in time and second order in space without any restrictions on λ. By energy arguments, the maximum norm of the error for the Crank-Nicolson method was showed second order in both space and time. Lin, Xu and Yin [9] studied the finite difference approximations to the solution of the two dimensional heat equation with nonlocal boundary conditions on the unit rectangular. They presented a semi-implicit and a fully implicit backward Euler difference schemes. It is proved that both schemes preserve the maximum principle and monotonicity of the solution of the original problem and are unconditionally convergent of first order in time and second order in space. In [5, 7 ,8], Ionkin dealt with the stability of difference schemes approximating another nonlocal boundary conditions, namely,

$$u(0,t) = 0, \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t), \qquad 0 \leqslant t \leqslant T$$

for heat equation (1.1.1). It is well known (see, e.g.,[6, 11]) that a six-point difference scheme with second order in time and fourth order in space may be constructed for one-dimensional heat equations with constant coefficients and the first kind of boundary conditions. In [13], a similar difference scheme was derived for the variable coefficients case and proved that the difference scheme was unconditionally stable and convergent with the convergence order $O(\tau^2 + h^4)$ in the $L_\infty-$norm. In addition, an asymptotic expansion of the difference solution is gained and an $O(\tau^4+h^6)$ order in the $L_\infty-$norm accuracy approximation by single extrapolation is obtained. In this paper, we will provide a two-level implicit difference scheme for (1.1). The integrals in (1.1.2) and (1.1.3) are approximated by the composite Simpson rule. It is shown that the maximum norm of the error of the difference scheme is second order in temporal step size and fourth order in spatial step size without any restrictions on the mesh ratio.

We take a positive integer K and a positive even integer M. Let $\tau = T/K, h = 1/M$. Cover the domain $[0, 1] \times [0, T]$ by $\Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i | x_i = ih, 0 \leqslant i \leqslant M\}$ and $\Omega_\tau = \{t_k | t_k = k\tau, 0 \leqslant k \leqslant K\}$.

Let $U = \{U_i^k | 0 \leqslant i \leqslant M, 0 \leqslant k \leqslant K\}$ be a net function on $\Omega_h \times \Omega_\tau$. We introduce the following notation:

$$\begin{aligned}
U_{i-\frac{1}{2}}^k &= \frac{1}{2}(U_i^k + U_{i-1}^k), \quad \delta_x U_{i-\frac{1}{2}}^k = \frac{1}{h}(U_i^k - U_{i-1}^k),\\
U_i^{k-\frac{1}{2}} &= \frac{1}{2}(U_i^k + U_i^{k-1}), \quad \delta_t U_i^{k-\frac{1}{2}} = \frac{1}{\tau}(U_i^k - U_i^{k-1}),\\
\delta_x^2 U_i^k &= \frac{1}{h^2}(U_{i+1}^k - 2U_i^k + U_{i-1}^k),
\end{aligned}$$

$$\|U^k\| = \sqrt{h\left[\frac{1}{2}(U_0^k)^2 + \sum_{i=1}^{M-1}(U_i^k)^2 + \frac{1}{2}(U_M^k)^2\right]},$$

$$\|\delta_t U^{k+\frac{1}{2}}\| = \sqrt{h\left[\frac{1}{2}(\delta_t U_0^{k+\frac{1}{2}})^2 + \sum_{i=1}^{M-1}(\delta_t U_i^{k+\frac{1}{2}})^2 + \frac{1}{2}(\delta_t U_M^{k+\frac{1}{2}})^2\right]},$$

$$\|U^k\|_\infty = \max_{0\leqslant i\leqslant M}|U_i^k|, \quad \|\delta_x U^k\| = \sqrt{h\sum_{i=1}^{M}(\delta_x U_{i-\frac{1}{2}}^k)^2}.$$

In addition, if $g \in C[0,1]$, we denote

$$\|g\|_{L_2} = \sqrt{\int_0^1 g(s)^2 ds},$$

$$\langle g, U^k\rangle = \frac{h}{3}\sum_{j=0}^{\frac{M}{2}-1}\left(g(x_{2j})U_{2j}^k + 4g(x_{2j+1})U_{2j+1}^k + g(x_{2j+2})U_{2j+2}^k\right).$$

Our difference scheme for (1.1) is as follows:

$$\frac{1}{12}\left(\delta_t U_{i-1}^{k-\frac{1}{2}} + 10\delta_t U_i^{k-\frac{1}{2}} + \delta_t U_{i+1}^{k-\frac{1}{2}}\right) - \delta_x^2 U_i^{k-\frac{1}{2}}$$
$$= \frac{1}{12}\left(f_{i-1}^{k-\frac{1}{2}} + 10 f_i^{k-\frac{1}{2}} + f_{i+1}^{k-\frac{1}{2}}\right), \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant K, \tag{1.2.1}$$
$$U_0^k = \langle \alpha, U^k\rangle + g_0(t_k), \quad U_M^k = \langle \beta, U^k\rangle + g_1(t_k), \quad 1 \leqslant k \leqslant K, \tag{1.2.2}$$
$$U_i^0 = \phi(x_i) - [c_0(1-x_i) + c_1 x_i]h^4, \quad 0 \leqslant i \leqslant M, \tag{1.2.3}$$

where c_0 and c_1 are two constants such that

$$U_0^0 = \langle \alpha, U^0\rangle + g_0(t_0), \quad U_M^0 = \langle \beta, U^0\rangle + g_1(t_0). \tag{1.2.4}$$

Equations (1.2.4) is a similarity of the compatibility conditions (1.1.5). At the end of this section, we will point out how to choose c_0 and c_1.

Lemma 1.1 [12]**.** *Let M be an even integer, $h = 1/M, x_i = ih, 0 \leqslant i \leqslant M$.*
1) If $g(x) \in C^2[0,1]$, then

$$\int_0^1 g(x)\mathrm{d}x - \frac{h}{2}\sum_{i=0}^{M-1}[g(x_i) + g(x_{i+1})] = -\frac{1}{12}h^2\left.\frac{d^2 g(x)}{dx^2}\right|_{x=\xi}, \quad \xi \in (0,1).$$

2) If $g(x) \in C^2[0,1]$, then

$$\int_0^1 g(x)\mathrm{d}x - h\sum_{i=0}^{M-1} g(x_{i+\frac{1}{2}}) = \frac{1}{24}h^2\left.\frac{d^2 g(x)}{dx^2}\right|_{x=\eta}, \quad \eta \in (0,1).$$

3) If $g(x) \in C^4[0,1]$, then

$$\int_0^1 g(x)\mathrm{d}x - \frac{h}{3}\sum_{i=0}^{\frac{M}{2}-1}[g(x_{2i}) + 4g(x_{2i+1}) + g(x_{2i+2})] = -\frac{1}{180}h^4\left.\frac{d^4 g(x)}{dx^4}\right|_{x=\zeta}, \quad \zeta \in (0,1).$$

Lemma 1.2. *If $\alpha \in C^4[0,1]$ and $\beta \in C^4[0,1]$, then there exists a constant c_2 such that*

$$\sqrt{\langle \alpha, \alpha \rangle} \leqslant \|\alpha\|_{L_2} + c_2 h^2, \quad \sqrt{\langle \beta, \beta \rangle} \leqslant \|\beta\|_{L_2} + c_2 h^2.$$

Proof. It follows from Lemma 1.1 that

$$\begin{aligned}\|\alpha\|^2_{L_2([0,1])} - \langle \alpha, \alpha \rangle &= \int\limits_0^1 \alpha(x)^2 \mathrm{d}x - \frac{h}{3} \sum_{i=0}^{\frac{M}{2}-1} \left[\alpha^2_{2j} + 4\alpha^2_{2j+1} + \alpha^2_{2j+2}\right] \\ &= -\frac{1}{180} h^4 \left.\frac{d^4\alpha(x)^2}{dx^4}\right|_{x=\zeta}.\end{aligned}$$

Thus,

$$\langle \alpha, \alpha \rangle = \|\alpha\|^2_{L_2([0,1])} + \frac{1}{180} h^4 \left.\frac{d^4\alpha(x)^2}{dx^4}\right|_{x=\zeta} \leqslant \|\alpha\|^2_{L_2([0,1])} + c_2^2 h^4,$$

or

$$\sqrt{\langle \alpha, \alpha \rangle} \leqslant \|\alpha\|_{L_2} + c_2 h^2.$$

Similarly, we have

$$\sqrt{\langle \beta, \beta \rangle} \leqslant \|\beta\|_{L_2} + c_2 h^2.$$

□

Lemma 1.3. *If $\alpha \in C^4[0,1]$ and $\beta \in C^4[0,1]$, then, for any $U = \{U_i^k\}$ defined on $\Omega_h \times \Omega_\tau$, there exists a constant c_3 such that*

$$|\langle \alpha, U^k \rangle| \leqslant \frac{\sqrt{10}}{3}(\|\alpha\|_{L_2} + c_3 h)\|U^k\|, \quad |\langle \beta, U^k \rangle| \leqslant \frac{\sqrt{10}}{3}(\|\beta\|_{L_2} + c_3 h)\|U^k\|$$

Proof. Using the Cauchy-Schwartz inequality, we have

$$\begin{aligned}|\langle \alpha, U^k \rangle| &= \frac{h}{3} \sum_{j=0}^{\frac{M}{2}-1} \left(\alpha_{2j} U^k_{2j} + 4\alpha_{2j+1} U^k_{2j+1} + \alpha_{2j+2} U^k_{2j+2}\right) \\ &\leqslant \frac{1}{3} \sqrt{h \sum_{j=0}^{\frac{M}{2}-1} \left(2\alpha^2_{2j} + 16\alpha^2_{2j+1} + 2\alpha^2_{2j+2}\right)} \sqrt{h \sum_{j=0}^{\frac{M}{2}-1} \left(\frac{1}{2}(U^k_{2j})^2 + (U^k_{2j+1})^2 + \frac{1}{2}(U^k_{2j+2})^2\right)} \\ &= \frac{\sqrt{2}}{3} \sqrt{\frac{(2h)}{2} \sum_{j=0}^{\frac{M}{2}-1} \left(\alpha^2_{2j} + \alpha^2_{2j+2}\right) + 4 \times (2h) \sum_{j=0}^{\frac{M}{2}-1} \alpha^2_{2j+1}}\, \|U^k\|.\end{aligned}$$

According to Lemma 1.1, we have

$$\int\limits_0^1 \alpha(x)^2 dx = \frac{(2h)}{2} \sum_{j=0}^{\frac{M}{2}-1} \left(\alpha^2_{2j} + \alpha^2_{2j+2}\right) - \frac{1}{12}(2h)^2 \left.\frac{d^2\alpha(x)^2}{dx^2}\right|_{x=\zeta_1}, \ \zeta_1 \in (0,1),$$

$$\int\limits_0^1 \alpha(x)^2 dx = (2h) \sum_{j=0}^{\frac{M}{2}-1} (\alpha_{2j+1})^2 + \frac{(2h)^2}{24} \left.\frac{d^2\alpha(x)^2}{dx^2}\right|_{x=\zeta_2}, \ \zeta_2 \in (0,1).$$

Thus,

$$|\langle\alpha, U^k\rangle| \leqslant \frac{\sqrt{2}}{3}\sqrt{5(\|\alpha\|_{L_2}^2 + c_3^2h^2)}\|U^k\| \leqslant \frac{\sqrt{10}}{3}\left(\|\alpha\|_{L_2} + c_3h\right)\|U^k\|.$$

Similarly, we can obtain

$$|\langle\beta, U^k\rangle| \leqslant \frac{\sqrt{10}}{3}(\|\beta\|_{L_2} + c_3h)\|U^k\|.$$

□

Let

$$r_0 = [\phi(0) - \langle\alpha,\phi\rangle - (g_0)(t_0)]/(h^4), \quad r_M = [\phi(1) - \langle\beta,\phi\rangle - (g_1)(t_0)]/(h^4). \tag{1.3}$$

According to Lemma 1.1 and (1.1.5), we have

$$r_0 = -\frac{1}{180}\frac{d^4}{dx^4}[\alpha(x)\phi(x)]|_{x=\zeta_1}, \quad r_M = -\frac{1}{180}\frac{d^4}{dx^4}[\beta(x)\phi(x)]|_{x=\zeta_2}, \quad \zeta_1,\zeta_2 \in (0,1).$$

Thus, if $\alpha \in C^4([0,1])$ and $\beta \in C^4([0,1])$, there exists a constant c_4 such that

$$|r_0| \leqslant c_4, \qquad |r_M| \leqslant c_4. \tag{1.4}$$

Consider the system of linear algebraic equations

$$\begin{aligned} \left(1 - \langle\alpha, 1-x\rangle\right)c_0 - \langle\alpha, x\rangle c_1 &= r_0, \\ -\langle\beta, 1-x\rangle c_0 + \left(1 - \langle\beta, x\rangle\right)c_1 &= r_M. \end{aligned} \tag{1.5}$$

According to Lemma 1.2, when $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < \sqrt{3}$, there exists a h_0, for $h \leqslant h_0$, the determinant of the coefficient matrix of (1.5)

$$\begin{aligned} D = & \begin{vmatrix} 1 - \langle\alpha, 1-x\rangle & -\langle\alpha, x\rangle \\ -\langle\beta, 1-x\rangle & 1 - \langle\beta, x\rangle \end{vmatrix} \\ = & \left(1 - \langle\alpha, 1-x\rangle\right)\left(1 - \langle\beta, x\rangle\right) - \langle\alpha, x\rangle\langle\beta, 1-x\rangle \\ \geqslant & \left(1 - \sqrt{\langle\alpha,\alpha\rangle}\sqrt{\langle 1-x, 1-x\rangle}\right)\left(1 - \sqrt{\langle\beta,\beta\rangle}\sqrt{\langle x, x\rangle}\right) \\ & - \sqrt{\langle\alpha,\alpha\rangle}\sqrt{\langle x,x\rangle}\sqrt{\langle\beta,\beta\rangle}\sqrt{\langle 1-x, 1-x\rangle} \\ = & \left(1 - \frac{1}{\sqrt{3}}\sqrt{\langle\alpha,\alpha\rangle}\right)\left(1 - \frac{1}{\sqrt{3}}\sqrt{\langle\beta,\beta\rangle}\right) \\ & - \frac{1}{\sqrt{3}}\sqrt{\langle\alpha,\alpha\rangle}\,\frac{1}{\sqrt{3}}\sqrt{\langle\beta,\beta\rangle} \\ = & \; 1 - \frac{1}{\sqrt{3}}\left(\sqrt{\langle\alpha,\alpha\rangle} + \sqrt{\langle\beta,\beta\rangle}\right) \\ \geqslant & \; 1 - \frac{1}{\sqrt{3}}\left(\|\alpha\|_{L_2} + \|\beta\|_{L_2} + 2c_2h^2\right) \\ > & \; 0. \end{aligned}$$

Thus, (1.5) has a unique solution

$$c_0 = \frac{1}{D}\begin{vmatrix} r_0 & -\langle\alpha, x\rangle \\ r_M & 1 - \langle\beta, x\rangle \end{vmatrix} = \frac{1 - \langle\beta, x\rangle}{D}r_0 + \frac{\langle\alpha, x\rangle}{D}r_M, \tag{1.6.1}$$

$$c_1 = \frac{1}{D}\begin{vmatrix} 1 - \langle\alpha, 1-x\rangle & r_0 \\ -\langle\beta, 1-x\rangle & r_M \end{vmatrix} = \frac{\langle\beta, 1-x\rangle}{D}r_0 + \frac{1 - \langle\alpha, 1-x\rangle}{D}r_M. \tag{1.6.2}$$

Taking these c_0, c_1 in (1.2.3), we have

$$\begin{aligned} & U_0^0 - \langle \alpha, U^0 \rangle - g_0(t_0) \\ &= \phi(0) - c_0 h^4 - \langle \alpha, \phi - (c_0(1-x) + c_1 x) h^4 \rangle - g_0(t_0) \\ &= \phi(0) - \langle \alpha, \phi \rangle - g_0(0) - ((1 - \langle \alpha, 1-x \rangle) c_0 - \langle \alpha, x \rangle c_1)\, h^4 \\ &= \{ r_0 - ((1 - \langle \alpha, 1-x \rangle) c_0 - \langle \alpha, x \rangle c_1) \}\, h^4 \\ &= 0. \end{aligned}$$

Similarly

$$U_M^0 - \langle \beta, U^0 \rangle - g_1(t_0) = 0.$$

That is, (1.2.4) is valid.

Letting $\bar{c}_0 = c_0 h^4, \bar{c}_1 = c_1 h^4$ and multiplying (1.5) by h^4, we obtain

$$\begin{aligned} (1 - \langle \alpha, 1-x \rangle)\, \bar{c}_0 - \langle \alpha, x \rangle \bar{c}_1 &= \phi(0) - \langle \alpha, \phi \rangle - g_0(t_0), \\ -\langle \beta, 1-x \rangle \bar{c}_0 + (1 - \langle \beta, x \rangle)\, \bar{c}_1 &= \phi(1) - \langle \beta, \phi \rangle - g_1(t_0). \end{aligned} \tag{1.5$'$}$$

Then (1.2.3) can be written as

$$U_i^0 = \phi(x_i) - (\bar{c}_0(1 - x_i) + \bar{c}_1 x_i)\,, \quad 0 \leqslant i \leqslant M. \tag{1.2.3$'$}$$

2. The solvability

Theorem 2.1. *Suppose $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < 6/5\sqrt{3/10}\,(= \sqrt{0.432})$. Then there exists a constant h_0, when $h \leqslant h_0$, the difference scheme (1.2.1)-(1.2.3) has a unique solution.*

Proof. At first, we notice that $\{U_i^0 \mid 0 \leqslant i \leqslant M\}$ is given by (1.2.3).

Now, suppose that $\{U_i^{k-1} \mid 0 \leqslant i \leqslant M\}$ is uniquely determined by (1.2). Then (1.2.1)-(1.2.2) is a system, with respect to $\{U_i^k \mid 0 \leqslant i \leqslant M\}$, of linear algebraic equations

$$\begin{aligned} & \frac{1}{12}\left(\delta_t U_{i-1}^{k-\frac{1}{2}} + 10\delta_t U_i^{k-\frac{1}{2}} + \delta_t U_{i+1}^{k-\frac{1}{2}}\right) - \delta_x^2 U_i^{k-\frac{1}{2}}, \\ &= \frac{1}{12}\left(f_{i-1}^{k-\frac{1}{2}} + 10 f_i^{k-\frac{1}{2}} + f_{i+1}^{k-\frac{1}{2}}\right), \quad 1 \leqslant i \leqslant M-1, \end{aligned} \tag{2.1.1}$$

$$U_0^k = \langle \alpha, U^k \rangle + g_0(t_k), \quad U_M^k = \langle \beta, U^k \rangle + g_1(t_k). \tag{2.1.2}$$

In order to prove that it has a unique solution, one only needs to prove that the homogeneous equation

$$\frac{1}{12\tau}\left(U_{i-1}^k + 10U_i^k + U_{i+1}^k\right) - \frac{1}{2}\delta_x^2 U_i^k = 0, \quad 1 \leqslant i \leqslant M-1, \tag{2.2.1}$$

$$U_0^k = \langle \alpha, U^k \rangle, \quad U_M^k = \langle \beta, U^k \rangle \tag{2.2.2}$$

has a trivial solution.

Let

$$W_i^k = U_i^k - (1 - x_i)\langle \alpha, U^k \rangle - x_i \langle \beta, U^k \rangle. \tag{2.3}$$

Then, we have

$$U_i^k = W_i^k + (1-x_i)\langle \alpha, U^k\rangle + x_i\langle \beta, U^k\rangle, \tag{2.4.1}$$

$$\frac{1}{12}\left(U_{i-1}^k + 10U_i^k + U_{i+1}^k\right)$$
$$= \frac{1}{12}\left(W_{i-1}^k + 10W_i^k + W_{i+1}^k\right) + (1-x_i)\langle \alpha, U^k\rangle + x_i\langle \beta, U^k\rangle, \tag{2.4.2}$$

$$\delta_x^2 U_i^k = \delta_x^2 W_i^k. \tag{2.4.3}$$

Substituting (2.4.2) and (2.4.3) into (2.2) and multiplying the result by τ, we know that W_i^k satisfies

$$\frac{1}{12}\left(W_{i-1}^k + 10W_i^k + W_{i+1}^k\right) - \frac{\tau}{2}\delta_x^2 W_i^k$$
$$= -(1-x_i)\langle \alpha, U^k\rangle - x_i\langle \beta, U^k\rangle, \quad 1 \leqslant i \leqslant M-1, \tag{2.5.1}$$

$$W_0^k = 0, \quad W_M^k = 0. \tag{2.5.2}$$

Multiplying (2.5.1) by hW_i^k and summing up for i from 1 to $M-1$, we have

$$\begin{aligned} &\frac{1}{12}h\sum_{i=1}^{M-1}\left(W_{i-1}^k + 10W_i^k + W_{i+1}^k\right)W_i^k - \frac{\tau}{2}h\sum_{i=1}^{M-1}(\delta_x^2 W_i^k)W_i^k \\ &= -h\sum_{i=1}^{M-1}\left[(1-x_i)\langle \alpha, U^k\rangle + x_i\langle \beta, U^k\rangle\right) W_i^k. \end{aligned} \tag{2.6}$$

Estimate each term in (2.6) as follows: using (2.5), we have

$$\begin{aligned} &\frac{1}{12}h\sum_{i=1}^{M-1}\left(W_{i-1}^k + 10W_i^k + W_{i+1}^k\right)W_i^k \\ =\ & \frac{1}{12}h\sum_{i=1}^{M-1}\left(10(W_i^k)^2 + W_{i-1}^k W_i^k + W_{i+1}^k W_i^k\right) \\ =\ & \frac{1}{12}h\sum_{i=1}^{M-1}\left(10(W_i^k)^2 - \frac{1}{2}(W_{i-1}^k)^2 - \frac{1}{2}(W_i^k)^2 - \frac{1}{2}(W_{i+1}^k)^2 - \frac{1}{2}(W_i^k)^2\right) \\ \geqslant\ & \frac{2}{3}\|W^k\|^2, \end{aligned} \tag{2.7}$$

$$-h\sum_{i=1}^{M-1}(\delta_x^2 W_i^k)W_i^k = h\sum_{i=1}^{M}\left(\frac{W_i^k - W_{i-1}^k}{h}\right)^2 = \|\delta_x W^k\|^2. \tag{2.8}$$

In the following, we denote

$$\gamma_h = \sqrt{\frac{1}{6}(2+h^2)}, \quad \rho_h = \frac{\sqrt{10}}{3}\left(\|\alpha\|_{L_2} + \|\beta\|_{L_2} + 2c_3 h\right).$$

Then, it follows from (2.4.1) and Lemma 1.3 that

$$\begin{aligned} \|U^k\| &\leqslant \|W^k\| + \|1-x\|\,|\langle \alpha, U^k\rangle| + \|x\|\,|\langle \beta, U^k\rangle| \\ &\leqslant \|W^k\| + \gamma_h\left(|\langle \alpha, U^k\rangle| + |\langle \beta, U^k\rangle|\right) \\ &\leqslant \|W^k\| + \gamma_h\left[\frac{\sqrt{10}}{3}\left(\|\alpha\|_{L_2} + c_3 h\right)\|U^k\| + \frac{\sqrt{10}}{3}\left(\|\beta\|_{L_2} + c_3 h\right)\|U^k\|\right] \\ &\leqslant \|W^k\| + \gamma_h\rho_h\|U^k\|, \end{aligned}$$

which gives

$$\|U^k\| \leqslant \frac{1}{1-\gamma_h\rho_h}\|W^k\|. \tag{2.9}$$

Consider the right-hand side of (2.6). Using (2.9), we have

$$\begin{aligned}
& -h\sum_{i=1}^{M-1}\left[(1-x_i)\langle\alpha,U^k\rangle + x_i\langle\beta,U^k\rangle\right)W_i^k \\
= \quad & -\langle\alpha,U^k\rangle h\sum_{i=1}^{M-1}(1-x_i)W_i^k - \langle\beta,U^k\rangle h\sum_{i=1}^{M-1}x_iW_i^k \\
\leqslant \quad & |\langle\alpha,U^k\rangle|\cdot\sqrt{h\sum_{i=1}^{M-1}(1-x_i)^2}\sqrt{h\sum_{i=1}^{M-1}(W_i^k)^2} \\
& +|\langle\beta,U^k\rangle|\cdot\sqrt{h\sum_{i=1}^{M-1}(x_i)^2}\sqrt{h\sum_{i=1}^{M-1}(W_i^k)^2} \\
\leqslant \quad & \sqrt{\frac{1}{3}}\left(|\langle\alpha,U^k\rangle| + |\langle\beta,U^k\rangle|\right)\|W^k\| \\
\leqslant \quad & \sqrt{\frac{1}{3}}\rho_h\|U^k\|\cdot\|W^k\| \leqslant \sqrt{\frac{1}{3}}\frac{\rho_h}{1-\gamma_h\rho_h}\|W^k\|^2.
\end{aligned} \tag{2.10}$$

Inserting (2.7), (2.8) and (2.10) into (2.6), we obtain

$$\frac{2}{3}\|W^k\|^2 + \frac{\tau}{2}\|\delta_x W^k\|^2 \leqslant \frac{1}{\sqrt{3}}\frac{\rho_h}{1-\gamma_h\rho_h}\|W^k\|^2. \tag{2.11}$$

Using the condition $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < \frac{6}{5}\sqrt{\frac{3}{10}}$, there exists a h_0, when $h \leqslant h_0$,

$$\frac{1}{\sqrt{3}}\,\frac{\rho_h}{1-\gamma_h\rho_h} \leqslant \frac{2}{3}.$$

Then, it follows from (2.11) that

$$\|\delta_x W^k\| = 0.$$

Noticing (2.5.2), we have

$$W_i^k = 0, \quad 0 \leqslant i \leqslant M.$$

Estimate (2.9) gives $\|U^k\| = 0$, i.e.,

$$U_i^k = 0.\ 0 \leqslant i \leqslant M.$$

This means that (2.2) has only a trivial solution. Therefore,(2.1) determines $\{U_i^k|0 \leqslant i \leqslant M\}$ uniquely. □

3. The convergence

Define the net functions on $\Omega_h \times \Omega_\tau$:

$$u_i^k = u(x_i, t_k), \qquad e_i^k = u_i^k - U_i^k.$$

Theorem 3.1. *Let $\{U_i^k\}$ be the solution of (1.2) and $u \in C^{6,3}([0,1]\times[0,T])$ be the solution of (1.1). Suppose $\alpha \in C^4([0,1])$, $\beta \in C^4([0,1])$, and $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < 6/5\sqrt{3/10}$. Then there exist two constants C and h_0, such that, when $h \leqslant h_0$,*

$$\|\delta_x e^k\| \leqslant C(\tau^2+h^4), \quad \|e^k\|_\infty \leqslant C(\tau^2+h^4), \quad 0 \leqslant k \leqslant K.$$

Proof. Using the Taylor expansion, Lemma 1.1 and (1.1), we have

$$\begin{aligned}
&\frac{1}{12}\left(\delta_t u_{i-1}^{k-\frac12} + 10\delta_t u_i^{k-\frac12} + \delta_t u_{i+1}^{k-\frac12}\right) - \delta_x^2 u_i^{k-\frac12} \\
&= \frac{1}{12}\left(f_{i-1}^{k-\frac12} + 10 f_i^{k-\frac12} + f_{i+1}^{k-\frac12}\right) + R_i^{k-\frac12}, \quad 1 \leqslant i \leqslant M-1, \ \ 1 \leqslant k \leqslant K, && (3.1.1)\\
&u_0^k = \langle \alpha, u^k\rangle + g_0(t_k) + r_0^k, \quad u_M^k = \langle \beta, u^k\rangle + g_1(t_k) + r_M^k, \ 1 \leqslant k \leqslant K, && (3.1.2)\\
&u_i^0 = \phi(x_i), \quad 0 \leqslant i \leqslant M && (3.1.3)
\end{aligned}$$

and

$$u_0^0 = \langle \alpha, u^0\rangle + g_0(t_0) + r_0^0, \quad u_M^0 = \langle \beta, u^0\rangle + g_1(t_0) + r_M^0, \tag{3.1.4}$$

where

$$\begin{aligned}
R_i^{k-\frac12} &= \left(\frac{1}{24}\frac{\partial^3 u(\xi_{1,i},\eta_{1,k})}{\partial t^3} + \frac18 \frac{\partial^4 u(\xi_{2,i},\eta_{2,k})}{\partial x^2 \partial t^2}\right)\tau^2 - \frac{1}{240}\frac{\partial^6 u(\xi_{3,i},\eta_{3,k})}{\partial x^6} h^4,\\
&\qquad \xi_{1,i},\ \xi_{2,i},\ \xi_{3,i} \in (x_{i-1}, x_{i+1}),\ \eta_{1,k},\ \eta_{2,k},\ \eta_{3,k} \in (t_{k-1}, t_k),\\
&\qquad 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant K, \qquad\qquad \text{(Ref. [10, 13])}\\
r_0^k &= -\frac{1}{180}h^4 \frac{\partial^4}{\partial x^4}(\alpha(x)u(x,t))|_{x=\xi(t_k)},\ \xi(t_k) \in (0,1),\ 0 \leqslant k \leqslant K,\\
r_M^k &= -\frac{1}{180}h^4 \frac{\partial^4}{\partial x^4}(\beta(x)u(x,t))|_{x=\eta(t_k)},\ \eta(t_k) \in (0,1),\ 0 \leqslant k \leqslant K.
\end{aligned}$$

There exists a constant c_4 such that

$$\begin{aligned}
&|R_i^{k-\frac12}| \leqslant c_4(\tau^2+h^4),\ 1 \leqslant i \leqslant M-1,\ 1 \leqslant k \leqslant K, && (3.2.1)\\
&|r_0^k| \leqslant c_4 h^4, \quad |r_M^k| \leqslant c_4 h^4, \quad 0 \leqslant k \leqslant K. && (3.2.2)
\end{aligned}$$

It follows from (3.1.2) and (3.1.4) that

$$\begin{aligned}
&\delta_t u_0^{k-\frac12} = \langle \alpha, \delta_t u^{k-\frac12}\rangle + \delta_t (g_0)^{k-\frac12} + \delta_t r_0^{k-\frac12}, \quad 1 \leqslant k \leqslant K, && (3.3.1)\\
&\delta_t u_M^{k-\frac12} = \langle \beta, \delta_t u^{k-\frac12}\rangle + \delta_t (g_1)^{k-\frac12} + \delta_t r_M^{k-\frac12}, \quad 1 \leqslant k \leqslant K, && (3.3.2)
\end{aligned}$$

where $\delta_t(g_0)^{k-1/2} = 1/\tau\,(g_0(t_k) - g_0(t_{k-1}))$ and $\delta_t(g_1)^{k-1/2}, \delta_t r_0^{k-1/2}, \delta_t r_1^{k-1/2}$ are similar.

Equation (3.3) may be regarded as the expansions of the following two equations:

$$u_t(0,t) = \int_0^1 \alpha(s) u_t(s,t)ds + (g_0)_t(t),\ u_t(1,t) = \int_0^1 \beta(s) u_t(s,t)ds + (g_1)_t(t),$$

which is the derivative of (1.1.2)-(1.1.3) with respect to t. Thus, there exists a constant c_5 such that

$$|\delta_t r_0^{k-\frac12}| \leqslant c_5(\tau^2+h^4), \quad |\delta_t r_M^{k-\frac12}| \leqslant c_5(\tau^2+h^4), \quad 1 \leqslant k \leqslant K. \tag{3.4}$$

Subtracting (1.2) from (3.1), we obtain the error equations

$$\frac{1}{12}\left(\delta_t e_{i-1}^{k-\frac{1}{2}} + 10\delta_t e_i^{k-\frac{1}{2}} + \delta_t e_{i+1}^{k-\frac{1}{2}}\right) - \delta_x^2 e_i^{k-\frac{1}{2}} = R_i^{k-\frac{1}{2}},$$
$$1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant K, \tag{3.5.1}$$
$$e_0^k = \langle \alpha, e^k\rangle + r_0^k, \quad e_M^k = \langle \beta, e^k\rangle + r_M^k, \quad 1 \leqslant k \leqslant K, \tag{3.5.2}$$
$$e_i^0 = [c_0(1-x_i) + c_1 x_i]h^4, \quad 0 \leqslant i \leqslant M, \tag{3.5.3}$$

and

$$e_0^0 = \langle \alpha, e^0\rangle + r_0^0, \quad e_M^0 = \langle \beta, e^0\rangle + r_M^0. \tag{3.5.4}$$

Let

$$\bar{e}_i^k = e_i^k - (1-x_i)(\langle \alpha, e^k\rangle + r_0^k) - x_i(\langle \beta, e^k\rangle + r_M^k). \tag{3.6}$$

Then we have

$$\delta_t e_i^{k-\frac{1}{2}} = \delta_t \bar{e}_i^{k-\frac{1}{2}} + (1-x_i)\left[\langle \alpha, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_0^{k-\frac{1}{2}}\right] + x_i\left[\langle \beta, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_M^{k-\frac{1}{2}}\right], \tag{3.7.1}$$
$$\delta_x^2 e_i^{k-\frac{1}{2}} = \delta_x^2 \bar{e}_i^{k-\frac{1}{2}}. \tag{3.7.2}$$

Substituting these two equalities into (3.5), we obtain

$$\frac{1}{12}\left(\delta_t \bar{e}_{i-1}^{k-\frac{1}{2}} + 10\delta_t \bar{e}_i^{k-\frac{1}{2}} + \delta_t \bar{e}_{i+1}^{k-\frac{1}{2}}\right) - \delta_x^2 \bar{e}_i^{k-\frac{1}{2}} = -(1-x_i)\left(\langle \alpha, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_0^{k-\frac{1}{2}}\right)$$
$$-x_i\left(\langle \beta, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_M^{k-\frac{1}{2}}\right) + R_i^{k-\frac{1}{2}}, \quad 1 \leqslant i \leqslant M-1, \quad 1 \leqslant k \leqslant K, \tag{3.8.1}$$
$$\bar{e}_0^k = 0, \quad \bar{e}_M^k = 0, \quad 1 \leqslant k \leqslant K, \tag{3.8.2}$$
$$\bar{e}_i^0 = e_i^0 - (1-x_i)(\langle \alpha, e^0\rangle + r_0^0) - x_i(\langle \beta, e^0\rangle + r_M^0), \; 0 \leqslant i \leqslant M \tag{3.8.3}$$

and

$$\bar{e}_0^0 = 0, \quad \bar{e}_M^0 = 0. \tag{3.8.4}$$

From (3.8.2) and (3.8.4), we have

$$\delta_t \bar{e}_0^{k-\frac{1}{2}} = 0, \quad \delta_t \bar{e}_M^{k-\frac{1}{2}} = 0, \quad 1 \leqslant k \leqslant K. \tag{3.9}$$

Similarly to the proof of (2.9), it follows from (3.7.1) that

$$\|\delta_t e^{k-\frac{1}{2}}\| \leqslant \frac{1}{1-\gamma_h \rho_h}\|\delta_t \bar{e}^{k-\frac{1}{2}}\| + \frac{\gamma_h}{1-\gamma_h \rho_h}\left(|\delta_t r_0^{k-\frac{1}{2}}| + |\delta_t r_M^{k-\frac{1}{2}}|\right), \quad 1 \leqslant k \leqslant K. \tag{3.10}$$

Multiplying (3.8.1) by $h\delta_t \bar{e}_i^{k-1/2}$ and summing up for i from 1 to $M-1$, we have

$$\frac{1}{12}h\sum_{i=1}^{M-1}\left(\delta_t \bar{e}_{i-1}^{k-\frac{1}{2}} + 10\delta_t \bar{e}_i^{k-\frac{1}{2}} + \delta_t \bar{e}_{i+1}^{k-\frac{1}{2}}\right)\delta_t \bar{e}_i^{k-\frac{1}{2}} - h\sum_{i=1}^{M-1}(\delta_x^2 \bar{e}_i^{k-\frac{1}{2}})\delta_t \bar{e}_i^{k-\frac{1}{2}}$$
$$= h\sum_{i=1}^{M-1}\left[-(1-x_i)(\langle \alpha, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_0^{k-\frac{1}{2}}) - x_i(\langle \beta, \delta_t e^{k-\frac{1}{2}}\rangle + \delta_t r_M^{k-\frac{1}{2}})\right]\delta_t \bar{e}_i^{k-\frac{1}{2}} \tag{3.11}$$
$$+ h\sum_{i=1}^{M-1} R_i^{k-\frac{1}{2}}\delta_t \bar{e}_i^{k-\frac{1}{2}}.$$

Now we estimate each term in the above equation. For the two terms on the left, we have

$$\frac{1}{12}h\sum_{i=1}^{M-1}\left(\delta_t\bar{e}_{i-1}^{k-\frac{1}{2}}+10\delta_t\bar{e}_i^{k-\frac{1}{2}}+\delta_t\bar{e}_{i+1}^{k-\frac{1}{2}}\right)\delta_t\bar{e}_i^{k-\frac{1}{2}}\geqslant\frac{2}{3}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2, \tag{3.12}$$

$$-h\sum_{i=1}^{M-1}\delta_x^2\bar{e}_i^{k-\frac{1}{2}}\delta_t\bar{e}_i^{k-\frac{1}{2}}=\frac{1}{2\tau}(\|\delta_x\bar{e}^k\|^2-\|\delta_x\bar{e}^{k-1}\|^2). \tag{3.13}$$

For the first term on the right, we have

$$\begin{aligned}
&\left|h\sum_{i=1}^{M-1}\left[-(1-x_i)(\langle\alpha,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_0^{k-\frac{1}{2}})-x_i(\langle\beta,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_M^{k-\frac{1}{2}})\right]\delta_t\bar{e}_i^{k-\frac{1}{2}}\right|\\
&\leqslant|\langle\alpha,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_0^{k-\frac{1}{2}}|\cdot|h\sum_{i=1}^{M-1}(1-x_i)\delta_t\bar{e}_i^{k-\frac{1}{2}}|\\
&\quad+|\langle\beta,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_M^{k-\frac{1}{2}}|\cdot|h\sum_{i=1}^{M-1}x_i\delta_t\bar{e}_i^{k-\frac{1}{2}}|\\
&\leqslant\frac{1}{\sqrt{3}}|\langle\alpha,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_0^{k-\frac{1}{2}}|\cdot\|\delta_t\bar{e}^{k-\frac{1}{2}}\|+\frac{1}{\sqrt{3}}|\langle\beta,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_M^{k-\frac{1}{2}}|\cdot\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&\leqslant\frac{1}{\sqrt{3}}\left(|\langle\alpha,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_0^{k-\frac{1}{2}}|+|\langle\beta,\delta_t e^{k-\frac{1}{2}}\rangle+\delta_t r_M^{k-\frac{1}{2}}|\right)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&\leqslant\frac{1}{\sqrt{3}}\left(|\langle\alpha,\delta_t e^{k-\frac{1}{2}}\rangle|+|\langle\beta,\delta_t e^{k-\frac{1}{2}}\rangle|+|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|\right)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&\leqslant\frac{1}{\sqrt{3}}\left(\rho_h\|\delta_t e^{k-\frac{1}{2}}\|+|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|\right)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&\leqslant\frac{1}{\sqrt{3}}\left(\frac{\rho_h}{1-\gamma_h\rho_h}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|+(1+\frac{\gamma_h\rho_h}{1-\gamma_h\rho_h})(|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|)\right)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&\leqslant\frac{1}{\sqrt{3}}\frac{\rho_h}{1-\gamma_h\rho_h}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2+\frac{1}{\sqrt{3}}(1+\frac{\gamma_h\rho_h}{1-\gamma_h\rho_h})(|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|.
\end{aligned} \tag{3.14}$$

In the last inequality, we have used (3.10). Inserting (3.12)—(3.14) into (3.11), we get

$$\begin{aligned}
&\frac{2}{3}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2+\frac{1}{2\tau}(\|\delta_x\bar{e}^k\|^2-\|\delta_x\bar{e}^{k-1}\|^2)\\
\leqslant\ &\frac{1}{\sqrt{3}}\frac{\rho_h}{1-\gamma_h\rho_h}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2+\frac{1}{\sqrt{3}}(1+\frac{\gamma_h\rho_h}{1-\gamma_h\rho_h})(|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|)\|\delta_t\bar{e}^{k-\frac{1}{2}}\|\\
&+h\sum_{i=1}^{M-1}R_i^{k-\frac{1}{2}}\delta_t\bar{e}_i^{k-\frac{1}{2}}\\
\leqslant\ &\frac{1}{\sqrt{3}}\frac{\rho_h}{1-\gamma_h\rho_h}\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2\\
&+\delta_1\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2+\frac{1}{4\delta_1}\left\{\frac{1}{\sqrt{3}}(1+\frac{\gamma_h\rho_h}{1-\gamma_h\rho_h})(|\delta_t r_0^{k-\frac{1}{2}}|+|\delta_t r_M^{k-\frac{1}{2}}|)\right\}^2\\
&+\delta_2\|\delta_t\bar{e}^{k-\frac{1}{2}}\|^2+\frac{1}{4\delta_2}h\sum_{i=1}^{M-1}(R_i^{k-\frac{1}{2}})^2,\quad(\delta_1>0,\delta_2>0).
\end{aligned}$$

Since $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < 6/5\sqrt{3/10}$, there exist constants h_0, δ_1, δ_2 such that, when $h \leqslant h_0$,

$$\frac{1}{\sqrt{3}}\frac{\rho_h}{1-\gamma_h\rho_h} + \delta_1 + \delta_2 \leqslant \frac{2}{3}.$$

Therefore, we have

$$\begin{aligned}&\frac{1}{2\tau}(\|\delta_x\bar{e}^k\|^2 - \|\delta_x\bar{e}^{k-1}\|^2)\\ \leqslant\ &\frac{1}{4\delta_1}\left\{\frac{1}{\sqrt{3}}(1+\frac{\gamma_h\rho_h}{1-\gamma_h\rho_h})(|\delta_t r_0^{k-\frac{1}{2}}| + |\delta_t r_M^{k-\frac{1}{2}}|)\right\}^2 + \frac{1}{4\delta_2}h\sum_{i=1}^{M-1}(R_i^{k-\frac{1}{2}})^2.\end{aligned}$$

Using (3.2.1) and (3.4), there exists a constant c_6 such that

$$\frac{1}{2\tau}(\|\delta_x\bar{e}^k\|^2 - \|\delta_x\bar{e}^{k-1}\|^2) \leqslant c_6(\tau^2+h^4)^2,$$

or,

$$\|\delta_x\bar{e}^k\|^2 \leqslant \|\delta_x\bar{e}^{k-1}\|^2 + 2c_6\tau(\tau^2+h^4)^2, \quad 1 \leqslant k \leqslant K,$$

which gives

$$\|\delta_x\bar{e}^k\|^2 \leqslant \|\delta_x\bar{e}^0\|^2 + 2c_6T(\tau^2+h^4)^2, \quad 1 \leqslant k \leqslant K.$$

Noticing the definition (3.6), we have

$$\begin{aligned}\|\delta_x\bar{e}^0\| &\leqslant \|\delta_x e^0\| + |\langle\alpha, e^0\rangle + r_0^0| + |\langle\beta, e^0\rangle + r_M^0|\\ &\leqslant \|\delta_x e^0\| + \big(|\langle\alpha, e^0\rangle| + |\langle\beta, e^0\rangle|\big) + |r_0^0| + |r_M^0|\\ &\leqslant \|\delta_x e^0\| + \rho_h\|e^0\| + |r_0^0| + |r_M^0|\\ &\leqslant |c_1 - c_0|h^4 + \gamma_h\rho_h(|c_0| + |c_1|)h^4 + |r_0^0| + |r_M^0|.\end{aligned}$$

Using this inequality and (3.2.2), there exists a constant c_7 such that

$$\|\delta_x\bar{e}^k\| \leqslant c_7(\tau^2+h^4), \quad 0 \leqslant k \leqslant K. \tag{3.16}$$

Consequently,

$$\|\bar{e}^k\| \leqslant \frac{1}{2}c_7(\tau^2+h^4), \quad \|\bar{e}^k\|_\infty \leqslant \frac{1}{2}c_7(\tau^2+h^4), \quad 0 \leqslant k \leqslant K. \tag{3.17}$$

Using (3.6), we may obtain

$$\begin{aligned}\|e^k\| &\leqslant \|\bar{e}^k\| + |\langle\alpha, e^k\rangle + r_0^k|\cdot\|1-x\| + |\langle\beta, e^k\rangle + r_M^k|\cdot\|x\|\\ &\leqslant \|\bar{e}^k\| + \gamma_h(\rho_h\|e^k\| + |r_0^k| + |r_M^k|),\end{aligned}$$

or,

$$\|e^k\| \leqslant \frac{1}{1-\gamma_h\rho_h}\left(\|\bar{e}^k\| + \gamma_h(|r_0^k| + |r_M^k|)\right).$$

Thus,

$$\begin{aligned}\|e^k\| &\leqslant \frac{1}{1-\gamma_h\rho_h}\left(\frac{1}{2}c_7(\tau^2+h^4) + 2c_4\gamma_h(\tau^2+h^4)\right)\\ &= \frac{1}{1-\gamma_h\rho_h}\left(\frac{1}{2}c_7 + 2c_4\gamma_h\right)(\tau^2+h^4) \leqslant c_8(\tau^2+h^4),\ 0 \leqslant k \leqslant K.\end{aligned} \tag{3.18}$$

Using (3.6), we obtain

$$\|\delta_x e^k\| \leqslant \|\delta_x \bar{e}^k\| + \rho_h \|e^k\| + |r_0^k| + |r_M^k| \leqslant (c_7 + \rho_h c_8 + 2c_4)(\tau^2 + h^4) \leqslant c_9(\tau^2 + h^4). \quad (3.19)$$

Using (3.6) again, we obtain

$$\begin{aligned} \|e^k\|_\infty &\leqslant \|\bar{e}^k\|_\infty + |\langle \alpha, e^k\rangle + r_0^k| + |\langle \beta, e^k\rangle + r_M^k| \\ &\leqslant \|\bar{e}^k\|_\infty + \rho_h \|e^k\| + |r_0^k| + |r_M^k| \\ &\leqslant \frac{1}{2} c_7(\tau^2 + h^4) + c_8(\tau^2 + h^4) + 2c_4(\tau^2 + h^4) \\ &\leqslant \left\{\frac{1}{2} c_7 + c_8 + 2c_4\right\}(\tau^2 + h^4) \\ &\leqslant c_{10}(\tau^2 + h^4), \quad 0 \leqslant k \leqslant K. \end{aligned} \quad (3.20)$$

□

4. Numerical examples

In this section, we test the proposed difference method on six examples, whose exact solutions are known to us. The right-hand side functions as well as the nonlocal boundary value conditions and initial value conditions are obtained from the exact solutions. The systems of linear algebraic equations have been solved by using the Gaussian pivot method.

Problem 1. The problem is to solve (1.1) with $\alpha = 0.2, \beta = 0.4$ and the exact solution $u(x,t) = \exp(2(x-t))$.

Problem 2. The problem is to solve (1.1) with $\alpha = 0.4, \beta = -0.6$ and the exact solution $u(x,t) = \exp(2(x-t))$.

Problem 3. The problem is to solve (1.1) with $\alpha = 1.4, \beta = 0.4$ and the exact solution $u(x,t) = \exp(2(x-t))$.

Problem 4. The problem is to solve (1.1) with $\alpha = 0.2, \beta = 0.4$ and the exact solution $u(x,t) = x^6 + t^6$.

Problem 5. The problem is to solve (1.1) with $\alpha = 0.4, \beta = -0.6$ and the exact solution $u(x,t) = x^6 + t^6$.

Problem 6. The problem is to solve (1.1) with $\alpha = 1.4, \beta = 0.4$ and the exact solution $u(x,t) = x^6 + t^6$.

Table 4.1. Some numerical results for Problem 1 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	.0896188	.0457165	.1343220	.0675746	.2170946	.0840996
.10	$.10^2$	.1353392	.0000039	.2019092	.0000127	.3012131	.0000189
.05	$.05^2$	.1353353	.0000000	.2018963	.0000002	.3011938	.0000004
exact solution		.1353353		.2018965		.3011942	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	.3553329	.0939961	.5738221	.0964980	.9085670	.0914330
.10	$.10^2$	.4493501	.0000211	.6703380	.0000179	1.0000080	.0000079
.05	$.05^2$	.4493283	.0000007	.6703195	.0000005	.9999999	.0000001
exact solution		.4493290		.6703200		1.0000000	

Table 4.2. Some numerical results for Problem 2 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	.0705301	.0648052	.1700984	.0317981	.3049023	.0037081
.10	$.10^2$	.1353398	.0000045	.2019057	.0000092	.3012058	.0000115
.05	$.05^2$	.1353354	.0000001	.2018966	.0000001	.3011943	.0000000
exact solution		.1353353		.2018965		.3011942	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	.4882091	.0388801	.7412245	.0709045	1.0972080	.0972078
.10	$.10^2$	.4493394	.0000105	.6703249	.0000049	.9999931	.0000069
.05	$.05^2$	.4493288	.0000001	.6703199	.0000002	1.0000000	.0000002
exact solution		.4493290		.6703200		1.0000000	

Table 4.3. Some numerical results for Problem 3 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	-3.2990080	3.4343430	-3.1176980	3.3195940	-2.6871880	2.9883830
.10	$.10^2$	.1357124	.0003771	.2022422	.0003457	.3014969	.0003026
.05	$.05^2$	.1353408	.0000055	.2019015	.0000050	.3011985	.0000043
exact solution		.1353353		.2018965		.3011942	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	-2.0144020	2.4637310	-1.1097680	1.7800880	.0187589	.9812410
.10	$.10^2$	.4495775	.0002486	.6705036	.0001836	1.0001080	.0001078
.05	$.05^2$	.4493322	.0000032	.6703224	.0000023	1.0000010	.0000014
exact solution		.4493290		.6703200		1.0000000	

Table 4.4. Some numerical results for Problem 4 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	.9244545	.0755455	.9288128	.0712512	.9283456	.0757504
.10	$.10^2$	1.0000030	.0000035	1.0000160	.0000482	1.0040240	.0000725
.05	$.05^2$	.9999999	.0000001	1.0000600	.0000042	1.0040900	.0000063
exact solution		1.0000000		1.0000640		1.0040960	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	.9577495	.0889065	1.1495510	.1125929	1.8489090	.1510909
.10	$.10^2$	1.0465840	.0000715	1.2620980	.0000457	2.0000070	.0000069
.05	$.05^2$	1.0466490	.0000068	1.2621390	.0000049	2.0000000	.0000000
exact solution		1.0466560		1.2621440		2.0000000	

Table 4.5. Some numerical results for Problem 5 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	.8728189	.1271811	.9378006	.0622634	.9976370	.0064591
.10	$.10^2$	1.0000030	.0000032	1.0000110	.0000529	1.0040140	.0000815
.05	$.05^2$	.9999999	.0000001	1.0000600	.0000039	1.0040900	.0000060
exact solution		1.0000000		1.0000640		1.0040960	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	1.0952570	.0486006	1.3732480	.1111040	2.1907710	.1907713
.10	$.10^2$	1.0465720	.0000840	1.2620850	.0000594	1.9999950	.0000048
.05	$.05^2$	1.0466500	.0000064	1.2621390	.0000045	2.0000000	.0000005
exact solution		1.0466560		1.2621440		2.0000000	

Table 4.6. Some numerical results for Problem 6 at $t = 1$

h	τ	$u_h^\tau(0.0)$	$e_h^\tau(0.0)$	$u_h^\tau(0.2)$	$e_h^\tau(0.2)$	$u_h^\tau(0.4)$	$e_h^\tau(0.4)$
.20	$.20^2$	-.4449051	1.4449050	-.1066642	1.1067280	.1447555	.8593405
.10	$.10^2$	1.0005460	.0005456	1.0005090	.0004450	1.0044490	.0003525
.05	$.05^2$	1.0000290	.0000288	1.0000870	.0000234	1.0041140	.0000180
exact solution		1.0000000		1.0000640		1.0040960	
h	τ	$u_h^\tau(0.6)$	$e_h^\tau(0.8)$	$u_h^\tau(0.8)$	$e_h^\tau(0.8)$	$u_h^\tau(1.0)$	$e_h^\tau(1.0)$
.20	$.20^2$	.3762318	.6704242	.7388541	.5232899	1.5871700	.4128300
.10	$.10^2$	1.0469270	.0002710	1.2623480	.0002038	2.0001560	.0001557
.05	$.05^2$	1.0466690	.0000129	1.2621530	.0000094	2.0000080	.0000083
exact solution		1.0466560		1.2621440		2.0000000	

Take $\tau = h^2$, or, $K = M^2$. Some numerical results at $t = 1$ are listed in Tables 4.1—4.6, respectively. In the tables, $u_h^\tau(x)$ represents the numerical solution with the space step-size h and the time step-size τ at the point $(x, 1.0)$ and $e_h^\tau(x) = |u(x, 1.0) - u_h^\tau(x)|$. The exact solutions at the points (x, 0.1) are listed in the last row.

5. Conclusions

We have presented a difference scheme (1.2) for the nonlocal problem (1.1) and proved the solvability and convergence in the order $O(\tau^2 + h^4)$. Numerical examples show us the coincidence with the theoretical results. The condition $\|\alpha\|_{L_2} + \|\beta\|_{L_2} < 6/5\sqrt{3/10}$ is sufficient for convergence, but is not necessary for it. This can be seen from the numerical examples.

For a given tolerance error, for example, $\epsilon = 10^{-8}$, if we use the difference scheme (1.2) in this paper, we take $h \sim 10^{-2}$ and $\tau \sim 10^{-4}$ (i.e., $M \sim 10^2$ and $N \sim 10^4$) and have $\max_{1 \leqslant k \leqslant N} \|e^k\| = O(\tau^2 + h^4) \sim \epsilon$. In other words, solving about $N = 10^4$ systems of linear algebraic equations, **each with $M = 10^2$ unknowns**, yields the satisfied numerical solution. But if we use the Crank-Nicolson scheme from [4], in order to obtain the same accuracy numerical solution, about $N = 10^4$ same type systems of linear algebraic equations, **each with $M = 10^4$ unknowns**, must be solved.

Instead of (1.2.3), we may take initial value of the difference scheme as

$$U_i^0 = \phi(x_i), \quad 0 \leqslant i \leqslant M. \tag{5.1}$$

In this way, we have

$$e_i^0 = 0, \quad 0 \leqslant i \leqslant M.$$

Consequently,

$$\delta_t e_0^{k-\frac{1}{2}} = \langle \alpha, \delta_t e^{k-\frac{1}{2}} \rangle + \frac{1}{\tau} r_0^1, \quad \delta_t e_M^{k-\frac{1}{2}} = \langle \beta, \delta_t e^{k-\frac{1}{2}} \rangle + \frac{1}{\tau} r_M^1,$$

which gives

$$\|e^k\|_\infty \leqslant C(\tau^2 + \frac{h^4}{\sqrt{\tau}}), \quad 0 \leqslant k \leqslant K.$$

Instead of (1.2.3), we may also take initial value of the difference scheme as

$$U_0^0 = \langle \alpha, U^0 \rangle + g_0(t_0), \quad U_i^0 = \phi(x_i),\ 1 \leqslant i \leqslant M-1, \quad U_M^0 = \langle \beta, U^0 \rangle + g_1(t_0). \tag{5.2}$$

In this way, we have

$$e_0^0 = \frac{h}{3}(\alpha_0 e_0^0 + \alpha_M e_M^0) + r_0^0, \quad e_i^0 = 0,\ 1 \leqslant i \leqslant M-1, e_M^0 = \frac{h}{3}(\beta_0 e_0^0 + \beta_M e_M^0) + r_M^0.$$

Consequently,

$$\delta_x e_{\frac{1}{2}}^0 = O(h^3), \quad \delta_x e_{i-\frac{1}{2}}^0 = 0,\ 2 \leqslant i \leqslant M-1, \quad \delta_x e_{M-\frac{1}{2}}^0 = O(h^3),$$

which gives

$$\|e^k\| \leqslant C(\tau^2 + h^{\frac{7}{2}}), \quad 0 \leqslant k \leqslant K.$$

Numerical examples show us that the difference scheme (1.2.1)-(1.2.2) with (5.1) or with (5.2) has the same accuracy as with (1.2.3). But we have not proved this so far.

It is an important topic in practice to consider the difference methods with high order accuracy for two-dimensional heat equation with nonlocal boundary conditions. Perhaps, the method in this paper is useful.

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