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NEWTON TYPE OPERATOR INTERPOLATION FORMULAS BASED ON INTERPOLATION BY MEANS OF RATIONAL FUNCTIONS

MARINA V. IGNATENKO

Belarusian State University
4 Skoriny Ave., 220050 Minsk, Belarus
E-mail: ignatenkomv@bsu.by

LEONID A. YANOVICH

Institute of Mathematics, National Academy of Sciences of Belarus
11 Surganov Str., 220072 Minsk, Belarus
E-mail: yanovich@im.bas-net.by

Abstract — This paper is devoted to the construction of Newton type interpolation formulas for operators differentiable or not in the Gateaux sense based on interpolation by means of rational functions. The formulas obtained are exact for operator polynomials typical for the considered systems of rational functions.

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1. Introduction

One of the techniques for constructing interpolation formulas for operators given in the functional spaces is based on the use of interpolation polynomials for numerical functions. Using such an approach, a family of operator interpolation formulas of the Lagrange, Hermite, and other types has been obtained [2, 5–7].

In this paper, novel Newton type interpolation formulas for operators defined on a set of continuous or differentiable functions have been constructed, using for this purpose interpolation polynomials in rational functions of a special form.

Let $X = X(T)$ be a given space of smooth functions, where $T \subseteq \mathbb{R}$, and $F : X \to Y$ is an operator differentiable in the Gateaux sense, where Y is also a certain functional space. In particular, the operator $F(x)$ can be integral, differential, or of some other type. We shall also consider the nondifferentiable operators $F(x)$.

For such a class of spaces X and Y, it is natural to consider the operator polynomials of degree n of the form

$$\mathcal{P}_{n,j}(x) = a_0(s) + \sum_{k=0}^{n} \int_T a_{jk}(s,t) \frac{d^j}{dt^j} \varphi_k(x(t))dt \quad (j = 0, 1, ...), \tag{1}$$

where $a_0(s)$, $a_{jk}(s,t)$ and $\varphi_k(u)$ are given functions ($s \in \mathbb{R}^m,\ t \in T, u \in \mathbb{R}$), and $x(t) \in X$. As $\varphi_k(u)$, the rational functions given below will be used.

The Newton formula for the operators is written in the conventional manner, as in case of the function interpolation, if one knows how to construct operators of divided differences.

Generally speaking, there is no unique method for constructing operators of divided differences, since their structure largely depends on the form of the required interpolation polynomial. These operators were obtained earlier in [4] for the general linear and many functional spaces, where the interpolation operator polynomial is an analog of the usual algebraic ones. In using other classes of scalar interpolation polynomials, the divided differences for the operators will have a different structure.

In this paper, we propose principles for constructing divided differences for the operator interpolation problem under consideration herein, and, as a consequence, we have obtained corresponding Newton type interpolation formulas.

2. First-order divided differences

We consider the rational functions of the form

$$\varphi_k(t) = (t+c)^{-k} \ (k = 0, 1, 2, ...),$$

where c is a fixed number or a function of s ($s \in \mathbb{R}^m$). In what follows, we assume that the expression $x(t) + c \neq 0$ for $x(t) \in X$ on T, $[0,1] \subseteq T$.

Denote by $\delta F[x;h]$ the Gateaux differential at the point $x = x(t)$ with respect to the direction $h = h(t)$ ($x(t), h(t) \in X$) and by $\chi(\tau, t)$ – the numerical function of the form

$$\chi(\tau,t) = \begin{cases} 1, & \tau \geqslant t; \\ 0, & \tau < t, \end{cases} \tag{2}$$

$$0 < \tau < 1, \quad \chi(0,t) \equiv 0, \quad \chi(1,t) \equiv 1.$$

Lemma 1. *Assume that there exist the integrals*

$$F[x_0,x_1]h = \int_0^1 \delta F[x_0 + \tau(x_1 - x_0); (x_1 + c)h] d\tau, \tag{3}$$

$$F[x_0,x_1]h = \int_0^1 \frac{(x_1(\tau) + c)h(\tau)}{x_1(\tau) - x_0(\tau)} d_\tau F[x_0(\cdot) + \chi(\tau,\cdot)(x_1(\cdot) - x_0(\cdot))]. \tag{4}$$

Then each of them defines the first-order divided difference with respect to the nodes x_0 and x_1 for the operator $F(x)$.

Proof. Operators (3) and (4) are linear in h and vanish for the constant $F(x)$ in X.

At $h(\tau) = \dfrac{x_1(\tau) - x_0(\tau)}{x_1(\tau) + c}$ the integrals defining the operators $F[x_0,x_1]h$ are easy to calculate, and $F[x_0,x_1]h = F(x_1) - F(x_0)$.

For

$$F(x) = \int_T a_{j1}(s,t) \frac{d^j}{dt^j} \frac{1}{x(t)+c} dt \ \ (j = 0, 1, ...), \tag{5}$$

the Gateaux differential in any direction $h(t) \in X$ has the form

$$\delta F[x;h] = -\int_T a_{j1}(s,t) \frac{d^j}{dt^j} \frac{h(t)}{(x(t)+c)^2} dt \ \ (j = 0, 1, ...).$$

Computing integral (3) for the given operator $F(x)$ at $h = \dfrac{x - x_0}{x + c}$, we obtain that the equation $F[x_0, x_1] = F(x) - F(x_0)$ holds.

If $F(x) = \varphi_1(x) = \dfrac{1}{x(t)+c}$, then from (2) it follows that

$$F[x_0(t) + \chi(\tau,t)(x_1(t) - x_0(t))] = \begin{cases} \dfrac{1}{x_1(t)+c}, & \tau > t; \\ \dfrac{1}{x_0(t)+c}, & \tau < t. \end{cases}$$

Therefore, the Stieltjes integral (4) for operator (5), where $T = [0,1]$ and $j = 0$, is computable and also equal to the difference $F(x) - F(x_0)$ at $h = \dfrac{x - x_0}{x + c}$. □

The corresponding first–degree interpolation polynomials are

$$L_1(x) = F(x_0) + \int_0^1 \delta F[x_0 + \tau(x_1 - x_0); \frac{x_1 + c}{x + c}(x - x_0)] d\tau,$$

$$L_1(x) = F(x_0) + \int_0^1 \frac{[x(\tau) - x_0(\tau)][x_1(\tau) + c]}{[x_1(\tau) - x_0(\tau)][x(\tau) + c]} d_\tau F[x_0(\cdot) + \chi(\tau,\cdot)(x_1(\cdot) - x_0(\cdot))].$$

The first of the formulas obtained is exact for the operator polynomials $\mathcal{P}_{1,j}(x)$, where $[0,1] \subseteq T$, $j = 0, 1, ...$, and the second one is exact for $\mathcal{P}_{1,0}(x)$ with $T = [0,1]$.

3. Formulas for divided differences of arbitrary order

We introduce the following notation:

$$h_{n,k} = \frac{(x_k + c)^{n-1}(x_n + c)}{\omega_n'(x_k)}(x_k - x_0),$$

where $\omega_n(x) = (x - x_0)(x - x_1)\dots(x - x_n)$, $x_k = x_k(t)$ are the given interpolation nodes $(k = 0, 1, \dots, n)$.

Theorem 1. *Each of the formulas*

$$F[x_0, x_1, ..., x_n] h_n h_{n-1} ... h_1 = \sum_{k=1}^{n} \int_0^1 \delta F[x_0 + \tau(x_k - x_0); h_{n,k} h_1 h_2 \dots h_n] d\tau \tag{6}$$

and

$$F[x_0, x_1, ..., x_n]h_n h_{n-1}...h_1 =$$

$$= \sum_{k=1}^{n} \int_0^1 \frac{h_{n,k}(\tau)h_1(\tau)h_2(\tau)\ldots h_n(\tau)}{x_k(\tau) - x_0(\tau)} d_\tau F[x_0(\cdot) + \chi(\tau, \cdot)(x_k(\cdot) - x_0(\cdot))] \qquad (7)$$

defines the n-th order divided difference with respect to the nodes $x_0, x_1, ..., x_n$ $(n \geqslant 1)$ *for the operator* $F(x)$ *in* X.

Proof. Let $n = 2$. Then for $h_2 = \dfrac{x_2 - x_1}{x_2 + c}$ formulas (6) and (7) take on respectively, the form

$$\begin{aligned} F[x_0, x_1, x_2]\frac{x_2 - x_1}{x_2 + c}h_1 &= -\int_0^1 \delta F[x_0 + \tau(x_1 - x_0); (x_1 + c)h_1]d\tau \\ &+ \int_0^1 \delta F[x_0 + \tau(x_2 - x_0); (x_2 + c)h_1]d\tau \\ &= F[x_0, x_2]h_1 - F[x_0, x_1]h_1 \end{aligned}$$

and

$$\begin{aligned} F[x_0, x_1, x_2]\frac{x_2 - x_1}{x_2 + c}h_1 &= -\int_0^1 \frac{x_1(\tau) + c}{x_1(\tau) - x_0(\tau)} h_1(\tau) d_\tau F[x_0(\cdot) + \chi(\tau, \cdot)(x_1(\cdot) - x_0(\cdot))] \\ &+ \int_0^1 \frac{x_2(\tau) + c}{x_2(\tau) - x_0(\tau)} h_1(\tau) d_\tau F[x_0(\cdot) + \chi(\tau, \cdot)(x_2(\cdot) - x_0(\cdot))] \\ &= F[x_0, x_2]h_1 - F[x_0, x_1]h_1, \end{aligned}$$

i.e. the main condition for the second-order divided differences is fulfilled.

For $n = 2$ and, further, at $n > 2$, we see whether or not the divided differences (6) and (7) satisfy the conditions that provide exactness of the corresponding interpolation formulas for the polynomials $\mathcal{P}_{n,0}(x)$ with $T = [0, 1]$.

Simple calculations show that $F[x_0, x_1, x_2]h_2h_1$ for the operator $F(x) = \varphi_1(x)$ vanishes $(x_i, h_i \in X)$. At the same time, if $F(x) = \varphi_2(x)$, then

$$F[x_0, x_1, x_2]h_2h_1 = \sum_{k=0}^{2} \frac{(x_k + c)(x_2 + c)}{\omega_2'(x_k)}[F(x_k) - F(x_0)]h_1h_2.$$

Performing the necessary calculations, for $h_2 = \dfrac{x - x_1}{x + c}$, we finally obtain

$$F[x_0, x_1, x_2]\frac{x - x_1}{x + c}h_1 = F[x_0, x]h_1 - F[x_0, x_1]h_1,$$

i.e. the right-hand side of this identity does not depend on the node x_2.

Let $n > 2$. The expression $h_{n,k}h_n$, where $h_n = \dfrac{x_n - x_{n-1}}{x_n + c}$, can be written in the form

$$h_{n,k}h_n = \frac{(x_k + c)^{n-2}(x_k - x_0)}{\omega_n'(x_k)}[(x_k - x_{n-1})(x_n + c) - (x_k - x_n)(x_{n-1} + c)]. \tag{8}$$

Accordingly, for this h_n, taking into account equation (8), formulas (6) and (7) will take on the form

$$F[x_0, x_1, ..., x_n]\frac{x_n - x_{n-1}}{x_n + c}h_{n-1}...h_2h_1 =$$
$$= F[x_0, x_1, ..., x_{n-2}, x_n]h_{n-1}...h_2h_1 - F[x_0, x_1, ..., x_{n-2}, x_{n-1}]h_{n-1}...h_2h_1.$$

Thus, the main requirement of the divided differences for $F[x_0, x_1, ..., x_n]h_nh_{n-1}...h_1$ defined by (6) and (7) is also satisfied for $n > 2$.

Let us show that for the operators $F_\nu(x) = \varphi_\nu(x)$ the following equality holds for any $h_i \in X$:

$$F_\nu[x_0, x_1, ..., x_n]h_nh_{n-1}...h_1 = 0 \quad (\nu = 0, 1, \dots, n-1). \tag{9}$$

Indeed, computing the integrals of (6), (7) for $F(x) = F_\nu(x)$, we obtain

$$F_\nu[x_0, x_1, ..., x_n]h_nh_{n-1}...h_1 = h_1h_2\dots h_n \sum_{k=0}^{n} \frac{(x_k + c)^{n-1}(x_n + c)}{\omega_n'(x_k)}[F_\nu(x_k) - F_\nu(x_0)].$$

Then, taking into account the relation $(x_n + c)(x - x_k) = (x_n - x_k)(x + c) - (x_n - x)(x_k + c)$, we can represent the latter sum in the form of the difference

$$\sum_{k=0}^{n} \frac{(x_k + c)^{n-1}(x_n - x_k)(x + c)}{(x - x_k)\omega_n'(x_k)}[F_\nu(x_k) - F_\nu(x_0)] - $$
$$- \sum_{k=0}^{n} \frac{(x_k + c)^{n}(x_n - x)}{(x - x_k)\omega_n'(x_k)}[F_\nu(x_k) - F_\nu(x_0)]. \tag{10}$$

From Lagrange's interpolation formula it follows that

$$\frac{x + c}{\omega_n(x)} \sum_{k=0}^{n} \frac{\omega_n(x)(x_k + c)^{n-1}(x_n - x_k)}{(x - x_k)\omega_n'(x_k)} = \frac{(x + c)^n(x_n - x)}{\omega_n(x)}$$

and

$$\frac{x_n - x}{\omega_n(x)} \sum_{k=0}^{n} \frac{\omega_n(x)(x_k + c)^n}{(x - x_k)\omega_n'(x_k)} = \frac{(x_n - x)(x + c)^n}{\omega_n(x)},$$

from which we conclude that the coefficient at $F_\nu(x_0)$ in expression (10) will vanish for any ν.

Analogously, for $0 \leqslant \nu \leqslant n-1$, we have

$$(x + c)\sum_{k=0}^{n} \frac{(x_k + c)^{n-1}(x_n - x_k)}{(x - x_k)\omega_n'(x_k)}F_\nu(x_k) = \frac{(x + c)}{\omega_n(x)} \sum_{k=0}^{n} \frac{\omega_n(x)(x_k + c)^{n-1-\nu}(x_n - x_k)}{(x - x_k)\omega_n'(x_k)}$$
$$= \frac{(x + c)^{n-\nu}(x_n - x)}{\omega_n(x)}.$$

The second sum of (10) with co-factors $F_\nu(x_k)$ is equal to the same expression, namely,

$$\sum_{k=0}^{n} \frac{(x_k+c)^n(x_n-x)}{(x-x_k)\omega_n'(x_k)} F_\nu(x_k) = \frac{(x_n-x)(x+c)^{n-\nu}}{\omega_n(x)}.$$

From this it follows that $F_\nu[x_0,x_1,...,x_n]h_n h_{n-1}...h_1 = 0$ for $\nu = 0,1,\ldots,n-1$.

At $\nu = n$ and $h_n = \dfrac{x-x_{n-1}}{x+c}$ for $F(x) = F_n(x)$ the equality

$$\begin{aligned} F[x_0,x_1,...,x_n]\frac{x-x_{n-1}}{x+c}h_{n-1}h_{n-2}...h_1 &= \\ &= \{F[x_0,x_1,...,x_{n-2},x] - F[x_0,x_1,...,x_{n-1}]\}h_{n-1}...h_2h_1, \end{aligned} \tag{11}$$

holds true. Let us prove it. To this end, we make use of the equality

$$\frac{\omega_n(x)}{(x+c)^n} \sum_{k=0}^{n} \frac{(x_k+c)^n f(x)}{(x-x_k)\omega_n'(x_k)} \equiv f(x),$$

where $f(x) = F_\nu(x)$ ($\nu = 0,1,\ldots,n$). From this equality, which is well-known from the theory of interpolation in the system of rational functions being considered, and taking into account the sign, the second term in (10) reduces to the form

$$\frac{(x+c)^n}{\omega_{n-1}(x)}[F(x) - F(x_0)], \tag{12}$$

for $F_n(x) \equiv F(x)$.

In the first sum of equality (10), the term at $k = n$ vanishes, and the term at $k = n-1$, which we consider separately, is

$$\frac{(x_{n-1}+c)^{n-1}(x+c)}{(x_{n-1}-x)\omega_{n-1}'(x_{n-1})}[F(x_{n-1}) - F(x_0)]. \tag{13}$$

Using the relation

$$\begin{aligned} &\frac{(x_k+c)^{n-1}(x_n-x_k)(x+c)}{(x-x_k)\omega_n'(x_k)} = \\ &= \frac{(x_k+c)^{n-2}(x_k-x_{n-1})(x+c)}{(x-x_{n-1})\omega_{n-1}'(x_k)}\left[\frac{x+c}{x_k-x} - \frac{x_{n-1}+c}{x_k-x_{n-1}}\right] \quad (k = 0,1,\ldots,n-2), \end{aligned}$$

we represent the rest of the first sum in (10) in the form of the difference

$$\begin{aligned} &\sum_{k=0}^{n-2} \frac{(x_k+c)^{n-2}(x+c)^2}{(x-x_{n-1})(x_k-x)\omega_{n-2}'(x_k)}[F(x_k) - F(x_0)] - \\ &- \sum_{k=0}^{n-2} \frac{(x_k+c)^{n-2}(x+c)(x_{n-1}+c)}{(x-x_{n-1})\omega_{n-1}'(x_k)}[F(x_k) - F(x_0)]. \end{aligned} \tag{14}$$

Adding term (12) to the first sum in (14) and term (13) to the second one and multiplying the obtained identity by $\dfrac{x-x_{n-1}}{x+c}h_{n-1}h_{n-2}\ldots h_1$, we obtain relation (11). □

The recurrent equalities (9) and (11) to be satisfied for $F[x_0, x_1, ..., x_n]h_n h_{n-1}...h_1$, are called additional conditions imposed on the operators of divided differences. These conditions provide exactness of the corresponding interpolation formulas for the operator polynomials $\mathcal{P}_{n,0}(x)$ of the form of (1), with $T = [0, 1]$.

Note that for the operators of divided differences given by (6) the additional conditions are also satisfied at $F(x) = \mathcal{P}_{n,j}(x)$, where $[0, 1] \subset T$ and $j > 0$. We give proof only for the case of $n = 2$.

Indeed, in the case of $F(x) = \int_T a_{j1}(s,t)\frac{d^j}{dt^j}\varphi_1(x(t))dt$, $j > 0$, for the second-order divided difference $F[x_0, x_1, x_2]h_2h_1$, we obtain

$$F[x_0, x_1, x_2]h_2h_1 = \int_T a_{j1}(s,t)\frac{d^j}{dt^j}\left[h_1(t)h_2(t)y_1(t)\right]dt, \quad y_1 = \sum_{k=1}^{2}\frac{h_{2,k}}{x_k - x_0}\left[\frac{1}{x_k + c} - \frac{1}{x_0 + c}\right].$$

Since the expression $y_1 \equiv y_1(t)$ is identical to zero, we obtain equality (9) for $n = 2$.

On the other hand, if the operator $F(x) = \int_T a_{j2}(s,t)\frac{d^j}{dt^j}\varphi_2(x(t))dt$, $j > 0$, then the second-order divided difference has the form

$$F[x_0, x_1, x_2]h_2h_1 = \int_T a_{j2}(s,t)\frac{d^j}{dt^j}\left[h_1(t)h_2(t)y_2(t)\right]dt,$$

where

$$y_2 = \sum_{k=1}^{2}\frac{(x_k + c)(x_2 + c)}{\omega_2'(x_k)}\left[\frac{1}{(x_k + c)^2} - \frac{1}{(x_0 + c)^2}\right].$$

Taking into account the equality

$$F[x_0, x_1]h = \int_T a_{j2}(s,t)\frac{d^j}{dt^j}\left\{\frac{h(t)(x_1(t) + c)}{x_1(t) - x_0(t)}\left[\frac{1}{(x_1(t) + c)^2} - \frac{1}{(x_0(t) + c)^2}\right]\right\}dt,$$

fulfilled for the operators $F(x)$ of this class, in the case of $h_2 = \dfrac{x - x_1}{x + c}$, we finally have

$$F[x_0, x_1, x_2]\frac{x - x_1}{x + c}h_1 = F[x_0, x]h_1 - F[x_0, x_1]h_1\ ,$$

which coincides with (11) at $n = 2$. The validity of the additional conditions (9) and (11) for polynomials (1) is proved in the same manner for $n > 2$.

Thus, Newton's interpolation formula

$$L_n(x) = F(x_0) + \sum_{k=1}^{n} F[x_0, x_1, \dots, x_k]\frac{x - x_{k-1}}{x + c}\cdots\frac{x - x_1}{x + c}\,\frac{x - x_0}{x + c}, \tag{15}$$

constructed on the basis of the divided differences $F[x_0, x_1, ..., x_k]h_k h_{k-1}...h_1$ given by equalities (6) and (7), will assume the following form, respectively:

$$L_n(x) = F(x_0) + \sum_{k=1}^{n}\sum_{\nu=1}^{k}\int_0^1 \delta F\left[x_0 + \tau(x_\nu - x_0);\ h_{k,\nu}\frac{\omega_{k-1}(x)}{(x + c)^k}\right]d\tau \tag{16}$$

and

$$L_n(x) = F(x_0)+\sum_{k=1}^{n}\sum_{\nu=1}^{k}\int_0^1 \frac{h_{k,\nu}(\tau)\omega_{k-1}(x(\tau))}{[x_\nu(\tau)-x_0(\tau)][x(\tau)+c]^k} d_\tau F[x_0(\cdot)+\chi(\tau,\cdot)(x_\nu(\cdot)-x_0(\cdot))], \quad (17)$$

where, as before, $\omega_{k-1}(x) = (x-x_0)(x-x_1)\dots(x-x_{k-1})$.

The interpolation formula (16) is exact for the operator polynomials $\mathcal{P}_{n,j}(x)$, where $[0,1] \subseteq T$, $j = 0, 1, \dots$, and formula (17) is exact for $\mathcal{P}_{n,0}(x)$, with $T = [0,1]$.

In particular, in the case of the quadratic interpolation we have

$$L_2(x) = L_1(x) + \int_0^1 \delta F\left[x_0+\tau(x_1-x_0);\ \frac{(x_1+c)(x_2+c)}{x_1-x_2}\frac{x-x_0}{x+c}\frac{x-x_1}{x+c}\right] d\tau + \\ + \int_0^1 \delta F\left[x_0+\tau(x_2-x_0);\ \frac{(x_2+c)^2}{x_2-x_1}\frac{x-x_0}{x+c}\frac{x-x_1}{x+c}\right] d\tau$$

and

$$L_2(x) = L_1(x) + \\ + \int_0^1 \frac{[x_1(\tau)+c][x_2(\tau)+c][x(\tau)-x_0(\tau)][x(\tau)-x_1(\tau)]}{[x_1(\tau)-x_0(\tau)][x_1(\tau)-x_2(\tau)][x(\tau)+c]^2} d_\tau F[x_0(\cdot)+\chi(\tau,\cdot)(x_1(\cdot)-x_0(\cdot))] \\ + \int_0^1 \frac{[x_2(\tau)+c]^2[x(\tau)-x_0(\tau)][x(\tau)-x_1(\tau)]}{[x_2(\tau)-x_0(\tau)][x_2(\tau)-x_1(\tau)][x(\tau)+c]^2} d_\tau F[x_0(\cdot)+\chi(\tau,\cdot)(x_2(\cdot)-x_0(\cdot))].$$

Note that formula (16) obtained with the use of the divided differences (6) is another form of writing of Lagrange's interpolation formula (1.6) (see [2, p. 285]). The operators of the divided differences (6) and (7) constructed here allow one to write down the truncation errors of the corresponding interpolation formulas.

Thus, for the error $r_n(x) = F(x) - L_n(x)$, where $L_n(x)$ is the interpolation polynomial of the form of (16) or (17), the following equalities take place, respectively:

$$r_n(x) = F[x_0, x_1, \dots, x_{n+1}]\frac{x-x_n}{x+c}\dots\frac{x-x_1}{x+c}\frac{x-x_0}{x+c} \\ = \sum_{k=1}^{n+1}\int_0^1 \delta F\left[x_0+\tau(x_k-x_0);\ h_{n+1,k}\frac{\omega_n(x)}{(x+c)^{n+1}}\right] d\tau \quad (18)$$

and

$$r_n(x) = \sum_{k=1}^{n+1}\int_0^1 \frac{h_{n+1,k}(\tau)\omega_n(x(\tau))}{[x_k(\tau)-x_0(\tau)][x(\tau)+c]^{n+1}} d_\tau F[x_0(\cdot)+\chi(\tau,\cdot)(x_k(\cdot)-x_0(\cdot))], \quad (19)$$

where $x_{n+1} = x$.

Further, note that formulas (16) and (17) use the values of the interpolating operator F from a continual manifold of functions. For (16), such functions are the fragmentons

$x^\nu(t,\xi) = x_0(t)+\xi(x_\nu(t)-x_0(t))$, and for (17) – the functions $x^\nu(t,\xi) = x_0(t)+\chi(\xi,t)(x_\nu(t)-x_0(t))$, $0 \leqslant \xi \leqslant 1$, $t \in T$, $\nu = 0,1,\dots,n$. It is therefore desirable that the interpolation conditions should also be met at the points $x(t) = x^\nu(t,\xi)$. The functional polynomials satisfying this property have been established in [1].

Formula (17) is an interpolation formula in the above sense. Indeed, when $x(\tau) = x^\nu(\tau,\xi)$, the factor $\omega_n(x(\tau))$ of error (19) transforms to the form

$$\omega_n(x^\nu(\tau,\xi)) = \chi(\xi,\tau)(\chi(\xi,\tau)-1)(x_\nu(\tau)-x_0(\tau))^2 \prod_{k=1,k\neq\nu}^{n} [x^\nu(\tau,\xi) - x_k(\tau)].$$

Since $\chi(\xi,\tau)(\chi(\xi,\tau)-1) = 0$ for any $\xi \in [0,1]$, the identity $r_n(x^\nu(\tau,\xi)) = F(x^\nu(\tau,\xi)) - L_n(x^\nu(\tau,\xi)) \equiv 0$ holds.

Note that the interpolation polynomial (16) has no analogous property. On the other hand, if one uses the functions $x^\nu(t,\xi) = x_0(t) + \xi(x_\nu(t) - x_0(t)) \in X$ $(0 \leqslant \xi \leqslant 1)$ as the nodes $x_\nu(t)$, then the conditions $L_n(x^\nu(\cdot,\xi)) = F(x^\nu(\cdot,\xi))$ $(\nu = 0,1,...,n)$ are fulfilled. In this case, the Gateaux differential of the interpolated operator of (16) should be defined on a set of the functions $x_0(t) + \tau\xi(x_\nu(t) - x_0(t))$, $0 \leqslant \nu \leqslant n$ having the same structure.

Example 1. Let $F(x) = \exp\{\alpha \int_0^1 f[\beta x(t)]\,dt\}$, where α and β are fixed numbers. Then $\delta F[x;h] = \alpha\beta F(x) \int_0^1 f'[\beta x(t)]h(t)\,dt$ and, consequently,

$$L_n(x) = F(x_0)+$$

$$+\alpha\beta \sum_{k=1}^{n}\sum_{\nu=1}^{k} \int_0^1\int_0^1 F[x_0+\tau(x_\nu-x_0)]f'[\beta(x_0(t)+\tau(x_\nu(t)-x_0(t))]h_{k,\nu}(t)\frac{\omega_{k-1}(x(t))}{(x(t)+c)^k}\,d\tau dt,$$

or, in a somewhat different form

$$L_n(x) = L_{n-1}(x)+$$

$$+\alpha\beta \sum_{k=1}^{n} \int_0^1\int_0^1 F[x_0+\tau(x_k-x_0)]f'[\beta(x_0(t)+\tau(x_k(t)-x_0(t))]h_{n,k}(t)\frac{\omega_{n-1}(x(t))}{(x(t)+c)^n}\,d\tau dt.$$

Example 2. Consider the operator of the form $F(x) = \int_0^1 f[\beta x(t)]\,dt$, where $f(u)$ is a function differentiable in $\mathbb{R}$. In this case, $\delta F[x;h] = \beta \int_0^1 f'[\beta x(t)]h(t)\,dt$ and, accordingly,

$$L_n(x) = F(x_0) + \sum_{k=1}^{n}\sum_{\nu=1}^{k} \int_0^1 \frac{h_{k,\nu}(t)\omega_{k-1}(x(t))}{(x_\nu(t)-x_0(t))(x(t)+c)^k}[f(\beta x_\nu(t) - f(\beta x_0(t))]\,dt.$$

In particular, if $f(u) = \sum_{k=0}^{n} b_k(\frac{u}{\beta}+c)^{-k}$, where b_k is any arbitrary real or complex number, then $L_n(x) \equiv \int_0^1 f(\beta x(t))\,dt$.

In conclusion, note that operator interpolation, as one of the methods of operator approximation, can be used in many problems of approximate numerical analysis. In particular, it find wide use in identifying complex nonlinear phenomena [3].

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