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FINITE DIFFERENCE SCHEMES FOR PORO-ELASTIC PROBLEMS [1]

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Abstract — In this paper, we present a finite difference analysis of the consolidation problem for saturated porous media. In the classical model, the behaviour of the porous environment – fluid system is described by a set of equations for the unknown vector displacements of the matrix skeleton and the fluid pressure. For simplicity we consider a model problem with constant coefficients in a rectangular domain. A priori estimates for the difference solution of the problem are obtained and on their basis the convergence of two-level difference schemes is investigated.

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Introduction

Filtration and consolidation problems are concerned with the response of a saturated porous material under some modification of the equilibrium conditions. For instance, pumping water from an aquifer produces a reduction of the water pressure and, therefore, a growth in the effective stress in the solid skeleton, which results in a consolidation process. Also, an increase in the loads on the ground surface leads to an increase in the stress, which results in a deformation and time-dependent consolidation process associated with the drainage of the pore fluid [1, 14].

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A simple mathematical model was first proposed in [18] with a one-dimensional dissipation equation for the pore pressure without simultaneous consideration of the strain in the solid matrix. Another important approach is the one proposed by Biot ([2–4]). Here the strain and the flow are considered simultaneously in a three-dimensional domain and the resulting evolutionary problem consists of a set of equations for the unknown vector of displacements of the solid skeleton and the pore pressure of the fluid. Models for the description of the poro-elastic problems under more general conditions were proposed, for example, in [7], where the problems with the nonlinear filtration law and rheological properties are considered.

The existence and uniqueness of the solution for the initial-boundary value problem for the system of equations given by Biot's model have been investigated in several papers [17,22]. The problem of the well-posedness for nonlinear models is considered, for example, in [5].

The numerical solution of the poro-elastic problems by the finite element method is considered for the incompressible Biot's model in several papers, see for example [9, 21]. For a more comprehensive analysis see [11–13]; these papers discuss the error estimates for Euler-Galerkin discretizations using stable and unstable combinations of finite element spaces of displacement and pore pressure fields and the short- and long-time behaviour of such approximations. Finite difference methods for the flow and deformation problems are used in [10], without analyzing the convergence. For the most common nonlinear problems, with incompressible fluids, the difference schemes are investigated in [6].

In this paper, the construction of difference schemes for the numerical solution of the poro-elastic problems for a slightly compressible fluid is considered. For technical simplicity in the finite difference analysis, the proposed model considers only constant coefficient equations in a rectangular domain. A priori estimates (stability estimates) for the semi-discrete and fully discrete difference solution of the problem are obtained using the appropriate Hilbert norms. These estimates and the approximation error bounds of the proposed schemes lead to convergence.

1. Continuous problem

Let us consider a domain $\Omega \subset R^2$ (a two-dimensional skeleton matrix) with a piecewise smooth boundary $\partial\Omega$ and unit outward normal $\mathbf{n}$. We use the notation $\mathbf{x} = (x_1, x_2)$ for the generic point in Ω, $\mathbf{v} = (v_1, v_2)$ for the displacement vector, and p for the pore pressure of the fluid.

We shall consider the elementary consolidation problem for a saturated, homogeneous and isotropic porous material. The classical quasi-static Biot model is based on the assumption of incompressibility for the filtering fluid. Here, we consider a different situation where the fluid is slightly compressible. Under these conditions, the process of filtration and consolidation is described in [1, 17]. Neglecting the body forces, the problem is governed by the set of equations

$$-\mu\tilde{\Delta}\mathbf{v} - (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v} + \operatorname{grad} p = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(a\,p + \operatorname{div}\mathbf{v}) - \chi\Delta p = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad 0 < t \leqslant T\,, \tag{2}$$

where λ and μ are the Lamé coefficients; $a = n_f\beta \neq 0$, with n_f the porosity and β the compressibility coefficient of the fluid; $\chi = k/\mu_f$, being k the permeability of the porous environment, μ_f the viscosity of the fluid and

$$\tilde{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}.$$

The source term $f(\mathbf{x}, t)$ is used, for example, when a forced filtration process is present.

For simplicity, we assume that $\partial\Omega$ is rigid and permeable (free drainage condition) so that we can impose the homogeneous Dirichlet boundary conditions

$$\mathbf{v}(\mathbf{x}, t) = 0\,, \quad p(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

The initial condition can be given by

$$a\,p(\mathbf{x}, 0) + \operatorname{div} \mathbf{v}\,(\mathbf{x}, 0) = \mathbf{s}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{4}$$

that is just the fluid content in the system at the initial time.

To formulate the operator for (1)–(4), we first introduce appropriate functional spaces and operators. Let $\mathcal{H} = L_2(\Omega)$ be the set of square-integrable scalar valued functions defined on Ω, with the scalar product and the corresponding norm

$$(u, v) = \int\limits_{\Omega} u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \quad \|u\| = (u, u)^{1/2}.$$

Let $\mathcal{V}$ denote the usual Sobolev space $H_0^1(\Omega)$ of functions vanishing at the boundary $\partial\Omega$ equipped with the inner product and norm

$$(u, v)_1 = \sum_{\alpha=1}^{2} \int\limits_{\Omega} \frac{\partial u}{\partial x_\alpha} \frac{\partial v}{\partial x_\alpha}\, d\mathbf{x} \quad , \quad \|u\|_1 = (u, u)^{1/2}.$$

For the two-dimensional vector-valued functions $\mathbf{u}$, $\mathbf{v}$ we shall set the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ with the inner product and norm

$$(\mathbf{u}, \mathbf{v}) = (u_1, v_1) + (u_2, v_2), \quad \|\mathbf{u}\| = (\|u_1\|^2 + \|u_2\|^2)^{1/2},$$

and the Hilbert space $\widetilde{\mathcal{V}} = \mathcal{V} \oplus \mathcal{V}$, with

$$(\mathbf{u}, \mathbf{v})_1 = (u_1, v_1)_1 + (u_2, v_2)_1, \quad \|\mathbf{u}\|_1 = (\|u_1\|_1^2 + \|u_2\|_1^2)^{1/2}.$$

In $\widetilde{\mathcal{H}}$ we consider the elasticity operator $\mathcal{A}$ by

$$\mathcal{A}\mathbf{v} = -\mu\widetilde{\Delta}\mathbf{v} - (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v}, \tag{5}$$

with the domain $D(\mathcal{A}) = \{\mathbf{v} \in \widetilde{\mathcal{V}} \,|\, \mathcal{A}\mathbf{v} \in \widetilde{\mathcal{H}}\}$. The operator $\mathcal{A}$ is positive and self-adjoint ($\mathcal{A} > 0$, $\mathcal{A} = \mathcal{A}^*$). Also, the following energetic equivalence holds:

$$-\mu(\tilde{\Delta}\mathbf{v}, \mathbf{v}) \leqslant (\mathcal{A}\mathbf{v}, \mathbf{v}) \leqslant -(\lambda + 2\mu)(\tilde{\Delta}\mathbf{v}, \mathbf{v}). \tag{6}$$

In view of this $\mathcal{A} = \mathcal{A}^* \geqslant \mu\delta_0\widetilde{\mathcal{E}}$, where $\widetilde{\mathcal{E}}$ is the identity operator on $\widetilde{\mathcal{V}}$ and δ_0 is the minimum eigenvalue of $-\Delta$.

Similarly, in $\mathcal{H}$ we define

$$\mathcal{B}p = -\chi \Delta p, \tag{7}$$

with the domain $D(\mathcal{B}) = \{p \in \mathcal{V} \,|\, \mathcal{B}p \in \mathcal{H}\}$. The operator $\mathcal{B}$ is positive and satisfies $\mathcal{B} = \mathcal{B}^* \geqslant \chi\delta_0\mathcal{E}$, where $\mathcal{E}$ now denotes the identity operator on $\mathcal{H}$. In constructing discrete analogs for $\mathcal{A}$ and $\mathcal{B}$, we will be oriented to the fulfillment of the same properties.

Problem (1)–(4) can be written in differential operator form as the abstract initial value problem in $\widetilde{\mathcal{H}} \times \mathcal{H}$

$$\mathcal{A}\mathbf{v} + \operatorname{grad} p = 0, \tag{8}$$

$$\frac{d}{dt}(a\, p + \operatorname{div} \mathbf{v}) + \mathcal{B}p = f(t), \tag{9}$$

$$a\, p(0) + \operatorname{div} \mathbf{v}(0) = s, \tag{10}$$

with $s \in \mathcal{H}$ and $f(t) \in \mathcal{H}$, $\forall\, t \leqslant T$. We assume that problem (8)-(10) has a solution $(\mathbf{v}, p)$, for which $ap(\cdot) + \operatorname{div}\mathbf{v}(\cdot) \in C^0([0,T], \mathcal{H}) \bigcap C^1((0,T], \mathcal{H})$. Moreover in the analysis of the numerical approximation below, we take the classical solution as regular as we need.

We associate to the operators $\mathcal{A}$ and $\mathcal{B}$ the inner products

$$(\mathbf{y}, \mathbf{w})_{\mathcal{A}} = (\mathcal{A}\mathbf{y}, \mathbf{w}), \quad \mathbf{y}, \mathbf{w} \in \widetilde{\mathcal{H}},$$
$$(y, w)_{\mathcal{B}} = (\mathcal{B}y, w), \quad\quad y, w \in \mathcal{H},$$

and the corresponding norms $\|\mathbf{y}\|_{\mathcal{A}}$ and $\|y\|_{\mathcal{B}}$. We use the same association for the self-adjoint and definite positive operator $\mathcal{B}^{-1}$.

By the properties of $\mathcal{A}$ and $\mathcal{B}$ it is easy to prove the following a priori estimate for the solution of (8), (10):

$$\|\mathbf{v}(t)\|_{\mathcal{A}}^2 + a\|p(t)\|^2 \leqslant \|\mathbf{v}(0)\|_{\mathcal{A}}^2 + a\|p(0)\|^2 + \frac{1}{2}\int_0^t \|f(\theta)\|_{\mathcal{B}^{-1}}^2 d\theta, \tag{11}$$

for $t \leqslant T$, which ensures stability with respect to the initial data and the right-hand side.

To obtain estimates for the pressure gradient field, we use the equations

$$(\mathcal{A}u_t, u_t) + (\mathcal{G}p_t, u_t) = 0, \tag{12}$$

$$a(p_t, p_t) + (\mathcal{D}u_t, p_t) + (\mathcal{B}p, p_t) = (f, p_t), \tag{13}$$

from which it follows that

$$\|u_t\|_A^2 + a\|p_t\|^2 + \frac{1}{2}\frac{d}{dt}\|p\|_{\mathcal{B}}^2 \leqslant \frac{1}{2\alpha}\|f\|^2 + \frac{\alpha}{2}\|p_t\|, \quad \forall \alpha \geqslant 0.$$

As $\|u_t\|_{\mathcal{A}} \geqslant 0$, taking $\alpha = 2a$, we have the stability estimate

$$\| \, p(t) \, \|_{\mathcal{B}}^2 \leqslant \| \, p(0) \, \|_{\mathcal{B}}^2 + \frac{1}{4a}\int_0^t \|f(x,\theta)\|^2 d\theta. \tag{14}$$

At the initial stage, the displacements and the pressure satisfy the system of equations

$$-\mu\tilde{\Delta}\mathbf{v}_0 - (\lambda + \mu)\operatorname{grad}\operatorname{div}\mathbf{v}_0 + \operatorname{grad} p_0 = 0, \tag{15}$$

$$a\, p_0 + \operatorname{div}\mathbf{v}_0 = \mathbf{s}(\mathbf{x}), \quad \mathbf{x} \in \Omega, \tag{16}$$

with the boundary condition $\mathbf{v}_0(\mathbf{x}) = 0$ on $\partial\Omega$. The solution of this problem gives the values for the displacements and pressure at $t = 0$ required in estimate (11).

2. Space Discretization

In considering of difference schemes for the solution of problem (1) – (4), we begin with making space approximation. We consider the problem on the rectangle

$$\Omega = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2),\ 0 < x_\alpha < l_\alpha,\ \alpha = 1, 2\}$$

discretized by a uniform rectangular grid with mesh steps $h_\alpha, \alpha = 1, 2$. Let ω be the set of internal nodes of the grid

$$\begin{aligned}\omega =&\{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), \quad x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 1, 2..., N_\alpha - 1,\\ &N_\alpha h_\alpha = l_\alpha, \quad \alpha = 1, 2\},\end{aligned}$$

and $\partial\omega$ – the set of boundary nodes. The finite difference solution of problem (1) - (4) will be denoted by $\mathbf{v}_h(\mathbf{x}, t), p_h(\mathbf{x}, t)\,, \mathbf{x} \in \omega \cup \partial\omega,\ 0 < t \leqslant T$. Using the standard index-free notation of the theory of difference schemes [16], for the right and left difference derivatives we write

$$w_x = \frac{w(x+h) - w(x)}{h}, \quad w_{\bar{x}} = \frac{w(x) - w(x-h)}{h},$$

and the second difference derivative is given by the expression

$$w_{\bar{x}x} = \frac{1}{h}(w_x - w_{\bar{x}}) = \frac{w(x+h) - 2w(x) + w(x-h)}{h^2}.$$

For the grid functions vanishing at $\partial\omega$, we define the Hilbert space $H = L_2(\omega)$, with the scalar product and norm given by

$$(y, w) = \sum_{x \in \omega} y w h_1 h_2, \quad \|y\| = (y, y)^{1/2}.$$

Similarly to the differential case, we introduce $\widetilde{H} = H \oplus H$ for vector-valued functions that are null at $\partial\omega$, with

$$(\mathbf{y}, \mathbf{w}) = (y_1, w_1) + (y_2, w_2), \quad \|\mathbf{y}\| = (\mathbf{y}, \mathbf{y})^{1/2}.$$

Given a self-adjoint and definite positive operator C , H_C denotes the space H provided by the scalar product $(y, w)_C = (Cy, w)$ and norm $\|y\| = (Cy, y)^{1/2}$.

We approximate the differential operator $\mathcal{A}$ by the difference operator

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{17}$$

$$\begin{aligned} A_{11} y &= -\mu \Delta_h y - (\lambda + \mu)\, y_{\bar{x}_1 x_1}, \\ A_{12} y &= A_{21} y = -\frac{\lambda + \mu}{2}(y_{\bar{x}_1 x_2} + y_{x_1 \bar{x}_2}), \\ A_{22} y &= -\mu \Delta_h y - (\lambda + \mu)\, y_{\bar{x}_2 x_2}, \end{aligned}$$

where the usual five-point stencil approximation of the Laplace operator

$$\Delta_h y = y_{\bar{x}_1 x_1} + y_{\bar{x}_2 x_2}$$

is used.

For $\mathbf{y}(\mathbf{x}), \mathbf{w}(\mathbf{x}) \in \widetilde{H}$ we have

$$(A\mathbf{y}, \mathbf{w}) = (\mathbf{y}, A\mathbf{w}),$$

i.e., A is self-adjoint in $\widetilde{H}$. In addition, we have

$$-\mu\tilde{\Delta}_h \leqslant A \leqslant -(\lambda + 2\mu)\tilde{\Delta}_h, \tag{18}$$

where

$$\tilde{\Delta}_h = \begin{pmatrix} \Delta_h & 0 \\ 0 & \Delta_h \end{pmatrix}.$$

Relation (18) is a discrete analog of (6) given for the differential operator $\mathcal{A}$. Due to this $A = A^* \geqslant \mu\delta_h\widetilde{E}$, where $\delta_h > 0$ is the minimum eigenvalue of operator Δ_h and $\widetilde{E}$ is the identity operator in $\widetilde{H}$, i.e. the difference operator A is, like the differential operator $\mathcal{A}$, a self-adjoint and positive definite operator.

To approximate the diffusion operator $\mathcal{B}$, we use

$$By = -\chi\Delta_h y, \tag{19}$$

so that $B = B^* \geqslant \chi\delta_h\, E$ where E is the identity operator in H.

The approximation of the coupling terms deserves separate consideration. For the functions $\mathbf{v} \in \widetilde{\mathcal{H}}$ and $p \in \mathcal{H}$,

$$(\mathbf{v}, \operatorname{grad} p) = -(p, \operatorname{div} \mathbf{v}),$$

i.e. the divergence operator coincides with the opposite conjugate of the gradient operator. Therefore, we will approximate these operators so that the same property takes place at the difference level

$$(\mathbf{v}_h, \operatorname{grad}_h p_h) = -(p_h, \operatorname{div}_h \mathbf{v}_h). \tag{20}$$

Several types of this kind of approximations for the gradient and divergence are widely used in the numerical simulation of viscous incompressible flows (see, for example [8, 15, 19, 20]). We now report the main possibilities to be used when the same grid is used for components of the displacement vector and pressure.

The first variant is connected with one-sided differences. For example, if forward differences are used for the gradient

$$\operatorname{grad}_h y = (y_{x_1}, y_{x_2}), \quad \mathbf{x} \in \omega, \quad \text{for} \quad y \in H,$$

the discrete divergence is

$$\operatorname{div}_h \mathbf{w} = (w_1)_{\bar{x}_1} + (w_2)_{\bar{x}_2}, \quad \mathbf{x} \in \omega, \quad \text{for} \quad \mathbf{w} \in \widetilde{H}.$$

Also, it is possible to use second-order approximations for the gradient and divergence by taking

$$\operatorname{grad}_h y = (y_{\overset{\circ}{x}_1}, y_{\overset{\circ}{x}_2}), \qquad \mathbf{x} \in \omega, \quad \text{for} \quad y \in H,$$
$$\operatorname{div}_h \mathbf{w} = (w_1)_{\overset{\circ}{x}_1} + (w_2)_{\overset{\circ}{x}_2}, \quad \mathbf{x} \in \omega, \quad \text{for} \quad \mathbf{w} \in \widetilde{H}.$$

After the space approximation, from (8), (9) we arrive at the Cauchy problem for the system of differential-difference equations

$$A\mathbf{v}_h + \operatorname{grad}_h p_h = 0, \tag{21}$$

$$\frac{d}{dt}(ap_h + \mathrm{div}_h \mathbf{v}_h) + Bp_h = f_h(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \tag{22}$$

with the initial condition

$$a\, p_h(0) + \mathrm{div}_h\, \mathbf{v}_h(0) = \mathbf{s}_h(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{23}$$

We now construct a simple difference scheme for the approximation of the solution $\{\mathbf{v}_h\,, p_h\}$ of the Cauchy problem for system (21), (22).

3. Fully Discrete Approximation

We use a uniform grid for time discretization with a step-size $\tau > 0$. Let $y^n(\mathbf{x}) = y(\mathbf{x}, t_n)$, where $t_n = n\tau,\ n = 0, 1, \ldots, N,\ N\tau = T$.

We consider the two-level scheme with weight σ

$$A\mathbf{v}_h^{n+1} + \mathrm{grad}_h p_h^{n+1} = 0, \quad n = 0, 1, \ldots, N-1, \tag{24}$$

$$\begin{aligned} a\frac{p_h^{n+1} - p_h^n}{\tau} + \frac{\mathrm{div}_h \mathbf{v}_h^{n+1} - \mathrm{div}_h \mathbf{v}_h^n}{\tau} + B(\sigma p_h^{n+1} + (1-\sigma)p_h^n) \\ = f_h(\mathbf{x}, \sigma t_{n+1} + (1-\sigma)t_n), \quad n = 0, 1, 2, \ldots, N-1, \quad \mathbf{x} \in \omega. \end{aligned} \tag{25}$$

Under standard restrictions for σ, stability of the difference scheme (24), (25) can be established. More precisely, the following result holds.
Proposition 1. *For $\sigma \geqslant 0.5$ the solution of the difference scheme (24), (25) satisfies the a priori estimate*

$$\|\mathbf{v}_h^{n+1}\|_A^2 + a\|p_h^{n+1}\|^2 \leqslant \|\mathbf{v}_h^n\|_A^2 + a\|p_h^n\|^2 + \frac{\tau}{2}\|f_h(\mathbf{x}, \sigma t_{n+1} + (1-\sigma)t_n)\|_{B^{-1}}^2. \tag{26}$$

Proof. Let us denote

$$p_{h,\sigma}^{n+1} = \sigma p_h^{n+1} + (1-\sigma)p_h^n$$

and

$$\mathbf{v}_{h,\sigma}^{n+1} = \sigma \mathbf{v}_h^{n+1} + (1-\sigma)\mathbf{v}_h^n.$$

From (24), (25) we have for $0 \leqslant n < N$

$$\left(A\mathbf{v}_{h,\sigma}^{n+1}, \frac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\tau}\right) + \left(\mathrm{grad}_h p_{h,\sigma}^{n+1}, \frac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\tau}\right) = 0, \tag{27}$$

and

$$\begin{aligned} a\left(\frac{p_h^{n+1} - p_h^n}{\tau}, p_{h,\sigma}^{n+1}\right) + \left(\frac{\mathrm{div}\,\mathbf{v}_h^{n+1} - \mathrm{div}\,\mathbf{v}_h^n}{\tau}, p_{h,\sigma}^{n+1}\right) \\ + (Bp_{h,\sigma}^{n+1}, p_{h,\sigma}^{n+1}) = (f_{h,\sigma}^{n+1}, p_{h,\sigma}^{n+1}). \end{aligned} \tag{28}$$

The addition of (27) and (28) yields

$$\left(A\mathbf{v}_{h,\sigma}^{n+1}, \frac{\mathbf{v}_h^{n+1} - \mathbf{v}_h^n}{\tau}\right) + a\left(\frac{p_h^{n+1} - p_h^n}{\tau}, p_{h,\sigma}^{n+1}\right) + (Bp_{h,\sigma}^{n+1}, p_{h,\sigma}^{n+1}) = (f_{h,\sigma}^{n+1}, p_{h,\sigma}^{n+1})$$

and by

$$(f_{h,\sigma}^{n+1}, p_{h,\sigma}^{n+1}) \leqslant \|p_{h,\sigma}^{n+1}\|_B + \frac{1}{4}\|f_{h,\sigma}^{n+1}\|_{B^{-1}},$$

we obtain

$$(A\mathbf{v}_{h,\sigma}^{n+1}, \mathbf{v}_h^{n+1} - \mathbf{v}_h^n) + a(p_{h,\sigma}^{n+1}, p_h^{n+1} - p_h^n) \leqslant \frac{\tau}{4} \|f_{h,\sigma}^{n+1}\|_{B^{-1}}. \tag{29}$$

Now, using the identity $\sigma\xi + (1-\sigma)\zeta = 1/2(\xi+\zeta) + (\sigma - 1/2)(\xi - \zeta)$ in the expression of $\mathbf{v}_{h,\sigma}^{n+1}$ and of $p_{h,\sigma}^{n+1}$, we have the inequality

$$\begin{aligned} \frac{1}{2}(\|\mathbf{v}_h^{n+1}\|_A^2 - \|\mathbf{v}_h^n\|_A^2) + \frac{a}{2}(\|p_h^{n+1}\|^2 - \|p_h^n\|^2) \\ + \left(\sigma - \frac{1}{2}\right)(\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_A^2 + a\|p_h^{n+1} - p_h^n\|^2) \leqslant \frac{\tau}{2}\|f_{h,\sigma}^{n+1}\|_{B^{-1}}, \end{aligned} \tag{30}$$

and if $\sigma \geqslant 0.5$, estimate (26) follows. □

Estimate (26) provides a difference analogue of (11) for the solution of problem (21), (22).

To derive the error bounds for the difference scheme, we apply the methodology of analyzing the of finite difference methods. We consider the difference equations for the error in the displacements and pressure

$$\begin{aligned} \delta\mathbf{v}_h^n(\mathbf{x}) &= \mathbf{v}_h^n(\mathbf{x}) - \mathbf{v}(\mathbf{x}, t_n), \\ \delta p_h^n(\mathbf{x}) &= p^n(\mathbf{x}) - p(\mathbf{x}, t_n), \quad \mathbf{x} \in \omega, \end{aligned}$$

$$A\delta\mathbf{v}_h^{n+1} + \mathrm{grad}_h \delta p_h^{n+1} = \psi_1^{n+1}(\mathbf{x}), \quad n = 0, 1, \ldots, N-1, \tag{31}$$

$$\begin{aligned} a\frac{\delta p_h^{n+1} - \delta p_h^n}{\tau} + \frac{\mathrm{div}_h \delta\mathbf{v}_h^{n+1} - \mathrm{div}_h \delta\mathbf{v}_h^n}{\tau} \\ + B(\sigma \delta p_h^{n+1} + (1-\sigma)\delta p_h^n) = \psi_2^{n+1}(\mathbf{x}), \quad n = 0, 1, \ldots, N-1, \quad \mathbf{x} \in \omega. \end{aligned} \tag{32}$$

The right-hand side terms in equations (31) and (32) are respectively the approximation errors for equations (1) and (2). For smooth solutions, we have

$$\psi_1^{n+1}(\mathbf{x}) = -A\mathbf{v}(\mathbf{x}, t_{n+1}) - \mathrm{grad}_h p(\mathbf{x}, t_{n+1}) = O(|h|^\alpha) \tag{33}$$

and

$$\begin{aligned} \psi_2^{n+1}(\mathbf{x}) =& f(\mathbf{x}, \sigma t_{n+1} + (1-\sigma)t_n) - a\frac{p(\mathbf{x}, t_{n+1}) - p(\mathbf{x}, t_n)}{\tau} \\ &- \frac{\mathrm{div}_h \mathbf{v}(\mathbf{x}, t_{n+1}) - \mathrm{div}_h \mathbf{v}(\mathbf{x}, t_n)}{\tau} \\ &- B(\sigma p(\mathbf{x}, t_{n+1}) + (1-\sigma)p(\mathbf{x}, t_{n+1})) = O(\tau^\nu + |h|^\alpha), \end{aligned} \tag{34}$$

where $|h| = \sqrt{h_1^2 + h_2^2}$, $\alpha = 1$ if the gradient and divergence operators are approximated by the one-sided differences or $\alpha = 2$ if the central differences are used, and $\nu = 2$ if $\sigma = 0.5$ or $\nu = 1$ if $\sigma \neq 0.5$.

Attempting to apply (25) in order to estimate errors $\delta\mathbf{v}_h^n(\mathbf{x})$ and $\delta p^n(\mathbf{x})$, $1 \leqslant n \leqslant N$, we split the displacement error $\delta\mathbf{v}_h^n(\mathbf{x}) = \mathbf{w}_{1,h}^n(\mathbf{x}) + \mathbf{w}_{2,h}^n(\mathbf{x})$, where $\mathbf{w}_{1,h}^0(\mathbf{x}) = 0$ and $\mathbf{w}_{1,h}^n(\mathbf{x})$, $1 \leqslant n \leqslant N$ is the solution of

$$A\mathbf{w}_{1,h}^n(\mathbf{x}) = \psi_1^n(\mathbf{x}). \tag{35}$$

This part of the error satisfies

$$\|\mathbf{w}_{1,h}^n\|_A \leqslant \|\psi_1^n\|_{A^{-1}}, \quad n = 0, 1, \ldots, N. \tag{36}$$

In this situation, $\mathbf{w}_{2,h}^n(\mathbf{x})$ and $\delta p^n(\mathbf{x})$ are the solutions of problem

$$A\mathbf{w}_{2,h}^{n+1}(\mathbf{x}) + \mathrm{grad}_h \delta p_h^{n+1} = 0, \quad n = 0, 1, \ldots, N-1, \tag{37}$$

$$\begin{aligned} a\frac{\delta p_h^{n+1} - \delta p_h^n}{\tau} + \frac{\mathrm{div}_h \mathbf{w}_{2,h}^{n+1}(\mathbf{x}) - \mathrm{div}_h \mathbf{w}_{2,h}^n(\mathbf{x})}{\tau} + B(\sigma \delta p_h^{n+1} + (1-\sigma)\delta p_h^n) \\ = \psi_2^{n+1}(\mathbf{x}) - \frac{\mathrm{div}_h \mathbf{w}_{1,h}^{n+1}(\mathbf{x}) - \mathrm{div}_h \mathbf{w}_{1,h}^n(\mathbf{x})}{\tau}, \\ n = 0, 1, \ldots, N-1, \quad \mathbf{x} \in \omega, \end{aligned} \tag{38}$$

where the right-hand side is null in equation (37). If $\sigma \geqslant 0.5$, from (26) we obtain

$$\begin{aligned} \|\mathbf{w}_{2,h}^{n+1}\|_A^2 + a\|\delta p_h^{n+1}\|^2 \leqslant \|\mathbf{w}_{2,h}^0\|_A^2 + a\|p_h^0\|^2 \\ + \frac{\tau}{2}\sum_{k=0}^{n}\left(\|\psi_2^{k+1}\|_{B^{-1}}^2 + \left\| \frac{\mathrm{div}_h \mathbf{w}_{1,h}^{k+1} - \mathrm{div}_h \mathbf{w}_{1,h}^k}{\tau} \right\|_{B^{-1}}^2 \right), \\ n = 0, 1, \ldots, N-1, \quad \mathbf{x} \in \omega. \end{aligned} \tag{39}$$

From (34) one can straightforward by obtain $\|\psi_2^{k+1}\|_{B^{-1}} = O(\tau^\nu + |h|^\alpha)$. The estimate for

$$D^{k+1} = \left\| \frac{\mathrm{div}_h \mathbf{w}_{1,h}^{k+1} - \mathrm{div}_h \mathbf{w}_{1,h}^k}{\tau} \right\|_{B^{-1}}$$

is given by the following lemma.

Lemma 1. *With the previous notation we have*

$$D^{n+1} \leqslant \frac{1}{\chi \delta_h}\sqrt{\frac{2}{\mu}} \left\| \frac{\psi_1^{n+1} - \psi_1^n}{\tau} \right\|_{A^{-1}}, \quad n = 0, \ldots N-1 \tag{40}$$

and consequently $D^n = O(\tau^\nu + |h|^\alpha)$, *where* $\alpha = 1$ *if the gradient and divergence operators are approximated by one-sided differences or* $\alpha = 2$ *if the central differences are used and* $\nu = 2$ *if* $\sigma = 0.5$ *or* $\nu = 1$ *if* $\sigma \neq 0.5$.

Proof. Let us denote by $\mathbf{s}_n = (s_n^1, s_n^2)$ the difference derivative $\dfrac{\mathbf{w}_{1,h}^{n+1} - \mathbf{w}_{1,h}^n}{\tau}$. From (19) we have

$$D^{n+1} = \|\mathrm{div}_h \mathbf{s}_n\|_{B^{-1}} \leqslant \frac{1}{\chi\,\delta_h}\|\mathrm{div}_h \mathbf{s}_n\|.$$

Now, taking into account that

$$\|\mathrm{div}_h \mathbf{s}_n\|^2 \leqslant 2\|s_n^1\|_{-\Delta_h}^2 + 2\|s_n^2\|_{-\Delta_h}^2 = 2\|\mathbf{s}_n\|_{-\widetilde{\Delta}_h}^2$$

and using (18), we obtain $\|\mathrm{div}_h \mathbf{s}_n\|^2 \leqslant \dfrac{2}{\mu}\|\mathbf{s}_n\|_A^2$. Then

$$D^{n+1} \leqslant \frac{1}{\chi \delta_{0,h}}\sqrt{\frac{2}{\mu}}\,\|\mathbf{s}_n\|_A\,.$$

By definition,

$$\|\mathbf{s}_n\|_A \;\leqslant \left\| \frac{\psi_1^{n+1} - \psi_1^n}{\tau} \right\|_{A^{-1}}$$

and (40) follows. □

Proposition 2. *Let $\mathbf{v}_h^0(\mathbf{x})$ and $p_h^0(\mathbf{x})$, $O(|h|^\alpha)$be approximations of $\mathbf{v}(\mathbf{x},0)$ and $p(\mathbf{x},0)$. For $\sigma \geqslant 0.5$ the difference solution of the Cauchy problem (21)–(23) converges to smooth enough solutions of problem (1)–(4).More precisely*

$$\| \delta\mathbf{v}_h^n \|_A + a \| \delta p_h^n \| = O(\tau^\nu + h^\alpha)\,,$$

where the parameters ν and α are specified in the previous Lemma .

Proof. Note that

$$\frac{1}{2}\|\delta\mathbf{v}_h^{n+1}\|_A^2 + a\|p_h^{n+1}\|^2 \leqslant \|\mathbf{w}_{1,h}^{n+1}\|_A^2 + \|\mathbf{w}_{2,h}^{n+1}\|_A^2 + a\|\delta p_h^{n+1}\|^2.$$

If $\sigma \geqslant 0.5$, using (36) and (39), we have

$$\frac{1}{2}\|\delta\mathbf{v}_h^{n+1}\|_A^2 + a\|\delta p_h^{n+1}\|^2 \leqslant \|\mathbf{w}_{2,h}^0\|_A^2 + a\|\delta p_h^0\|^2 + \|\psi_1^{n+1}\|_{A^{-1}}^2$$
$$+ \tau\sum_{k=0}^{n}\left(\|\psi_2^{k+1}\|_{B^{-1}}^2 + \frac{2\tau}{\chi^2\delta_h^2\mu}\sum_{k=0}^{n}\left\|\frac{\psi_1^{n+1}-\psi_1^n}{\tau}\right\|_{A^{-1}}^2\right),$$

and now it is sufficient to use the order estimates for the approximation errors ψ_1^n, ψ_2^n and for initial error, to obtain the result. □

From the previous proposition, the convergence properties for the displacements in the A–norm and for the pressure in the discrete L_2–norm follows. We now consider the convergence of the pressure gradients or equivalently convergence of pressure in the B–norm. From (24)–(25) we have

$$\tau\left\|\frac{\mathbf{v}_h^{n+1}-\mathbf{v}_h^n}{\tau}\right\|_A^2 + a\tau\left\|\frac{p_h^{n+1}-p_h^n}{\tau}\right\|^2 + (Bp_{h,\sigma}^{n+1}, p_h^{n+1}-p_h^n) = (f_{h,\sigma}^{n+1}, p_h^{n+1}-p_h^n), \tag{41}$$

which provides the inequality

$$\frac{1}{\tau}\left\|\frac{\mathbf{v}_h^{n+1}-\mathbf{v}_h^n}{\tau}\right\|_A^2 + \frac{a}{\tau}\| p_h^{n+1}-p_h^n \|^2 + \frac{1}{2}(\| p_h^{n+1} \|_B^2 - \| p_h^n \|_B^2)$$
$$+\left(\sigma-\frac{1}{2}\right)\| p_h^{n+1}-p_h^n \|_B^2 \leqslant \frac{1}{2\alpha}\| f_{h,\sigma}^{n+1} \|^2 + \frac{\alpha}{2}\| p_h^{n+1}-p_h^n \|^2\,. \tag{42}$$

If $\sigma \geqslant 0.5$, taking $\alpha = \dfrac{2a}{\tau}$ we have

$$\| p_h^{n+1} \|_B^2 \leqslant \| p_h^n \|_B^2 + \frac{\tau}{2a}\| f_{h,\sigma}^{n+1} \|^2\,.$$

Proposition 3. *Let us assume that the same conditions as in proposition 2 hold. For $\sigma \geqslant 0.5$ we have the estimate*

$$\| \delta p_h^n \|_B = O(\tau^\nu + h^\alpha),$$

where $\nu = 2$ if $\sigma = 0.5$ or $\nu = 1$ if $\sigma > 0.5$ and $\alpha = 1$ if the gradient and divergence operators are approximated by the one-sided differences or $\alpha = 2$ if the central differences are used.

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