COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 2 (2002), No. 2, pp. 113–124

NUMERICAL SOLUTIONS OF FUZZY DIFFERENTIAL EQUATIONS BY TAYLOR METHOD

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Abstract — In this paper, numerical algorithms for solving "fuzzy ordinary differential equations" are considered. A scheme based on the Taylor method of order p is discussed in detail and this is followed by a complete error analysis. The algorithm is illustrated by solving some linear and nonlinear fuzzy Cauchy problems.

2000 Mathematics Subject Classification: 34A12; 65L05.

Keywords: fuzzy differential equation, p-order Taylor method, fuzzy Cauchy problem.

1. Introduction

Knowledge about dynamical systems modeled by differential equations is often incomplete or vague. It concerns, for example, parameter values, functional relationships, or initial conditions. The well-known methods for solving analytically or numerically initial value problems can only be used for finding a selected system behavior, e.g., by fixing the unknown parameters to some plausible values. However, in this case, it is not possible to describe the whole set of system behaviors compatible with our partial knowledge. We may set that the fuzzy input is somehow transformed into the fuzzy output defined by the corresponding crisp systems. This reasons us to refer such systems to as Fuzzy Input – Fuzzy Output (FIFO) systems. Here, we are going to "operationalize" our approach, i.e., to propose a method for computing the approximate solution for a fuzzy differential equation using numerical methods. Since finding this set of solutions analytically does only work with trivial examples, a numerical approach seems to be the only way to "solve" such problems.

The topics of fuzzy differential equations, which attracted a growing interest for some time, in particular, in relation to the fuzzy control, have been rapidly developed recent years. The concept of a fuzzy derivative was first introduced by S. L. Chang, L. A. Zadeh in [4]. It was followed up by D. Dubois, H. Prade in [5], who defined and used the extension principle.

Other methods have been discussed by M. L. Puri, D. A. Ralescu in [10] and R. Goetschel, W. Voxman in [6]. Fuzzy differential equations and initial value problems were regularly treated by O. Kaleva in [7] and [8], S. Seikkala in [11]. A numerical method for solving fuzzy differential equations has been introduced by M. Ma, M. Friedman, A. Kandel in [9] via the standard Euler method.

The structure of this paper is organized as follows. In section 2, some basic results on fuzzy numbers and definition of a fuzzy derivative, which have been discussed by S. Seikkala in [9], are given. In section 3, we define the problem that is a fuzzy Cauchy one. Its numerical solution is of the main interest of this work. Solving numerically the fuzzy differential equation by the Taylor method of order p is discussed in section 4. The proposed algorithm is illustrated by some examples in section 5 and the conclusion is in section 6.

2. Preliminaries

Consider the initial value problem

$$\begin{cases} x'(t) &= f(t, x(t)); \qquad a \leqslant t \leqslant b, \\ x(a) &= \alpha. \end{cases} \tag{1}$$

Let $Y(t)$ be an exact solution of (1) and $Y(t_i)$ be approximated by $y_i = y(t_i)$, which in the p-order Taylor method is as follows:

$$y_{i+1} = y_i + hT(t_i, y_i), \quad i = 0, 1, \ldots, N-1, \tag{2}$$

and

$$T(t_i, y_i) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f^{(i)}(t_i, y_i), \tag{3}$$

where

$$a = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_N = b \ \text{ and } \ h = \frac{(b-a)}{N} = t_{i+1} - t_i. \tag{4}$$

A triangular fuzzy number v is defined by three numbers $a_1 < a_2 < a_3$, where the graph of $v(x)$, the member function of the fuzzy number v, is a triangle with the base on the interval $[a_1, a_3]$ and the vertex at $x = a_2$. We specify v as $(a_1/a_2/a_3)$. We will write: (1) $v > 0$ if $a_1 > 0$; (2) $v \geqslant 0$ if $a_1 \geqslant 0$; (3) $v < 0$ if $a_3 < 0$; and (4) $v \leqslant 0$ if $a_3 \leqslant 0$.

Let E be a set of all the upper semicontinuous normal convex fuzzy numbers with bounded r-level sets. It means that if $v \in E$, then the r-level set

$$[v]_r = \{s |\ v(s) \geqslant r\}, \quad 0 < r \leqslant 1,$$

is a closed bounded interval which is denoted by

$$[v]_r = [v_1(r), v_2(r)].$$

Let I be a real interval. The mapping $x : I \to E$ is called a fuzzy process and its r-level set is denoted by

$$[x(t)]_r = [x_1(t;r), x_2(t;r)], \quad t \in I, \quad r \in (0,1].$$

The derivative $x'(t)$ of the fuzzy process x is defined by

$$[x'(t)]_r = [x_1'(t;r), x_2'(t;r)], \quad t \in I, \quad r \in (0,1],$$

provided that this equation determines the fuzzy number, according to Seikkala [11].

Denote by κ the set of all nonempty compact subsets of R and by κ_c the subset of κ consisting of nonempty convex compact sets. Recall that

$$\rho(x, A) = \min_{a \in A} \|x - a\|$$

is a distance of the point $x \in$ R from $A \in \kappa$ and that the *Hausdorff separation* $\rho(A, B)$ of $A, B \in \kappa$ is defined as

$$\rho(A, B) = \max_{a \in A} \rho(a, B).$$

Note that the notation is consistent, since $\rho(a, B) = \rho(\{a\}, B)$. Now, ρ is not a metric. In fact, $\rho(A, B) = 0$ if and only if $A \subseteq B$. The *Hausdorff metric* d_H on κ is defined by

$$d_H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

The metric d_H on E is as follows:

$$d_\infty(u, v) = \sup\{d_H([u]_r, [v]_r) : 0 \leqslant r \leqslant 1\}, \quad u, v \in E.$$

3. Fuzzy Cauchy problem

Consider the fuzzy initial value problem

$$\begin{cases} x'(t) &= f(t, x(t)), \quad t \in I = [0, T], \\ x(0) &= x_0, \end{cases} \tag{5}$$

where f is a continuous mapping from $R_+ \times R$ into R and $x_0 \in E$ with the r-level set

$$[x_0]_r = [x_1(0; r), x_2(0; r)], \quad r \in (0, 1].$$

The extension principle of Zadeh leads to the following definition of $f(t, x)$ when $x = x(t)$ is a fuzzy number:

$$f(t, x)(s) = \sup\{x(\tau) |\ s = f(t, \tau)\}, \quad s \in R.$$

From this it follows that

$$[f(t, x)]_r = [f_1(t, x; r), f_2(t, x; r)], \quad r \in (0, 1],$$

where

$$\begin{aligned} f_1(t, x; r) = &\quad \min\{f(t, u) |\ u \in [x_1(t; r), x_2(t; r)]\}, \\ f_2(t, x; r) = &\quad \max\{f(t, u) |\ u \in [x_1(t; r), x_2(t; r)]\}. \end{aligned} \tag{6}$$

The mapping $f(t, x)$ is a fuzzy process and the derivative $f^{(i)}(t, x),\ i = 1, \ldots, p$ is defined by

$$[f^{(i)}(t, x)]_r = [f_1^{(i)}(t, x; r), f_2^{(i)}(t, x; r)], \quad t \in I, \quad r \in (0, 1],$$

provided that this equation determines the fuzzy number $f^{(i)}(t, x) \in E$, where

$$\begin{aligned} f_1^{(i)}(t, x; r) = &\quad \min\{f^{(i)}(t, u) |\ u \in [x_1(t; r), x_2(t; r)]\}, \\ f_2^{(i)}(t, x; r) = &\quad \max\{f^{(i)}(t, u) |\ u \in [x_1(t; r), x_2(t; r)]\}. \end{aligned} \tag{7}$$

Theorem 3.1. *Let f satisfy*

$$|f(t,v) - f(t,\overline{v})| \leqslant g(t, |v - \overline{v}|); \quad t \geqslant 0; \quad v, \overline{v} \in R,$$

where $g : R_+ \times R_+ \to R_+$ is a continuous mapping such that $r \to g(t,r)$ is nondecreasing. Let the initial value problem

$$u'(t) = g(t, u(t)), \quad u(0) = u_0, \tag{8}$$

have a solution on R_+ for $u_0 > 0$ and $u(t) = 0$ be the only solution of (8) for $u_0 = 0$. Then the fuzzy initial value problem (5) has a unique fuzzy solution.

Proof. [11] Since

$$x^{(i)}(t) = f^{(i-1)}(t, x(t)) = \frac{\partial f^{(i-2)}}{\partial t}(t, x(t)) + \frac{\partial f^{(i-2)}}{\partial x}(t, x(t)) f^{(i-2)}(t, x(t)), \tag{9}$$

from (7) it follows that

$$\begin{aligned} f_1^{(i-1)}(t,x;r) &= \min\left\{ \frac{\partial f^{(i-2)}}{\partial t}(t,u) + \frac{\partial f^{(i-2)}}{\partial u}(t,u) f^{(i-2)}(t,u) |\ u \in [x_1(t;r), x_2(t;r)] \right\}, \\ f_2^{(i-1)}(t,x;r) &= \max\left\{ \frac{\partial f^{(i-2)}}{\partial t}(t,u) + \frac{\partial f^{(i-2)}}{\partial u}(t,u) f^{(i-2)}(t,u) |\ u \in [x_1(t;r), x_2(t;r)] \right\}. \end{aligned} \tag{10}$$

□

4. Taylor method of order p

Let the exact solution

$$[Y(t)]_r = [Y_1(t;r), Y_2(t;r)]$$

be approximated by

$$[y(t)]_r = [y_1(t;r), y_2(t;r)].$$

The Taylor method of order p is based on the expansion

$$x(t+h;r) = \sum_{i=0}^{p} \frac{h^i}{i!} x^{(i)}(t;r), \tag{11}$$

where $x(t;r)$ is Y_1 or Y_2. We define

$$\begin{aligned} F[t,x;r] &= \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_1^{(i)}(t,x;r), \\ G[t,x;r] &= \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} f_2^{(i)}(t,x;r). \end{aligned} \tag{12}$$

The exact and approximate solutions at t_n, $0 \leqslant n \leqslant N$ are denoted by

$$[Y(t_n)]_r = [Y_1(t_n;r), Y_2(t_n;r)] \text{ and } [y(t_n)]_r = [y_1(t_n;r), y_2(t_n;r)],$$

respectively. The solution is calculated at the grid points of (4). Using the Taylor method of order p and substituting Y_1, Y_2 into (11) and considering (12), we have

$$\begin{aligned} Y_1(t_{n+1};r) &\approx Y_1(t_n;r) + hF[t_n, Y(t_n);r], \\ Y_2(t_{n+1};r) &\approx Y_2(t_n;r) + hG[t_n, Y(t_n);r]. \end{aligned} \tag{13}$$

Hence, we get

$$\begin{aligned} y_1(t_{n+1};r) &= y_1(t_n;r) + hF[t_n, y(t_n);r], \\ y_2(t_{n+1};r) &= y_2(t_n;r) + hG[t_n, y(t_n);r], \end{aligned} \tag{14}$$

where

$$y_1(0;r) = x_1(0;r), \quad y_2(0;r) = x_2(0;r).$$

The following lemmas will be applied to show convergence of these approximations, i.e.,

$$\lim_{h\to 0} y_1(t;r) = Y_1(t;r),$$
$$\lim_{h\to 0} y_2(t;r) = Y_2(t;r).$$

Lemma 4.1. *Let the sequence of numbers* $\{W_n\}_{n=0}^{N}$ *satisfy*

$$|W_{n+1}| \leqslant A|W_n| + B, \quad 0 \leqslant n \leqslant N-1,$$

for the given positive constants A *and* B*. Then*

$$|W_n| \leqslant A^n|W_0| + B\frac{A^n - 1}{A-1}, \quad 0 \leqslant n \leqslant N.$$

Proof. See [9]. □

Lemma 4.2. *Let the sequences of numbers* $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ *satisfy*

$$\begin{aligned} |W_{n+1}| &\leqslant |W_n| + A\max\{|W_n|, |V_n|\} + B, \\ |V_{n+1}| &\leqslant |V_n| + A\max\{|W_n|, |V_n|\} + B, \end{aligned}$$

for the given positive constants A *and* B*. Then, denoting*

$$U_n = |W_n| + |V_n|, \quad 0 \leqslant n \leqslant N,$$

we have

$$U_n \leqslant \bar{A}^n U_0 + \bar{B}\frac{\bar{A}^n - 1}{\bar{A} - 1}, \quad 0 \leqslant n \leqslant N,$$

where $\bar{A} = 1 + 2A$ *and* $\bar{B} = 2B$*.*

Proof. See [9]. □

Let $F^*(t,u,v)$ and $G^*(t,u,v)$ be the functions F and G in (12), where u and v are constants and $u \leqslant v$. In other words,

$$F^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \min\left\{ f_1^{(i)}(t,\tau) | \tau \in [u,v] \right\},$$
$$G^*(t,u,v) = \sum_{i=0}^{p-1} \frac{h^i}{(i+1)!} \max\left\{ f_1^{(i)}(t,\tau) | \tau \in [u,v] \right\},$$

or $F^*(t,u,v)$ and $G^*(t,u,v)$ are obtained by substituting $[x(t)]_r = [u,v]$ into (12). The domain of F^* and G^* is

$$K = \{(t,u,v)|\ 0 \leqslant t \leqslant T, \quad -\infty < v < \infty, \quad -\infty < u \leqslant v\}.$$

Theorem 4.1. *Let $F^*(t,u,v)$ and $G^*(t,u,v)$ belong to $C^{p-1}(K)$ and the partial derivatives of F^* and G^* be bounded over K. Then, for arbitrarily fixed $0 \leqslant r \leqslant 1$, the numerical solutions of (14) converge to the exact solutions $Y_1(t;r)$ and $Y_2(t;r)$ uniformly in t.*

Proof. It is sufficient to show

$$\lim_{h\to 0} y_1(t_N;r) = Y_1(t_N;r), \quad \lim_{h\to 0} y_2(t_N;r) = Y_2(t_N;r),$$

where $t_N = T$. For $n = 0,1,\ldots,N-1$, using the Taylor theorem, we get

$$\begin{aligned} Y_1(t_{n+1};r) &= Y_1(t_n;r) + hF^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \frac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_{n,1}), \\ Y_2(t_{n+1};r) &= Y_2(t_n;r) + hG^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] + \frac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_{n,2}), \end{aligned} \tag{15}$$

where $\xi_{n,1}, \xi_{n,2} \in (t_n, t_{n+1})$. Denoting

$$W_n = Y_1(t_n;r) - y_1(t_n;r), \quad V_n = Y_2(t_n;r) - y_2(t_n;r),$$

from (14) and (15) it follows that

$$W_{n+1} = W_n + h\{F^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - F^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^{p+1}}{(p+1)!}Y_1^{(p+1)}(\xi_{n,1}),$$

$$V_{n+1} = V_n + h\{G^*[t_n, Y_1(t_n;r), Y_2(t_n;r)] - G^*[t_n, y_1(t_n;r), y_2(t_n;r)]\} + \frac{h^{p+1}}{(p+1)!}Y_2^{(p+1)}(\xi_{n,2}).$$

Hence,

$$\begin{aligned} |W_{n+1}| &\leqslant |W_n| + 2Lh\max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M, \\ |V_{n+1}| &\leqslant |V_n| + 2Lh\max\{|W_n|, |V_n|\} + \frac{h^{p+1}}{(p+1)!}M, \end{aligned}$$

where

$$M_1 = \max|Y_1^{(p+1)}(t;r)|, \quad M_2 = \max|Y_2^{(p+1)}(t;r)| \quad \text{for } t \in [0,T],$$

$M = \max\{M_1, M_2\}$, and $L > 0$ is a bound for the partial derivatives of F^* and G^*. Therefore, from Lemma 4.2, we obtain

$$\begin{aligned} |W_n| &\leqslant (1+4Lh)^n|U_0| + \frac{2h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^n - 1}{4Lh}, \\ |V_n| &\leqslant (1+4Lh)^n|U_0| + \frac{2h^{p+1}}{(p+1)!}M\frac{(1+4Lh)^n - 1}{4Lh}, \end{aligned}$$

where $|U_0| = |W_0| + |V_0|$. In particular,

$$|W_N| \leqslant (1+4Lh)^N |U_0| + \frac{h^{p+1}}{(p+1)!} M \frac{(1+4Lh)^{\frac{T}{h}} - 1}{2Lh},$$
$$|V_N| \leqslant (1+4Lh)^N |U_0| + \frac{h^{p+1}}{(p+1)!} M \frac{(1+4Lh)^{\frac{T}{h}} - 1}{2Lh}.$$

Since $W_0 = V_0 = 0$, we have

$$|W_N| \leqslant M \frac{e^{4LT} - 1}{2L(p+1)!} h^p, \quad |V_N| \leqslant M \frac{e^{4LT} - 1}{2L(p+1)!} h^p.$$

Thus, if $h \to 0$, we get $W_N \to 0$ and $V_N \to 0$, which completes the proof. □

5. Examples

Example 5.1. Consider the fuzzy initial value problem [9]

$$\begin{cases} y'(t) = y(t), \quad t \in I = [0,1], \\ y(0) = (0.75 + 0.25r,\ 1.125 - 0.125r), \quad 0 < r \leqslant 1. \end{cases}$$

Using the Taylor method of order p, we have

$$y_1(t_{n+1}; r) = y_1(t_n; r) \sum_{i=0}^{p} \frac{h^i}{i!}, \quad y_2(t_{n+1}; r) = y_2(t_n; r) \sum_{i=0}^{p} \frac{h^i}{i!}.$$

The exact solution is given by $Y_1(t; r) = y_1(0; r)e^t$, $Y_2(t; r) = y_2(0; r)e^t$, which at $t = 1$ is

$$Y(1; r) = [(0.75 + 0.25r)e,\ (1.125 - 0.125r)e], \quad 0 < r \leqslant 1.$$

The exact and approximate solutions for $p = 2$ and $p = 4$ are compared and plotted at $t = 1$ in Figs. 1 and 2.

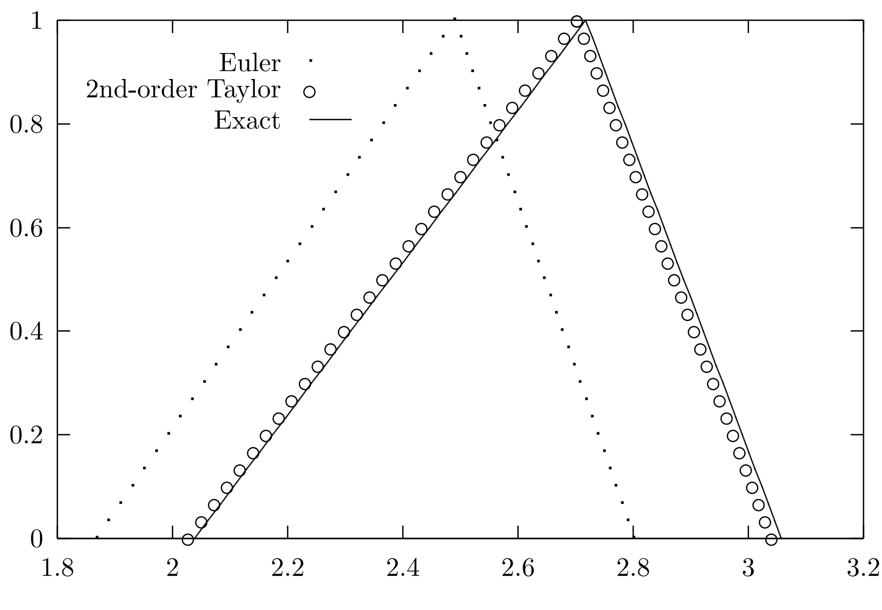

Figure 1. $h = 0.2$

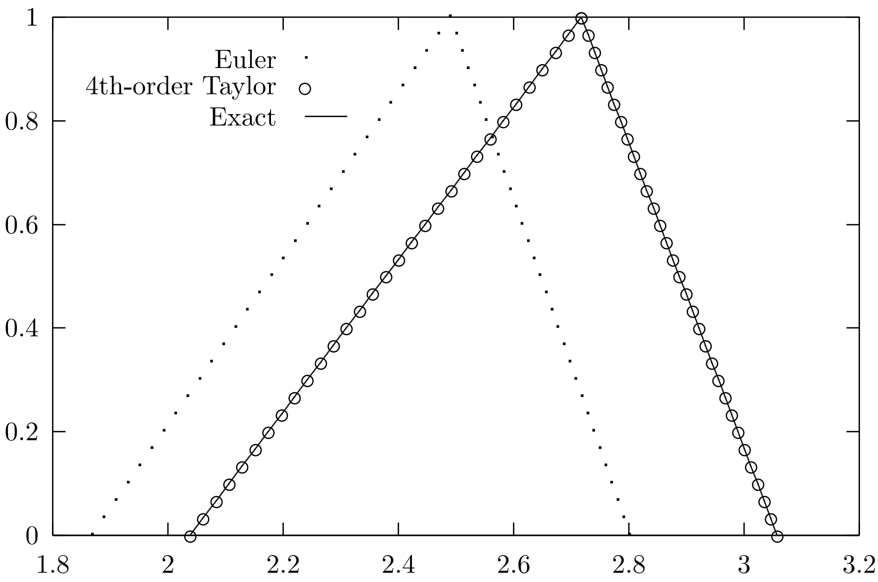

Figure 2. $h = 0.2$

The Hausdorff distances of the exact solution from the Euler solution, the 2nd- and 4th-order Taylor ones are $d_\infty = 0.2587$, $d_\infty = 0.0175$ and $d_\infty = 3.4528e - 005$, respectively.

Example 5.2. Consider the fuzzy initial value problem [9]

$$\begin{cases} y'(t) = ty(t), \quad [a,b] = [-1,1], \\ y(-1) = (\sqrt{e} - .5(1-r),\ \sqrt{e} + .5(1-r)), \quad 0 < r \leqslant 1. \end{cases}$$

We separate the problem into two steps.

(a) $t < 0$. The parametric form in this case is

$$\begin{aligned} y_1'(t;r) &= ty_2(t;r), \quad y_2'(t;r) = ty_1(t;r), \\ y_1''(t;r) &= (1+t^2)y_2(t;r), \quad y_2''(t;r) = (1+t^2)y_1(t;r), \\ y_1^{(3)}(t;r) &= \min\{(1+2t+2t^2+t^4)u |\ u \in [y_1(t;r), y_2(t;r)]\}, \\ y_2^{(3)}(t;r) &= \max\{(1+2t+2t^2+t^4)u |\ u \in [y_1(t;r), y_2(t;r)]\}, \\ y_1^{(4)}(t;r) &= \min\{(2+4t^2+4t(1+t^2)+4t(1+t^2)^2+(1+t^2)^4)u |\ u \in [y_1(t;r), y_2(t;r)]\}, \\ y_2^{(4)}(t;r) &= \max\{(2+4t^2+4t(1+t^2)+4t(1+t^2)^2+(1+t^2)^4)u |\ u \in [y_1(t;r), y_2(t;r)]\}, \end{aligned}$$

with the initial conditions given. The unique exact solution is

$$Y_1(t;r) = \frac{A-B}{2} y_2(0;r) + \frac{A+B}{2} y_1(0;r),$$

$$Y_2(t;r) = \frac{A+B}{2} y_2(0;r) + \frac{A-B}{2} y_1(0;r),$$

where $A = e^{\frac{(t^2-a^2)}{2}}$, $B = \dfrac{1}{A}$.

(b) $t \geqslant 0$. The parametric equations are

$$y_1'(t;r) = ty_1(t;r), \qquad y_2'(t;r) = ty_2(t;r),$$
$$y_1''(t;r) = (1+t^2)y_1(t;r), \quad y_2''(t;r) = (1+t^2)y_2(t;r),$$

and $[y^{(3)}]_r$, $[y^{(4)}]_r$ are calculated as in part (a) with the initial conditions given. The unique exact solution at $t > 0$ is

$$Y_1(t;r) = y_1(0;r)e^{\frac{t^2}{2}}, \quad Y_2(t;r) = y_2(0;r)e^{\frac{t^2}{2}}.$$

The exact and approximate solutions for $p = 2$ and $p = 4$ are compared and plotted in Figs. 3 and 4.

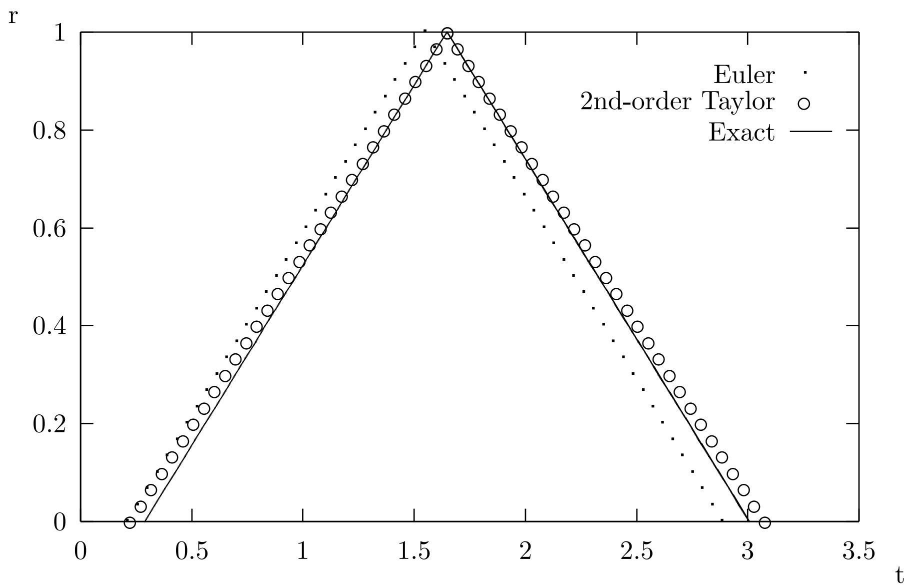

Figure 3. $h = 0.05$

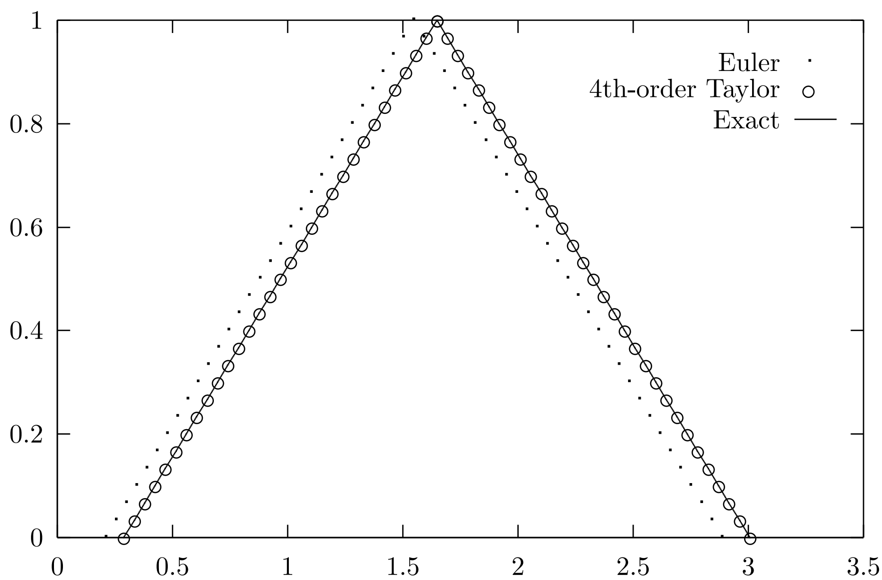

Figure 4. $h = 0.05$

The Hausdorff distances of the exact solution from the Euler solution, the 2nd- and 4th-order Taylor ones are $d_\infty = 0.1284$, $d_\infty = 0.0677$ and $d_\infty = 2.8099e-008$, respectively.

Example 5.3. Consider the fuzzy initial value problem

$$y'(t) = k_1 y^2(t) + k_2, \quad y(0) = 0,$$

where $k_j > 0$ $(j = 1, 2)$ are triangular fuzzy numbers [2].

The exact solution is given by

$$\begin{aligned} Y_1(t;r) &= l_1(r)\tan(w_1(r)t), \\ Y_2(t;r) &= l_2(r)\tan(w_2(r)t), \end{aligned}$$

with

$$l_1(r) = \sqrt{k_{2,1}(r)/k_{1,1}(r)}, \quad l_2(r) = \sqrt{k_{2,2}(r)/k_{1,2}(r)},$$
$$w_1(r) = \sqrt{k_{1,1}(r)k_{2,1}(r)}, \quad w_2(r) = \sqrt{k_{1,2}(r)k_{2,2}(r)},$$

where

$$[k_1]_r = [k_{1,1}(r), k_{1,2}(r)] \text{ and } [k_2]_r = [k_{2,1}(r), k_{2,2}(r)],$$

$$k_{1,1}(r) = .5 + .5r, \quad k_{1,2}(r) = 1.5 - .5r, \quad k_{2,1}(r) = .75 + .25r \quad k_{2,2}(r) = 1.25 - .25r.$$

The r-level sets of $y'(t)$ are

$$\begin{aligned} Y_1'(t;r) &= k_{2,1}(r)\sec^2(w_1(r)t), \\ Y_2'(t;r) &= k_{2,2}(r)\sec^2(w_2(r)t), \end{aligned}$$

which defines the fuzzy number. We have

$$\begin{aligned}
f_1(t,y;r) &= \min\{k_1u^2 + k_2 | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1, 2\}, \\
f_2(t,y;r) &= \max\{k_1u^2 + k_2 | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1, 2\}, \\
f_1'(t,y;r) &= \min\{2k_1^2u^3 + 2uk_1k_2 | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1, 2\}, \\
f_2'(t,y;r) &= \max\{2k_1^2u^3 + 2uk_1k_2 | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \ j = 1, 2\},
\end{aligned}$$

$$\begin{aligned}
f_1^{(i)}(t,y;r) = \min\{&\frac{\partial f^{(i-1)}(t,u)}{\partial u} f^{(i-1)}(t,u) | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \\
&j = 1, 2\}, \\
f_2^{(i)}(t,y;r) = \max\{&\frac{\partial f^{(i-1)}(t,u)}{\partial u} f^{(i-1)}(t,u) | \quad u \in [y_1(t;r), y_2(t;r)], \quad k_j \in [k_{j,1}(r), k_{j,2}(r)], \\
&j = 1, 2\},
\end{aligned}$$

for $i = 1, 2$.

There are two nonlinear programming problems. They can be solved by the GAMS (the acronym stands for General Algebraic Modeling Systems) software. GAMS is designed to make the construction and solution of large and complex mathematical programming models more straightforward for programmers and more comprehensible to users of models from other disciplines, e.g., economists [1].

Thus, the Taylor method suggested in this paper can be used. The exact and approximate solutions are shown in Figs. 5 and 6 at $t = 1$.

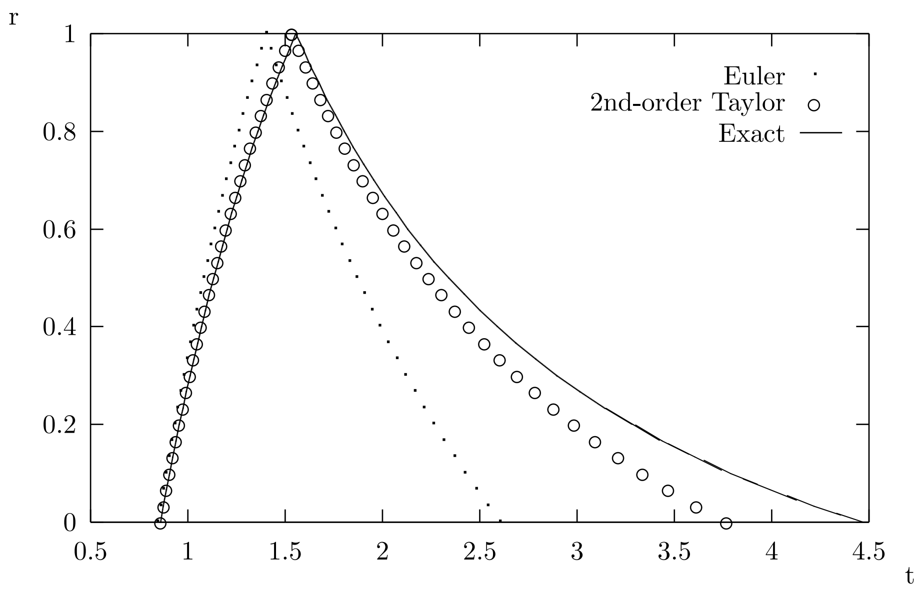

Figure 5. $h = 0.1$

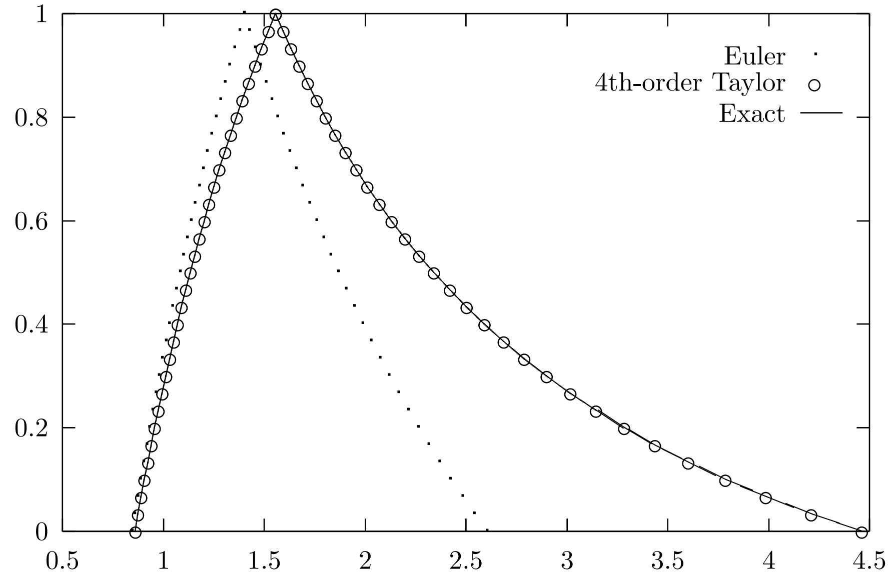

Figure 6. $h = 0.1$

The Hausdorff distances of the exact solution from the Euler solution, the 2nd- and 4th-order Taylor ones are $d_\infty = 1.8711$, $d_\infty = 0.7027$ and $d_\infty = 0.0073$, respectively.

6. Conclusion

It is shown that the convergence order of the method proposed is $O(h^p)$, while the Euler method from [9] converges with the rate $O(h)$ only. In comparison with [9] the solutions of Examples 5.1, 5.2 have the higher accuracy in this paper.

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Received 5 Dec. 2001
Revised 3 Aug. 2002