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ON THE STABILITY OF DIFFERENTIAL-OPERATOR EQUATIONS AND OPERATOR-DIFFERENCE SCHEMES AS $t \to \infty$

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Abstract — For the abstract Cauchy problem for a parabolic equation *a priori* estimates of the global and asymptotic stability in various energy norms have been obtained. Similar problems are also considered for the second-order equation. In the latter case, *a priori* estimates of the asymptotic stability by the initial data have been obtained. The corresponding estimates of the global stability for three-level operator difference schemes have been proved. Estimates of the asymptotic behavior of the solution for quasi-linear multidimensional equations with unbounded nonlinearity have been obtained. The corresponding mathematical apparatus permitting one to prove unconditional monotonicity of the difference schemes approximating nonlinear problems is presented.

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1. Introduction

The notion of stability is a component of the correctness of the differential problem and in the general case it points to a continuous dependence of the solution u on input data of the problem φ, i.e., there exists such a value of $\rho > 0$ independent of the solution and input data that for all φ, $\tilde{\varphi}$ from a certain allowable admissible set the following estimate holds:

$$\|\tilde{u} - u\|_1 \leqslant \rho(t)\|\tilde{\varphi} - \varphi\|_2, \tag{1.1}$$

where $\|\cdot\|_1$, $\|\cdot\|_2$ are certain norms and $\tilde{u}$ is the solution of the same differential problem with perturbed input data $\tilde{\varphi}$. The problem of stability becomes particularly urgent in the mathematical modeling of applied problems where input data can be given roughly (as a result of experimental measurements, observations etc).

As a constant ρ one most commonly chooses the following magnitudes:

$$\rho = e^{-Ct}, \quad C = \text{const} > 0, \tag{1.2}$$

$$\rho = 1, \tag{1.3}$$

$$\rho = e^{Ct}. \tag{1.4}$$

Further we shall use the following definition of global and asymptotic stability [17].

Definition 1.1. *Assume that for the solution of the differential or difference problem estimate* (1.1) *holds. This problem is called* stable for a long time *if* $\rho(t) \to \text{const}$ *when* $t \to +\infty$*, and* asymptotically stable *if* $\rho(t) \to 0$ *when* $t \to +\infty$.

It should be noted that in monograph [14] a more rigorous notion definition of asymptotic stability, where $\rho(t)$ for the difference problem approximates the corresponding magnitude for the differential problem with a good accuracy. In what follows we shall use a weakened notion of asymptotic stability given in Definition 1.1.

In the case of linear operator equations, estimate (1.1) is equivalent to the inequality

$$\|u\| \leqslant \rho\|\varphi\|. \tag{1.5}$$

Extensive literature is devoted to the obtaining of *a priori* estimates for linear evolutionary differential-operator equations [5, 7, 8, 12, 19]. Note that in the present paper we do not discuss the problem of stability of the solution to the perturbation of unbounded operator coefficients which also pertain to input data. This questions are considered in detail in [2–4, 9, 20, 21].

While for the first-order differential-operator equations the asymptotic behavior of the solution is a normal property, for the second-order equations such as second-order hyperbolic equations, a wave equation, etc. the asymptotic behavior of the solution is not observed. At present for such a kind of one frequently speaks of global stability [10], i.e. when at $t \to \infty$ the corresponding energy integral (norm of the solution in the Sobolev space) preserves its value or does not increase relative to the energy integral at the initial time. Note that monograph [18], for example, presents *a priori* estimates for the second-order hyperbolic equation that are not the estimates of the global stability even in the case of $\varphi(t) = \text{const}$. Therefore, it is tempting to obtain *a priori* estimates of the global (and for the second-order hyperbolic equations with dissipation, asymptotic) stability not only by the initial data but, in integral norms, by the right-hand side in the case of variable nonincreasing operator coefficients as well.

As for the study of the asymptotic properties of the solution for nonlinear problems, one should note in the first place, monographs [7,15]. Among the nonlinear problems for the first-order equations, the problems with *unbounded nonlinearity* are particularly notable [6,11,13]. Unlike the problems with bounded nonlinearity, they are characterized by the fact that certain properties (positiveness, boundedness) imposed on coefficients independent of the solution u are fulfilled not for all values of u but only for the range of values of the exact solution [1]. Firstly, this fact strongly narrows the class of allowable coefficients and, secondly, makes the statements of the problem approach actual applied problems. For example, for the heat conductivity coefficient in the form

$$k(x,t,u) = u - c_0, \quad c_0 = \text{const} > 0,$$

the necessary positivity condition $k > 0$ is met true only at $u > c_0$. For the difference problems approximating the differential problems with unbounded nonlinearity, the appropriate investigations become much more complicated. This is explained by the following fact. In order to prove the fulfillment of a corresponding condition (positiveness, boundedness) for the coefficients of the difference problem, it is necessary to prove that the approximate solution belong to the range of values of the exact solution, which leads to the obtaining of *a priori* information about the approximate solution in the C-norm.

In this paper, *a priori* estimates of global and asymptotic stability for the Cauchy problem for a parabolic equation have been obtained in different energy norms. The results obtained are also generalized to quasilinear multidimensional parabolic equations with unbounded nonlinearity. A mathematical apparatus that makes it possible to prove the unconditional monotonicity of the difference schemes approximating the nonlinear problems is presented. Investigations of the stability for the Cauchy problem for a abstract second-order hyperbolic equation are also carried out. In the latter case, the *a priori* estimates of the asymptotic stability at certain relations on the operators of the problem acting in Hilbert spaces have been obtained. Estimates of the global stability for three-level operator-difference schemes have been obtained. These estimates are in agreement with the corresponding estimates for the differential problem.

2. First-order equation

2.1. Asymptotic stability of the differential-operator equation

Let us assume that $\mathcal{H}$ is a real separable Hilbert space with an inner product $(\cdot,\cdot)$ and a norm $\|\cdot\|$ and $\mathcal{A}$ is an unbounded self-adjoint positively defined linear operator with the domain of definition $D(\mathcal{A})$ dense in $\mathcal{H}$. The expression $(u,v)_{\mathcal{A}} = (\mathcal{A}u, v)$, $u, v \in D(\mathcal{A})$, satisfies the axioms of the inner product. Supplementing $D(\mathcal{A})$ by the norm $\|u\|_{\mathcal{A}} = (u,\, u)_{\mathcal{A}}^{1/2}$, we obtain the so-called energy space $\mathcal{H}_{\mathcal{A}} \subset \mathcal{H}$. The relation $(u,\, v)_{\mathcal{A}^{-1}} = (\mathcal{A}^{-1}u,\, v)$, $u, v \in \mathcal{H}$, also satisfies the axioms of the inner product. Supplementing the space $\mathcal{H}$ by the norm $\|u\|_{\mathcal{A}^{-1}} = (u,\, u)_{\mathcal{A}^{-1}}^{1/2}$, we obtain the space $\mathcal{H}_{\mathcal{A}^{-1}} \supset \mathcal{H}$. In this case, $\mathcal{H}_{\mathcal{A}^{-1}} = \mathcal{H}_{\mathcal{A}}^*$ is an adjoint space for $\mathcal{H}_{\mathcal{A}}$, the inner product $(u,\, v)$ can be continuously extended on $\mathcal{H}_{\mathcal{A}^{-1}} \times \mathcal{H}_{\mathcal{A}}$, and the operator $\mathcal{A}$ can be extended to the mapping $\mathcal{A} : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_{\mathcal{A}^{-1}}$. There exists an unbounded self-adjoint positively defined linear operator $\mathcal{A}^{1/2}$ [8, 12]. In this case, $D(\mathcal{A}^{1/2}) = \mathcal{H}_{\mathcal{A}}$ and $(u,\, v)_{\mathcal{A}} = (\mathcal{A}u,\, v) = (\mathcal{A}^{1/2}u,\, \mathcal{A}^{1/2}v)$. The spaces $V_1 = \mathcal{H}_{\mathcal{A}}$, $\mathcal{H}$, and $V_1^* = \mathcal{H}_{\mathcal{A}^{-1}}$ represent a Gelfand triple: $V_1 \subset H \subset V_1^*$. Likewise, we define the spaces $V_2 = \mathcal{H}_{\mathcal{A}^2}$, $V_2^* = \mathcal{H}_{\mathcal{A}^{-2}}$, $V_3 = \mathcal{H}_{\mathcal{A}^3}$, $V_3^* = \mathcal{H}_{\mathcal{A}^{-3}}$ etc. In addition $\|u\|_{\mathcal{A}^2} = \|\mathcal{A}u\|$, $\|u\|_{\mathcal{A}^{-2}} = \|\mathcal{A}^{-1}u\|$, $\|u\|_{\mathcal{A}^3} = \|\mathcal{A}u\|_{\mathcal{A}}$, $\|u\|_{\mathcal{A}^{-3}} = \|\mathcal{A}^{-1}u\|_{\mathcal{A}^{-1}}$ etc and $\cdots \subset V_3 \subset V_2 \subset V_1 \subset H \subset V_1^* \subset V_2^* \subset V_3^* \subset \ldots$. It will be recalled that if we consider the operator $\mathcal{A}$ as the map $\mathcal{A} : V_1 \to V_1^*$, $\mathcal{A} : V_2 \to \mathcal{H}$ or $\mathcal{A} : H \to V_2^*$ then it naturally becomes bounded.

Let us also introduce Lebesgue space $L_2(0,T;\, \mathcal{H})$ of the functions $u = u(t)$ that take the segment $(0,T) \subset R$ to $\mathcal{H}$ [8, 19]. Let us assume that the inner product and the norm in

$L_2(0,T;\,\mathcal{H})$ are defined by the following relations:

$$(u,\,v)_{L_2(0,T;\,\mathcal{H})} = \int_0^T (u(t),\,v(t))\,dt\,, \qquad \|u\|_{L_2(0,T;\,\mathcal{H})} = (u,\,u)^{1/2}_{L_2(0,T;\,\mathcal{H})}.$$

Let us also introduce the Sobolev spaces $W_2^s(0,T;\,\mathcal{H})$ [8; Vol. 1, p. 52].

Consider the abstract Cauchy problem

$$\mathcal{B}u' + \mathcal{A}u = f(t), \quad t > 0; \qquad u(0) = u_0, \tag{2.1}$$

where $u'(t) = du/dt$, and $\mathcal{B}$ is the self-adjoint positively defined linear operator acting in $\mathcal{H}$, $\mathcal{B} \leqslant \mathcal{A}$, u_0 is the given element of $\mathcal{H}_\mathcal{B}$, $f(t) \in L_2(0,T;\,\mathcal{H}_{\mathcal{A}^{-1}})$ is the given function, and $u(t)$ is the desired function in $\mathcal{H}_\mathcal{A}$.

Taking the inner product of (2.1) with $2u$, we obtain

$$\Big(\|u(t)\|^2_\mathcal{B}\Big)' + 2\,\|u(t)\|^2_\mathcal{A} = 2\,(f(t),\,u(t)) \leqslant \|f(t)\|^2_{\mathcal{A}^{-1}} + \|u(t)\|^2_\mathcal{A}. \tag{2.2}$$

Consider the following eigenvalue problem:

$$\mathcal{A}u = \lambda\,\mathcal{B}u. \tag{2.3}$$

Problem (2.3) has a countable set of eigenvalues. All eigenvalues are positive. The eigenfunctions are orthogonal in $\mathcal{H}_\mathcal{B}$. For each $u \in \mathcal{H}_\mathcal{A}$ the following inequality takes place:

$$\|u\|^2_\mathcal{A} \geqslant \lambda_1 \|u\|^2_\mathcal{B}, \tag{2.4}$$

where λ_1 is the minimum eigenvalue of problem (2.3). From (2.2) and (2.4) it follows that

$$\Big(\|u(t)\|^2_\mathcal{B}\Big)' + \lambda_1\,\|u(t)\|^2_\mathcal{B} \leqslant \|f(t)\|^2_{\mathcal{A}^{-1}}. \tag{2.5}$$

From this, integrating the latter expression in t we obtain the estimate

$$\|u(t)\|^2_\mathcal{B} \leqslant e^{-\lambda_1 t}\bigg(\|u_0\|^2_\mathcal{B} + \int_0^t e^{\lambda_1 s}\,\|f(s)\|^2_{\mathcal{A}^{-1}}\,ds\bigg). \tag{2.6}$$

Premultiplying (2.1) by $\mathcal{A}\mathcal{B}^{-1}$, $(\mathcal{A}\mathcal{B}^{-1})^2$, $(\mathcal{A}\mathcal{B}^{-1})^3$ etc and applying estimate (2.6), we obtain the following *a priori* estimates:

$$\|\mathcal{B}u(t)\|^2_{\mathcal{A}^{-1}} \leqslant e^{-\lambda_1 t}\bigg(\|\mathcal{B}u_0\|^2_{\mathcal{A}^{-1}} + \int_0^t e^{\lambda_1 s}\,\|\mathcal{A}^{-1}f(s)\|^2_\mathcal{B}\,ds\bigg), \tag{2.7}$$

$$\|u(t)\|^2_\mathcal{A} \leqslant e^{-\lambda_1 t}\bigg(\|u_0\|^2_\mathcal{A} + \int_0^t e^{\lambda_1 s}\,\|f(s)\|^2_{\mathcal{B}^{-1}}\,ds\bigg), \tag{2.8}$$

$$\|\mathcal{A}u(t)\|^2_{\mathcal{B}^{-1}} \leqslant e^{-\lambda_1 t}\bigg(\|\mathcal{A}u_0\|^2_{\mathcal{B}^{-1}} + \int_0^t e^{\lambda_1 s}\,\|\mathcal{B}^{-1}f(s)\|^2_\mathcal{A}\,ds\bigg). \tag{2.9}$$

Estimates (2.6)–(2.9) express the asymptotic stability of problem (2.1). The coefficient at $\lambda_1 t$ can be increased to $2-\varepsilon$ by evaluating the inner product $(f(t),\, u(t))$ in (2.2) by means of the Cauchy inequality with $\varepsilon > 0$. Thus, instead of (2.5) we obtain

$$\left(\|u(t)\|^2_{\mathcal{B}}\right)' + (2-\varepsilon)\lambda_1 \|u(t)\|^2_{\mathcal{B}} \leqslant \frac{1}{\varepsilon}\, \|f(t)\|^2_{\mathcal{A}^{-1}}.$$

From this it follows that

$$\|u(t)\|^2_{\mathcal{B}} \leqslant e^{-(2-\varepsilon)\lambda_1 t}\left(\|u_0\|^2_{\mathcal{B}} + \frac{1}{\varepsilon}\int\limits_0^t e^{(2-\varepsilon)\lambda_1 s}\, \|f(s)\|^2_{\mathcal{A}^{-1}}\, ds\right).$$

Taking into account the relations

$$\left(\|u(t)\|^2_{\mathcal{B}}\right)' = 2\, \|u(t)\|_{\mathcal{B}} \left(\|u(t)\|_{\mathcal{B}}\right)', \quad (f(t),\, u(t)) \leqslant \|u(t)\|_{\mathcal{B}}\, \|f(t)\|_{\mathcal{B}^{-1}},$$

and inequality (2.4), from (2.2) we obtain the following estimate:

$$\left(\|u(t)\|_{\mathcal{B}}\right)' + \lambda_1 \|u(t)\|_{\mathcal{B}} \leqslant \|f(t)\|_{\mathcal{B}^{-1}}.$$

Integrating the latter expression in t, we obtain

$$\|u(t)\|_{\mathcal{B}} \leqslant e^{-\lambda_1 t}\left(\|u_0\|_{\mathcal{B}} + \int\limits_0^t e^{\lambda_1 s}\, \|f(s)\|_{\mathcal{B}^{-1}}\, ds\right). \tag{2.10}$$

Similarly we obtain the following estimates:

$$\|u(t)\|_{\mathcal{A}} \leqslant e^{-\lambda_1 t}\left(\|u_0\|_{\mathcal{A}} + \int\limits_0^t e^{\lambda_1 s}\, \|\mathcal{B}^{-1} f(s)\|_{\mathcal{A}}\, ds\right), \tag{2.11}$$

$$\|\mathcal{A}u(t)\|_{\mathcal{B}^{-1}} \leqslant e^{-\lambda_1 t}\left(\|\mathcal{A}u_0\|_{\mathcal{B}^{-1}} + \int\limits_0^t e^{\lambda_1 s}\, \|AB^{-1} f(s)\|_{\mathcal{B}^{-1}}\, ds\right), \tag{2.12}$$

$$\|\mathcal{B}^{-1}\mathcal{A}u(t)\|_{\mathcal{A}} \leqslant e^{-\lambda_1 t}\left(\|\mathcal{B}^{-1}\mathcal{A}u_0\|_{\mathcal{A}} + \int\limits_0^t e^{\lambda_1 s}\, \|\mathcal{B}^{-1}AB^{-1} f(s)\|_{\mathcal{A}}\, ds\right). \tag{2.13}$$

2.2. Asymptotic behavior of the solutions for the nonlinear problems with unbounded nonlinearity

2.2.1.* A priori *estimates for the differential problem. In the cylindrical domain $\bar{Q}_T = \big\{(\mathbf{x},t): \quad \mathbf{x} \in \bar{\Omega}, \quad t \geqslant 0\big\}$, $\mathbf{x} = (x_1, x_2, \ldots, x_p)$, where $\bar{\Omega} = \{0 \leqslant x_\alpha, \leqslant l_\alpha, \quad \alpha = 1, 2, \ldots, p\}$, $\bar{\Omega} = \Omega \cup \gamma$, γ is a boundary, we consider the first boundary-value problem for the p-dimensional heat conduction equation

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^{p} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(\mathbf{x}, t, u)\frac{\partial u}{\partial x_\alpha}\right), \quad (\mathbf{x}, t) \in Q_T, \tag{2.14}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\Omega}, \qquad u(\mathbf{x}, t) = \mu(\mathbf{x}, t), \quad \mathbf{x} \in \gamma, \quad t > 0. \tag{2.15}$$

Let us introduce the range of values of the exact solution by

$$\bar{\mathcal{D}}_u = \left\{u(\mathbf{x},t): \quad 0 < u_1 \leqslant u(\mathbf{x},t) \leqslant u_2, \quad (\mathbf{x},t) \in \bar{Q}_T\right\}.$$

A characteristic feature of the problems with unbounded nonlinearity is the fulfillment of the basic properties (*nonnegativeness*, *boundedness* etc) that are imposed on the nonlinear coefficients $k_\alpha(\mathbf{x},t,u)$, $\alpha = 1,2,\dots,p$, not for all values of u as a parameter, but only in the range of values of the exact solution $\bar{\mathcal{D}}_u$ or in its small neighborhood [6, 13]:

$$0 < k_1 \leqslant k_\alpha(\mathbf{x},t,u) \leqslant k_2, \quad \alpha = 1,2,\dots,p, \quad (\mathbf{x},t) \in \bar{Q}_T, \quad u \in \bar{\mathcal{D}}_u. \tag{2.16}$$

For instance, for the problems of radiant heat conductivity [15] the typical property is the power nonlinearity:

$$k_\alpha(\mathbf{x},t,u) = u^\sigma, \alpha = 1,2,\dots,p. \tag{2.17}$$

The condition of positiveness of coefficients is met only at $u > 0$. In another example

$$k_\alpha(\mathbf{x},t,u) = u - u^*, \quad u^* = \text{const} > 0, \tag{2.18}$$

the necessary condition of positiveness is met at $u > u^*$.

For the real-valued functions, we introduce in Ω the inner product

$$(u,v) = \int\limits_\Omega uv d\mathbf{x},$$

and the norm

$$\|u(t)\|^2 = \int\limits_\Omega u^2(\mathbf{x},t)d\mathbf{x}.$$

Further we shall assume the uniformity of the boundary conditions, i.e., $\mu(\mathbf{x},t) = 0$, $\mathbf{x} \in \gamma$. Following [7, 15], it is easy to obtain *a priori* estimates of the asymptotic behavior for the quasilinear problem (2.14)–(2.15). Indeed, taking the inner product in Ω of equation (2.14) with $2u$ and carrying out the elementary transformations, we obtain the following energy identity:

$$\frac{\partial}{\partial t}\|u(t)\|^2 + 2\sum_{\alpha=1}^{2}\left(k_\alpha(\mathbf{x},t,u), \left(\frac{\partial u}{\partial x_\alpha}\right)^2\right) = 0. \tag{2.19}$$

We use condition (2.16) and the Friedrichs inequality

$$\sum_{\alpha=1}^{p}\left(k_\alpha(\mathbf{x},t,u), \left(\frac{\partial u}{\partial x_\alpha}\right)^2\right) \geqslant c_1\|u(t)\|^2, \tag{2.20}$$

where $c_1 = k_1 p l_{\max}^{-1}$, k_1 is a constant from (2.16), and $l_{\max} = \max_{\alpha=1,2,\dots,p} l_\alpha$. From (2.19) and (2.20) we obtain the inequality

$$\frac{\partial}{\partial t}\|u(t)\|^2 + 2c_1\|u(t)\|^2 \leqslant 0. \tag{2.21}$$

Integrating the latter relation in t, we obtain the following *a priori* estimate of the asymptotic behavior of the solution of the differential problem:

$$\|u(t)\| \leqslant e^{-c_1 t}\|u_0\|.$$

2.2.2. Difference scheme. In the domain $\bar{Q}_T$ consider the uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$, $\bar{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, \dots\}$, $\bar{\omega}_h = \omega_h \cup \gamma_h$, where the set of interior nodes of the spatial grid is determined by the relation

$$\omega_h = \Big\{ \mathbf{x}_i = \left(x_1^{(i_1)}, x_2^{(i_2)}, \dots, x_p^{(i_p)}\right),\ x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\ i_\alpha = 1, 2, \dots, N_\alpha - 1,$$
$$h_\alpha N_\alpha = l_\alpha,\ \alpha = 1, 2, \dots, p \Big\},$$

and γ_h denotes the set of its boundary nodes. On $\bar{\omega}$ we approximate differential problem (2.14)–(2.15) by the difference scheme

$$y_t = \sum_{\alpha=1}^{p} \left(a_\alpha \hat{y}_{\bar{x}_\alpha}\right)_{x_\alpha}, \quad (\mathbf{x}, t) \in \omega, \tag{2.22}$$

$$y(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \bar{\omega}_h, \qquad \hat{y}\big|_{\gamma_h} = \mu(\mathbf{x}, t), \quad \mathbf{x} \in \gamma_h \subset \gamma, \quad t \in \omega_\tau. \tag{2.23}$$

Here we use the standard index-free notation [14]

$$\begin{aligned}
\left(a_\alpha y_{\bar{x}_\alpha}\right)_{x_\alpha} &= \frac{1}{h_\alpha}\left(a_{\alpha(+1_\alpha)} y_{x_\alpha} - a_\alpha y_{\bar{x}_\alpha}\right), \\
v_{\pm 1_\alpha} &= v\left(x_1^{(i_1)}, \dots, x_{\alpha-1}^{(i_{\alpha-1})}, x_\alpha^{(i_\alpha)} \pm h_\alpha, x_{\alpha+1}^{(i_\alpha+1)}, \dots, x_p^{(i_p)}\right), \\
y_{x_\alpha} &= \left(y_{(+1_\alpha)} - y\right)/h_\alpha, \quad y_{\bar{x}_\alpha} = \left(y - y_{(-1_\alpha)}\right)/h_\alpha, \\
a_\alpha &= a_\alpha^n = 0.5\left(k_\alpha(\mathbf{x}, t_n, y^n) + k_\alpha(\mathbf{x}_{(-1_\alpha)}, t_n, y^n_{(-1_\alpha)})\right).
\end{aligned} \tag{2.24}$$

Note that if $y^n \in \bar{\mathcal{D}}_u$, i.e.

$$0 < u_1 \leqslant y(\mathbf{x}, t_n) \leqslant u_2, \quad (\mathbf{x}, t_n) \in \bar{\omega}, \tag{2.25}$$

then by virtue of (2.16) we have

$$0 < k_1 \leqslant a_\alpha(\mathbf{x}, t, y) \leqslant k_2, \quad \alpha = 1, 2, \dots, p. \tag{2.26}$$

The proof of inequalities (2.25) and (2.26) is a very difficult issue in studying the properties of the difference schemes for nonlinear problems with unbounded nonlinearity. The belonging of the approximate solution to the range of values of the exact solution ($y^n \in \bar{\mathcal{D}}_u$) is proved by means of the maximum principle and its corollaries.

2.2.3. Maximum principle and its corollaries. In this section we shall prove the satisfiability of inequalities (2.25) and (2.26) for all $(\mathbf{x}, t) \in \omega$ and arbitrary τ and h_α. For the first time unconditional monotonicity of the difference scheme (2.22)–(2.23) for nonlinear problem will thus be established. We shall use the following canonical form of the difference scheme [14]:

$$A(\mathbf{x}) y(\mathbf{x}) = \sum_{\xi \in \mathcal{M}'(\mathbf{x})} B(\mathbf{x}, \xi) y(\xi) + F(\mathbf{x}), \quad \mathbf{x} \in \omega_h, \qquad y(\mathbf{x}) = \mu(\mathbf{x}), \quad \mathbf{x} \in \gamma_h, \tag{2.27}$$

where $\mathcal{M}'(\mathbf{x}) = \mathcal{M}(\mathbf{x}) \setminus \{\mathbf{x}\}$, $\mathcal{M}(\mathbf{x})$ is the stencil of the scheme. Further we shall assume that the following positiveness conditions of the coefficients are met:

$$A(\mathbf{x}) > 0, \quad B(\mathbf{x}, \xi) > 0, \quad D(\mathbf{x}) = A(\mathbf{x}) - \sum_{\xi \in \mathcal{M}'(\mathbf{x})} B(\mathbf{x}, \xi) \geqslant 0, \qquad \mathbf{x} \in \omega_h. \tag{2.28}$$

Let us formulate the maximum principle in the following form:

Theorem 2.1 [14]. *Let us assume that for equation* (2.27) *conditions* (2.28) *are fulfilled and*

$$F(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in \omega_h; \qquad \mu(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in \gamma_h.$$

Then the function $y(\mathbf{x})$ *is nonnegative, i.e.*

$$y(\mathbf{x}) \geqslant 0, \quad \mathbf{x} \in \bar{\omega}_h.$$

If $F(\mathbf{x}) \leqslant 0$ *(*$\mathbf{x} \in \omega_h$*),* $\mu(\mathbf{x}) \leqslant 0$ *(*$\mathbf{x} \in \gamma_h$*), then* $y(\mathbf{x}) \leqslant 0$ *for all* $\mathbf{x} \in \bar{\omega}_h$.

Corollary 2.1. *Let conditions* (2.28) *be met and*

$$F(\mathbf{x}) \geqslant u_1, \quad \mathbf{x} \in \omega_h; \qquad \mu(\mathbf{x}) \geqslant u_1, \quad \mathbf{x} \in \gamma_h, \quad u_1 = \text{const} \geqslant 0, \tag{2.29}$$

$$0 \leqslant D \leqslant 1. \tag{2.30}$$

Then the following inequality takes place:

$$y(\mathbf{x}) \geqslant u_1, \quad \mathbf{x} \in \bar{\omega}_h.$$

Indeed, substituting $y = \tilde{y} + u_1$ in (2.27), we obtain the problem for $\tilde{y}$

$$A(\mathbf{x})\tilde{y}(\mathbf{x}) = \sum_{\xi \in \mathcal{M}'(\mathbf{x})} B(\mathbf{x}, \xi)\tilde{y}(\xi) + \tilde{F}(\mathbf{x}), \quad \mathbf{x} \in \omega_h, \qquad \tilde{y}(\mathbf{x}) = \tilde{\mu}(\mathbf{x}), \quad \mathbf{x} \in \gamma_h,$$

where (by virtue of (2.28)–(2.30))

$$\tilde{F}(\mathbf{x}) = F(\mathbf{x}) - u_1 D(\mathbf{x}) \geqslant (1 - D)u_1 \geqslant 0, \quad x \in \omega_h; \qquad \tilde{\mu}(\mathbf{x}) = \mu(\mathbf{x}) - u_1 \geqslant 0, \quad \mathbf{x} \in \gamma_h.$$

Using Theorem 2.1, we obtain $\tilde{y}(\mathbf{x}) \geqslant 0$. Consequently, $y = \tilde{y} + u_1 \geqslant u_1$.

Let us introduce the grid norms

$$\| \cdot \|_{\bar{C}} = \max_{\mathbf{x} \in \bar{\omega}_h} | \cdot |, \quad \| \cdot \|_{C_\gamma} = \max_{\mathbf{x} \in \gamma_h} | \cdot |, \quad \| \cdot \|_{C} = \max_{\mathbf{x} \in \omega_h} | \cdot |.$$

Corollary 2.2 [14]. *Assume that the following stronger restrictions imposed on the coefficients are satisfied:*

$$A(\mathbf{x}) > 0, \quad B(\mathbf{x}, \xi) > 0, \quad D(\mathbf{x}) = A(\mathbf{x}) - \sum_{\xi \in \mathcal{M}'(\mathbf{x})} B(\mathbf{x}, \xi) > 0, \qquad \mathbf{x} \in \omega_h. \tag{2.31}$$

Then for the solution of problem (2.27) *the estimate*

$$\|y\|_{\bar{C}} \leqslant \max\left\{\|y\|_{C_\gamma}, \|F/D\|_C\right\} \tag{2.32}$$

takes place.

We now write the difference scheme (2.22)–(2.23) in the canonical form (2.27):

$$A^n y^{n+1} = \sum_{\alpha=1}^{p} \frac{\tau}{h_\alpha^2} \left(a^n_{\alpha(+1_\alpha)} y^{n+1}_{(+1_\alpha)} + a^n_\alpha y^{n+1}_{(-1_\alpha)} \right) + F^n. \tag{2.33}$$

Here

$$A^n = 1 + \sum_{\alpha=1}^{p} = \frac{\tau}{h_\alpha^2} \left(a^n_{\alpha(+1_\alpha)} + a^n_\alpha \right), \quad F^n = y^n. \tag{2.34}$$

Note that $y^0 = y(\mathbf{x}, 0) = u_0(\mathbf{x}) \in \bar{\mathcal{D}}_u$, i.e.

$$0 < u_1 \leqslant y^0 \leqslant u_2, \quad \mathbf{x} \in \omega_h.$$

Since $y^0 \in \bar{\mathcal{D}}_u$, we see that by virtue of (2.16) inequalities (2.26) are satisfied for $n = 0$ and for all $\alpha = 1, 2, \dots, p$ the following relations hold:

$$0 < k_1 \leqslant a_\alpha^0 \leqslant k_2.$$

Using expressions (2.33)–(2.34), we obtain the relations

$$F^0 = y^0 \geqslant u_1, \quad \mathbf{x} \in \omega_h, \qquad \mu(\mathbf{x}, t) \geqslant u_1, \quad \mathbf{x} \in \gamma_h, \quad D = 1.$$

On the basis of Corollary 2.1, we conclude that

$$y^1 \geqslant u_1 > 0.$$

Applying estimates (2.32) to problem (2.33)–(2.34) for $n = 0$, we obtain inequality

$$\|y^1\|_{\bar{C}} \leqslant \max\left\{\|\mu^1\|_{C_\gamma}, \|y^0\|_C\right\} \leqslant u_2.$$

We now assume that inequalities (2.25) and (2.26) are satisfied at $t = t_n$. We show that they also hold true for t_{n+1}. Indeed, by virtue of assumptions for problem (2.33)–(2.34) all conditions of Corollaries 2.1 and 2.2 are satisfied and, consequently,

$$0 < u_1 \leqslant y^{n+1} \leqslant \|y^{n+1}\|_{\bar{C}} \leqslant \max\left\{\max_{0<t\leqslant t_{n+1}} \|y(t)\|_{C_\gamma}, \|u_0\|_{\bar{C}}\right\}. \tag{2.35}$$

The latter estimate means that $y^{n+1} \in \bar{\mathcal{D}}_u$ and, consequently,

$$0 < k_1 \leqslant a_\alpha(\mathbf{x}, t, y^{n+1}) \leqslant k_2, \quad \alpha = 1, 2, \dots, p.$$

2.2.4. Asymptotic a priori estimates. We use the proved inequalities (2.25) and (2.26) to study the asymptotic properties of the solution of the difference scheme (2.22)–(2.23). Let us introduce the following inner product and norm [14]:

$$(u, v)_h = \sum_{\mathbf{x} \in \omega_h} u(\mathbf{x}, t_n) v(\mathbf{x}, t_n) h_1 \dots h_p, \quad \|v\|_h = \sqrt{(v, v)}.$$

As for the differential problem, we assume that the boundary conditions are uniform, i.e.

$$y\Big|_{\gamma_h} = 0, \quad \mathbf{x} \in \gamma_h, \quad t \in \omega_\tau.$$

Taking the inner product of the difference equation (2.22) with $2\tau\hat{y}$ and using the formulas of summation by parts, we obtain the energy identity

$$\|y^{n+1}\|_h^2 + \tau\|y_t\|_h^2 + 2\tau\|y^{n+1}\|_A^2 = \|y^n\|_h^2, \tag{2.36}$$

where

$$A = \sum_{\alpha=1}^{p} A_\alpha, \quad \|y\|_A^2 = \sum_{\alpha=1}^{p} \|y\|_{A_\alpha}^2, \quad \|y\|_{A_\alpha}^2 = \sum_{\mathbf{x} \in \omega_h} a_\alpha(\mathbf{x}, t, y) y_{\bar{x}_\alpha} h_1 \dots h_p.$$

Since $y \in \bar{\mathcal{D}}_u$, we see that $a_\alpha \geqslant k_1 > 0$ for all $\alpha = 1, 2, \dots, p$, and using the Friedrichs inequality, we obtain the estimate

$$\|y^{n+1}\|_h^2 \leqslant \frac{1}{1 + 2\tau c_{1h}} \|y^n\|_h^2 \leqslant e^{-c_1 t_{n+1}} \|u_0\|_h^2.$$

The latter estimate expresses the asymptotic behavior of the solution at $t \to \infty$.

3. Second-order equation

3.1. Global stability of the second-order equation

3.1.1. Stability with respect to initial data. Consider the Cauchy problem for the homogeneous differential-operator equation

$$\mathcal{D}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = 0, \quad t > 0; \qquad u(0) = u_0, \quad u'(0) = u_1. \tag{3.1}$$

Here $\mathcal{A} = \mathcal{A}^* > 0$, $\mathcal{D} = \mathcal{D}^* > 0$ are linear operators in $\mathcal{H}$, u_0, u_1 are given functions, and $\mathcal{B}$ is a nonnegative linear operator in $\mathcal{H}$. For problem (3.1) the following theorem holds.

Theorem 3.1. *Assume that the following conditions are met:*

$$\mathcal{A} = \mathcal{A}^* > 0, \quad \mathcal{D} = \mathcal{D}^* > 0, \quad \mathcal{B} \geqslant 0. \tag{3.2}$$

Then the solution of problem (3.1) *is stable with respect to initial data for all* $t > 0$ *and the following energy identity holds:*

$$\mathcal{E}(t) = \mathcal{E}(0), \quad t > 0, \tag{3.3}$$

where

$$\mathcal{E}(t) = \left(\|u'(t)\|_{\mathcal{D}}^2 + 2\int_0^t (\mathcal{B}u'(s), u'(s))ds + \|u(t)\|_{\mathcal{A}}^2 \right)^{1/2}. \tag{3.4}$$

Proof. Taking the inner product of equation (3.1) with $2u'$ and taking into account the equality

$$(\mathcal{B}u'(t), u'(t)) = \left(\int_0^t (\mathcal{B}u'(s), u'(s))ds \right)',$$

we obtain the energy identity

$$\left(\|u'(t)\|_{\mathcal{D}}^2 + 2\int_0^t (\mathcal{B}u'(s), u'(s))ds + \|u(t)\|_{\mathcal{A}}^2 \right)' = 0. \tag{3.5}$$

Integrating (3.5) in t, we obtain identity (3.3). □

Remark 3.1. Identity (3.3) expresses not only the global stability of the solution of problem (3.1) by the initial data but the practically important conservation law as well.

3.1.2. Stability with respect to the right-hand side. Consider the abstract Cauchy problem

$$\mathcal{D}u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = f(t), \quad t > 0; \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{3.6}$$

The following statement holds.

Theorem 3.2. *Assume that conditions* (3.2) *are fulfilled and $f(t)$ is bounded, and the integrals $\int_0^t \|f(s)\|_{\mathcal{D}^{-1}}$, $\int_0^t \|f'(s)\|_{\mathcal{A}^{-1}}$ converge when $t \to +\infty$. Then problem* (3.6) *stable for a global and the following a priori estimates holds*

$$\max_{s\in[0,t]} \Big(\|u'(s)\|_{\mathcal{D}} + \|u(s)\|_{\mathcal{A}} \Big) \leqslant \sqrt{2} \Big(\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{D}} + \int_0^t \|f(s)\|_{\mathcal{D}^{-1}}\, ds \Big), \tag{3.7}$$

$$\begin{aligned} \max_{s\in[0,t]} \Big(\|u'(s)\|_{\mathcal{D}} + \|u(s)\|_{\mathcal{A}} \Big) \leqslant \sqrt{2} \Big(&\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{D}} + \|f(0)\|_{\mathcal{A}^{-1}} \\ &+ \max_{s\in[0,t]} \|f(s)\|_{\mathcal{A}^{-1}} + \int_0^t \|f'(s)\|_{\mathcal{A}^{-1}}\, ds \Big), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \max_{s\in[0,t]} \Big(\|u'(s)\|_{\mathcal{D}} + \|u(s)\|_{\mathcal{A}} \Big) \leqslant \sqrt{2} \Big[&\|u_0\|_{\mathcal{A}} + \|u_1\|_{\mathcal{D}} + \|g(0)\|_{\mathcal{D}^{-1}} + \max_{s\in[0,t]} \|g(s)\|_{\mathcal{D}^{-1}} \\ &+ \int_0^t \Big(\|BD^{-1}g(s)\|_{\mathcal{D}^{-1}} + \|\mathcal{D}^{-1}g(s)\|_{\mathcal{A}} \Big)\, ds \Big]. \end{aligned} \tag{3.9}$$

Proof. Taking the inner product of (3.6) with $2u'$, we obtain the energy identity:

$$\Big(\|u'\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{A}}^2 \Big)' + 2\,\|u'\|_{\mathcal{B}}^2 = 2\,(f,\, u')\,. \tag{3.10}$$

Since $\mathcal{B} \geqslant 0$ it follows from (3.10) that

$$\Big(\|u'\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{A}}^2 \Big)' \leqslant 2\,\|u'\|_{\mathcal{D}}\,\|f\|_{\mathcal{D}^{-1}} \leqslant 2\,\Big(\|u'\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{A}}^2 \Big)^{1/2} \|f\|_{\mathcal{D}^{-1}}. \tag{3.11}$$

Integrating (3.11) in t we obtain the *a priori* estimate:

$$\Big(\|u'(t)\|_{\mathcal{D}}^2 + \|u(t)\|_{\mathcal{A}}^2 \Big)^{1/2} \leqslant \Big(\|u'(0)\|_{\mathcal{D}}^2 + \|u(0)\|_{\mathcal{A}}^2 \Big)^{1/2} + \int_0^t \|f(s)\|_{\mathcal{D}^{-1}}\, ds. \tag{3.12}$$

From (3.12) we obtain the desired estimate (3.7).

It also follows from (3.10) that

$$\begin{aligned} \Big(\|u'\|_{\mathcal{D}}^2 + \|u - \mathcal{A}^{-1}f\|_{\mathcal{A}}^2 \Big)' &\leqslant 2\,\|f'\|_{\mathcal{A}^{-1}}\,\|u - \mathcal{A}^{-1}f\|_{\mathcal{A}} \\ &\leqslant 2\,\|f'\|_{\mathcal{A}^{-1}} \Big(\|u'\|_{\mathcal{D}}^2 + \|u - \mathcal{A}^{-1}f\|_{\mathcal{A}}^2 \Big)^{1/2}. \end{aligned} \tag{3.13}$$

Integrating (3.13) in t, we obtain the *a priori* estimate

$$\begin{aligned} \Big(\|u'(t)\|_{\mathcal{D}}^2 + \|u(t) - \mathcal{A}^{-1}f(t)\|_{\mathcal{A}}^2 \Big)^{1/2} \leqslant &\Big(\|u'(0)\|_{\mathcal{D}}^2 + \|u(0) - \mathcal{A}^{-1}f(0)\|_{\mathcal{A}}^2 \Big)^{1/2} \\ &+ \int_0^t \|f'(s)\|_{\mathcal{A}^{-1}} ds. \end{aligned} \tag{3.14}$$

From (3.14) the desired estimate (3.8) follows.

Further, we rewrite equation (3.6) in the form

$$\mathcal{D}(u' - \mathcal{D}^{-1}g)' + \mathcal{B}(u' - \mathcal{D}^{-1}g) + \mathcal{A}u = -\mathcal{B}\mathcal{D}^{-1}g.$$

Taking the inner product of the latter equation with $u' - \mathcal{D}^{-1}g$, we obtain

$$\begin{aligned}\Big(\|u' - \mathcal{D}^{-1}g\|_{\mathcal{D}}^2 + \|u\|_{\mathcal{A}}^2\Big)' &+ 2\,\|u' - \mathcal{D}^{-1}g\|_{\mathcal{B}}^2 \\ &= 2\,(\mathcal{A}u,\, \mathcal{D}^{-1}g) - 2\,(BD^{-1}g,\, u' - \mathcal{D}^{-1}g) \\ &\leqslant 2\,\|u\|_{\mathcal{A}}\,\|\mathcal{D}^{-1}g\|_{\mathcal{A}} + 2\,\|u' - \mathcal{D}^{-1}g\|_{\mathcal{D}}\,\|BD^{-1}g\|_{\mathcal{D}^{-1}}\end{aligned}$$

and

$$\Big(\|u'-\mathcal{D}^{-1}g\|_{\mathcal{D}}^2+\|u\|_{\mathcal{A}}^2\Big)' \leqslant 2\,\Big(\|u'-\mathcal{D}^{-1}g\|_{\mathcal{D}}^2+\|u\|_{\mathcal{A}}^2\Big)^{1/2}\Big(\|\mathcal{D}^{-1}g\|_{\mathcal{A}}^2+\|BD^{-1}g\|_{\mathcal{D}^{-1}}^2\Big)^{1/2} \quad (3.15)$$

Integrating (3.15) in t, we obtain the relation

$$\begin{aligned}\Big(\|u'(t) - \mathcal{D}^{-1}g(t)\|_{\mathcal{D}}^2 + \|u(t)\|_{\mathcal{A}}^2\Big)^{1/2} &\leqslant \Big(\|u'(0) - \mathcal{D}^{-1}g(0)\|_{\mathcal{D}}^2 + \|u(0)\|_{\mathcal{A}}^2\Big)^{1/2} \\ &+ \int_0^t \Big(\|\mathcal{D}^{-1}g(s)\|_{\mathcal{A}}^2 + \|BD^{-1}g(s)\|_{\mathcal{D}^{-1}}^2\Big)^{1/2} ds\,.\end{aligned} \quad (3.16)$$

Inequality (3.16) yields the desired estimate (3.9). □

3.2. Global stability of the three-level operator-difference scheme

Let $H = H_h$ be the real finite-dimensional Hilbert space with the inner product $(\cdot,\cdot)_h$ and the norm $\|\cdot\|_h$. It is assumed that the following conditions are met:

1. there exists the operator $\mathcal{P}_h$ such that $\mathcal{P}_h u = u_h \in H$ for any $u \in \mathcal{H}$;
2. the norm $\|\cdot\|_h$ is consistent with the norm $\|\cdot\|$, i.e.

$$\lim_{|h|\to 0} \|\mathcal{P}_h u\|_h = \|u\|,$$

where $|h|$ is the norm of the parameter h.

Let A be the self-adjoint positively defined operator in H. We denote by H_A the Hilbert space consisting of elements of the space H and having by the inner product $(u,v)_A = (Au,v)$ and the norm $\|u\|_A = \sqrt{(u,u)_A}$, where $u,v \in H$. We also introduce the uniform time grid $\omega_\tau = \{t_n = n\tau,\ n = 0,1,\dots\}$. We shall use the following index-free notation [14]:

$$y = y_n = y(t_n), \quad \hat{y} = y_{n+1} = y(t_{n+1}), \quad \check{y} = y_{n-1} = y(t_{n-1}), \quad t_n \in \omega_\tau,$$
$$y_t = \frac{\hat{y} - y}{\tau}, \quad y_{\bar{t}} = \frac{y - \check{y}}{\tau}, \quad y_{\overset{\circ}{t}} = \frac{\hat{y} - \check{y}}{2\tau}, \quad y_{\bar{t}t} = \frac{y_t - y_{\bar{t}}}{\tau}, \quad y^{(0.5)} = 0.5(\hat{y} + y).$$

In this section we obtain *a priori* estimates of the global stability of the solution of the three-level operator-difference scheme that are consistent with the estimates of global stability obtained in Section 3.1.

3.2.1. Stability with respect to initial data. Consider the Cauchy problem for the homogeneous operator-difference equation

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = 0, \quad y(0) = u_0, \quad y_t(0) = u_1, \tag{3.17}$$

where $y = y(t) \in H$, H is a finite-dimensional Hilbert space; $A = A_{h\tau}$, $D = D_{h\tau}$ are self-adjoint positively defined operators in H; B is the nonnegative linear operator in H; u_0, $u_1 \in H$ are given; $t_n \in \omega_\tau$. The operators of the difference scheme (3.20) thereby approximate the operators of the original differential problem (3.6), i.e. $A_{h\tau} \to \mathcal{A}$, $B_{h\tau} \to \mathcal{B}$, $D_{h\tau} \to \mathcal{D}$ at $h \to 0$, $\tau \to 0$.

Let us formulate and prove the difference analogue of Theorem 3.1.

Theorem 3.3. *Assume that the operators of equation* (3.17) *satisfy the following conditions:*

$$A = A^* > 0, \quad D = D^* > 0, \quad D \geqslant \frac{\tau^2}{4} A, \quad B \geqslant 0. \tag{3.18}$$

Then the three-level difference scheme (3.17) *is stable with respect to the initial data for any $t_n \in \omega_\tau$, and the following energy identity holds:*

$$E(t_n) = E(0), \quad t_n \in \omega_\tau, \tag{3.19}$$

where

$$E(t_n) = E_{h\tau}(t_n) = \left(\|y_t(t_n)\|^2_{D - \frac{\tau^2}{4}A} + 2\sum_{s=1}^{n} \tau (By_{\overset{\circ}{t}}, y_{\overset{\circ}{t}}) + \|y^{(0.5)}\|^2_A \right)^{1/2}.$$

Proof. Taking the inner product of equation (3.17) with $2y_{\overset{\circ}{t}}$ and taking into account the relations

$$2(Dy_{\bar{t}t}(t_n), y_{\overset{\circ}{t}}(t_n)) = \left(\|y_t(t_n)\|^2_D\right)_{\bar{t}},$$
$$2(By_{\overset{\circ}{t}}(t_n), y_{\overset{\circ}{t}}(t_n)) = 2\left(\sum_{s=1}^{n} \tau (By_{\overset{\circ}{t}}(t_s), y_{\overset{\circ}{t}}(t_s))\right)_{\bar{t}},$$
$$2(Ay(t_n), y_{\overset{\circ}{t}}(t_n)) = \left(\|y^{(0.5)}(t_n)\|^2_A - \frac{\tau^2}{4}\|y_t(t_n)\|^2_A\right)_{\bar{t}},$$

we get the energy identity

$$\left(\|y_t(t_n)\|^2_{D - \frac{\tau^2}{4}A} + 2\sum_{s=1}^{n} \tau (By_{\overset{\circ}{t}}(t_s), y_{\overset{\circ}{t}}(t_s)) + \|y^{(0.5)}\|^2_A \right)_{\bar{t}} = 0,$$

from which the desirable estimate (3.19) follows. □

3.2.2. Stability with respect to the right-hand side. Consider the Cauchy problem for the operator-difference equation

$$Dy_{\bar{t}t} + By_{\overset{\circ}{t}} + Ay = \varphi, \quad y(0) = u_0, \quad y_t(0) = u_1. \tag{3.20}$$

The following statement holds.

Theorem 3.4. *Assume that conditions* (3.18) *are fulfilled, and* φ *is the bounded grid function, and the series* $\sum_{k=0}^{n}\tau\|\varphi_k\|_{D^{-1}}$ *and* $\sum_{k=1}^{n}\|\varphi_{\bar t,k}\|_{A^{-1}}$ *converge when* $n\to\infty$. *Then the operator difference scheme* (3.20) *is globally stable , and for its solution the following energy estimates holds*

$$\max_{0\leqslant k\leqslant n}\left(\|y_{t,k}\|_{D-\frac{\tau^2}{4}A}+\|y_{k+1}^{(0.5)}\|_A\right) \leqslant \sqrt{2}\left(\|y_{t,0}\|_{D-\frac{\tau^2}{4}A}+\|y_0^{(0.5)}\|_A+\sum_{k=0}^{n}\tau\|\varphi_k\|_{(D-\frac{\tau^2}{4}A)^{-1}}\right), \tag{3.21}$$

$$\max_{0\leqslant k\leqslant n}\left(\|y_{t,k}\|_{D-\frac{\tau^2}{4}A}+\|y_{k}^{(0.5)}\|_A\right) \leqslant \sqrt{2}\left(\|y_{t,0}\|_{D-\frac{\tau^2}{4}A}+\|y_0^{(0.5)}\|_A+\|\varphi_0\|_A^{-1}+\max_{0\leqslant k\leqslant n}\|\varphi_n\|_A^{-1}+2\sum_{k=1}^{n}\tau\|\varphi_{\bar t,k}\|_{A^{-1}}\right). \tag{3.22}$$

Proof. Taking the inner product of (3.20) with $2y_{\overset{\circ}{t}}$, we obtain the energy identity

$$\left(\|y_t\|^2_{D-\frac{\tau^2}{4}A}+\|y^{(0.5)}\|_A^2\right)_{\bar t}+2\|y_{\overset{\circ}{t}}\|_B^2=2\left(\varphi,y_{\overset{\circ}{t}}\right). \tag{3.23}$$

Since $B\geqslant 0$, it follows from (3.23) that

$$(G)_{\bar t}\leqslant(\varphi,y_t)+(\varphi,y_{\bar t})\leqslant\left(G^{1/2}+\check G^{1/2}\right)\|\varphi\|_{(D-\frac{\tau^2}{4}A)^{-1}},$$

where

$$G=G_{h\tau}=\|y_t\|^2_{D-\frac{\tau^2}{4}A}+\|y^{(0.5)}\|_A^2.$$

Consequently, taking into account the identity

$$(G^{1/2})_{\bar t}=\frac{G^{1/2}-\check G^{1/2}}{\tau}=\frac{G-\check G}{\tau(G^{1/2}+\check G^{1/2})}=\frac{G_{\bar t}}{G^{1/2}+\check G^{1/2}},$$

when the conditions of the theorem are met we obtain

$$\left(\|y_{t,n}\|_D^2+\|y_n^{(0.5)}\|_A^2\right)^{1/2}\leqslant\left(\|y_{t,0}\|_D^2+\|y_0^{(0.5)}\|_A^2\right)^{1/2}+\sum_{k=0}^{n}\tau\|\varphi_k\|_{(D-\frac{\tau^2}{4}A)^{-1}}. \tag{3.24}$$

Using the obvious relations

$$|a|+|b|\leqslant\sqrt{2(a^2+b^2)},\qquad \sqrt{a^2+b^2}\leqslant|a|+|b|,$$

from inequality (3.24) we obtain estimate (3.21).

Taking into account the obvious identity

$$(\varphi,(y^{(0.5)})_{\bar t})=(\varphi,y^{(0.5)})_{\bar t}-(\varphi_{\bar t},y^{(0.5)}),$$

from (3.23) we obtain

$$\left(\|y_t\|^2_{D-\frac{\tau^2}{4}A}+\|y^{(0.5)}\|_A^2\right)_{\bar t}\leqslant 2(\varphi,y^{(0.5)})_{\bar t}-2(\varphi_{\bar t},y^{(0.5)}),$$

$$\left(\|y_t\|^2_{D+\frac{\tau^2}{4}A}+\|y^{(0.5)}-A^{-1}\varphi\|_A^2\right)_{\bar t}\leqslant 2(\varphi_{\bar t},A^{-1}\varphi-y^{(0.5)}).$$

Hence,

$$\Big(\|y_t\|^2_{D-\frac{\tau^2}{4}A} + \|y^{(0.5)} - A^{-1}\varphi\|^2_A\Big)_{\bar t} \leqslant 2\|\varphi_{\bar t}\|_{A^{-1}}\|y^{(0.5)} - A^{-1}\varphi\|_A$$

$$\leqslant 2\|\varphi_{\bar t}\|_{A^{-1}}\Big(\|y_t\|^2_{D-\frac{\tau^2}{4}A} + \|y^{(0.5)} - A^{-1}\varphi\|^2_A\Big)^{1/2}.$$

From the latter relation it follows that

$$\Big(\|y_{t,n}\|^2_{D-\frac{\tau^2}{4}A} + \|y_n^{(0.5)} - A^{-1}\varphi_n\|^2_A\Big)^{1/2} \leqslant \Big(\|y_{t,0}\|^2_{D-\frac{\tau^2}{4}A} + \|y_0^{(0.5)} - A^{-1}\varphi_0\|^2_A\Big)^{1/2} + 2\sum_{k=1}^{n}\|\varphi_{\bar t,k}\|_{A^{-1}}.$$

The latter inequality yields estimate (3.22). □

3.3. Asymptotic stability of the second-order homogeneous differential-operator equation

Consider the abstract Cauchy problem

$$u''(t) + \mathcal{B}u'(t) + \mathcal{A}u(t) = 0, \quad t > 0; \qquad u(0) = u_0, \quad u'(0) = u_1, \tag{3.25}$$

where $\mathcal{A}$ and $\mathcal{B}$ are unbounded positive linear self-adjoint operators acting in the Hilbert space $\mathcal{H}$ with the domains of definition dense in $\mathcal{H}$; u_0 and u_1 are elements of $\mathcal{H}$; $u(t)$ is the desired function mapping $(0, +\infty)$ into $\mathcal{H}$. Assume also that that $AB = BA$ where this product is defined.

Taking the inner product of (3.25) with $2u'$ we obtain the energy identity

$$\Big(\|u'\|^2 + \|u\|^2_{\mathcal{A}}\Big)' + 2\,\|u'\|^2_{\mathcal{B}} = 0\,. \tag{3.26}$$

Further we consider two cases separately.

3.3.1. The case of $\mathcal{A} \geqslant \frac{1}{4}\mathcal{B}^2$.

Suppose that the following operator inequality is satisfied:

$$\mathcal{A} \geqslant \frac{1}{4}\mathcal{B}^2\,. \tag{3.27}$$

We transform the left-hand side of identity (3.26) as follows:

$$\Big(\|u'\|^2 + \|u\|^2_{\mathcal{A}}\Big)' + 2\,\|u'\|^2_{\mathcal{B}} = \Big(\Big\|u' + \frac{1}{2}\mathcal{B}u\Big\|^2 - (\mathcal{B}u, u') + \|u\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\Big)' + 2\,\|u'\|^2_{\mathcal{B}}$$

$$= \Big(\Big\|u' + \frac{1}{2}\mathcal{B}u\Big\|^2 + \|u\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\Big)' + \|u'\|^2_{\mathcal{B}} - (\mathcal{B}u, u'')\,.$$

Hence, using equation (3.25) and carrying out elementary manipulations, we obtain

$$\Big(\Big\|u' + \frac{1}{2}\mathcal{B}u\Big\|^2 + \|u\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\Big)' + \Big\|u' + \frac{1}{2}\mathcal{B}u\Big\|^2_{\mathcal{B}} + \|u\|^2_{\mathcal{B}(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)} = 0\,.$$

Then,

$$\|u\|_{\mathcal{B}}^2 = (\mathcal{B}u, u) \geqslant \lambda_1 \|u\|^2 ,$$

where $\lambda_1 > 0$ is the minimum eigenvalue of the operator $\mathcal{B}$. Thus, the inequality

$$\left(\left\|u' + \frac{1}{2}\mathcal{B}u\right\|^2 + \|u\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\right)' + \lambda_1\left(\left\|u' + \frac{1}{2}\mathcal{B}u\right\|^2 + \|u\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\right) \leqslant 0 .$$

takes place.

Integrating the latter relation in t we obtain

$$\left\|u'(t) + \frac{1}{2}\mathcal{B}u(t)\right\|^2 + \|u(t)\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2} \leqslant e^{-\lambda_1 t}\left(\left\|u_1 + \frac{1}{2}\mathcal{B}u_0\right\|^2 + \|u_0\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\right). \tag{3.28}$$

In the case where

$$\mathcal{A} > \frac{1}{4}\mathcal{B}^2,$$

from (3.26) it follows that

$$\left(\left\|\mathcal{A}u + \frac{1}{2}\mathcal{B}u'\right\|^2_{(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u'\|^2\right)' + \left\|\mathcal{A}u + \frac{1}{2}\mathcal{B}u'\right\|^2_{\mathcal{B}(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u'\|^2_{\mathcal{B}} = 0 .$$

Hence,

$$\left(\left\|\mathcal{A}u + \frac{1}{2}\mathcal{B}u'\right\|^2_{(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u'\|^2\right)' + \lambda_1\left(\left\|\mathcal{A}u + \frac{1}{2}\mathcal{B}u'\right\|^2_{(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u'\|^2\right) \leqslant 0$$

and

$$\left\|\mathcal{A}u(t) + \frac{1}{2}\mathcal{B}u'(t)\right\|^2_{(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u'(t)\|^2 \leqslant e^{-\lambda_1 t}\left(\left\|\mathcal{A}u_0 + \frac{1}{2}\mathcal{B}u_1\right\|^2_{(\mathcal{A}-\frac{1}{4}\mathcal{B}^2)^{-1}} + \|u_1\|^2\right). \tag{3.29}$$

Inequalities (3.28) and (3.29) express the asymptotic stability of the solution of equation (3.25). Assume that instead of (3.27) the following stronger inequality is satisfied:

$$\mathcal{A} - \frac{1}{4}\mathcal{B}^2 \geqslant c_0\,\mathcal{A}, \qquad 0 < c_0 < 1 . \tag{3.30}$$

From (3.28) and (3.30) we obtain the estimate

$$\begin{aligned}\|u(t)\|^2_{\mathcal{A}} \leqslant \frac{1}{c_0}\,\|u(t)\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2} &\leqslant \frac{1}{c_0}\,e^{-\lambda_1 t}\left(2\,\|u_1\|^2 + \frac{1}{2}\,\|\mathcal{B}u_0\|^2 + \|u_0\|^2_{\mathcal{A}-\frac{1}{4}\mathcal{B}^2}\right)\\ &\leqslant \frac{2}{c_0}\,e^{-\lambda_1 t}\left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right).\end{aligned} \tag{3.31}$$

Likewise, from (3.29) and (3.30) we obtain

$$\|u'(t)\|^2 \leqslant e^{-\lambda_1 t}\left(\frac{1}{c_0}\left\|\mathcal{A}u_0 + \frac{1}{2}\mathcal{B}u_1\right\|^2_{\mathcal{A}^{-1}} + \|u_1\|^2\right) \leqslant \frac{2}{c_0}\,e^{-\lambda_1 t}\left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right). \tag{3.32}$$

From (3.31) and (3.32) we obtain the following *a priori* estimate of the asymptotic stability:

$$\|u'(t)\|^2 + \|u(t)\|^2_{\mathcal{A}} \leqslant \frac{4}{c_0}\,e^{-\lambda_1 t}\left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right). \tag{3.33}$$

3.3.2. The case of $\mathcal{A} < \frac{1}{4}\mathcal{B}^2$. In the case where

$$\mathcal{A} < \frac{1}{4}\,\mathcal{B}^2\,, \tag{3.34}$$

we transform identity (3.26) to the following form:

$$\left(\left\|u' + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2\right)' + \left\|u' + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2_{\mathcal{B}-(\mathcal{B}^2-4\mathcal{A})^{1/2}} = 0\,.$$

Using inequality

$$\|u\|^2_{\mathcal{B}-(\mathcal{B}^2-4\mathcal{A})^{1/2}} \geqslant \mu_1\,\|u\|^2\,,$$

where $\mu_1 > 0$ is the minimum eigenvalue of the operator $\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}$, integrating the result in t, we obtain the *a priori* estimate

$$\left\|u'(t) + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u(t)\right\|^2 \leqslant e^{-\mu_1 t}\left\|u_1 + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u_0\right\|^2. \tag{3.35}$$

Similarly, from (3.26) we obtain the equality

$$\left(\left\|u' + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2\right)' + \left\|u' + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2_{\mathcal{B}+(\mathcal{B}^2-4\mathcal{A})^{1/2}} = 0.$$

The latter expression yields the following estimate:

$$\left\|u'(t) + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u(t)\right\|^2 \leqslant e^{-\widetilde{\mu}_1 t}\left\|u_1 + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u_0\right\|^2, \tag{3.36}$$

where $\widetilde{\mu}_1 > 0$ is the minimum eigenvalue of the operator $\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}$. In this case, $\widetilde{\mu}_1 > \mu_1 > 0$. Inequalities (3.35) and (3.36) express the asymptotic stability of the solution of equation (3.25) in the case where operator inequality (3.34) is fulfilled.

Assume that instead of (3.34) the following stronger inequality is fulfilled:

$$\frac{1}{4}\,\mathcal{B}^2 - \mathcal{A} \geqslant c_1\,\mathcal{B}^2\,, \qquad 0 < c_1 < \frac{1}{4}\,. \tag{3.37}$$

Then,

$$\begin{aligned}&\left\|u' + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2 + \left\|u' + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2\\ &\qquad = 2\left\|u' + \frac{1}{2}\,\mathcal{B}u\right\|^2 + 2\,\|u\|^2_{\frac{1}{4}\mathcal{B}^2-\mathcal{A}} \geqslant 2c_1\,\|\mathcal{B}u\|^2,\end{aligned}$$

$$\begin{aligned}&\left\|u' + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2 + \left\|u' + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2\\ &\qquad = \left\|(\mathcal{B}^2 - 2\mathcal{A})^{1/2}u + \mathcal{B}(\mathcal{B}^2 - 2\mathcal{A})^{-1/2}u'\right\|^2 + \|u'\|^2_{(\mathcal{B}^2-4\mathcal{A})(\mathcal{B}^2-2\mathcal{A})^{-1}} \geqslant 4c_1\,\|u'\|^2,\end{aligned}$$

$$\begin{aligned}&\left\|u' + \frac{1}{2}\left(\mathcal{B} + (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2 + \left\|u' + \frac{1}{2}\left(\mathcal{B} - (\mathcal{B}^2 - 4\mathcal{A})^{1/2}\right)u\right\|^2\\ &\qquad \leqslant \frac{\sqrt{5}+3}{2}\left(\|u'\|^2 + \|\mathcal{B}u\|^2\right).\end{aligned}$$

Taking into account the latter inequalities and inequalities (3.35) and (3.36), we obtain the following estimate of the asymptotic stability:

$$\|u'(t)\|^2 + \|\mathcal{B}u(t)\|^2 \leqslant \frac{2}{c_1}\,e^{-\mu_1 t}\left(\|u_1\|^2 + \|\mathcal{B}u_0\|^2\right). \tag{3.38}$$

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