ON THE RATE OF CONVERGENCE OF DIFFERENCE SCHEMES FOR THE POISSON EQUATION WITH DYNAMIC INTERFACE CONDITIONS

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Dedicated to Raytcho Lazarov on the occasion of his 60th birthday.

Abstract — The convergence of difference schemes for the two-dimensional weakly parabolic equation (elliptic equation with a dynamic interface condition) is studied. Estimates for the rate of convergence “almost” (except for the logarithmic factor) compatible with the smoothness of the differential problem solution in special discrete Sobolev norms are obtained.

2000 Mathematics Subject Classification: 65M15.

Keywords: weakly parabolic equation (elliptic equation with a dynamic interface condition), difference scheme, weak solution, rate of convergence.

1. Introduction

The modern theory of convergence of difference schemes for elliptic partial differential equations was started by A. A. Samarskii, R. D. Lazarov and V. L. Makarov [20]. According to [15], estimates for elliptic problems of the type

$$\|u - v\|_{W_p^k(\omega)} \leq Ch^{s-k} \|u\|_{W_p^s(\Omega)}, \quad k < s$$

are called estimates compatible with the smoothness of the solution u . Here v is a discrete solution, h is a discretization parameter, and $W_p^k(\omega)$ is a discrete Sobolev space. Such estimates can not be derived by the Taylor formula because the functions belong to Sobolev spaces and the Bramble-Hilbert lemma and its generalizations [2, 5] are often used.

The ideas of [20] have been further developed in [7] to obtain convergence rate estimates for parabolic and hyperbolic problems.

Interface problems model many phenomena in science and engineering and various forms of jump (interface) relations satisfied by the solution and its derivatives are known [6, 16]. In [3], a review of the recent results on numerical solutions of elliptic and parabolic two-dimensional interface problems is presented. In [11, 13, 23], the convergence of FDM for different elliptic, parabolic, and hyperbolic problems in which the solution is continuous on the interface curves, while the flow is discontinuous, has been studied.

In the present work, a weakly parabolic problem is considered i.e, a problem in which the solution satisfies an elliptic equation in two adjacent regions, and on the interface – a dynamic conjugation condition. The existence of a generalized solution for different assumptions of smoothness of the input data is proved in the next section. For the numerical solution the implicit difference scheme with an averaged right-hand side is analyzed in Section 3. The convergence of this scheme in special discrete Sobolev norms is proved in Section 4. Similar results for weighted difference schemes also hold.

2. Regularity of the continuous solution

Let $\Omega = (0, l_1) \times (0, l_2)$, $\Gamma = \partial\Omega$, $x_2^0 \in (0, l_2)$, $S = (0, l_1) \times \{x_2^0\}$, $\Omega^- = (0, l_1) \times (0, x_2^0)$ and $\Omega^+ = (0, l_1) \times (x_2^0, l_2)$. We denote $Q = \Omega \times (0, T)$, $Q^- = \Omega^- \times (0, T)$ and $Q^+ = \Omega^+ \times (0, T)$. We consider the initial boundary-value problem (IBVP) with the dynamic boundary condition

$$\begin{aligned} -\Delta u &= f(x, t), & x &= (x_1, x_2) \in \Omega^- \cup \Omega^+, & t &\in (0, T), \\ [u]_S &= 0, & \frac{\partial u}{\partial t} &= \left[\frac{\partial u}{\partial \nu} \right]_S + g(x, t), & x &\in S, & t &\in (0, T), \\ u(x, t) &= 0, & x &\in \Gamma, & t &\in (0, T), \\ u(x, 0) &= u_0(x), & x &\in S, \end{aligned} \quad (1)$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is the Laplace operator, $\partial/\partial \nu = \partial/\partial x_2$ is the normal derivative on S and $[u]_S = u(x_1, x_2^0 + 0, t) - u(x_1, x_2^0 - 0, t)$. A similar problem is considered in [21].

In a standard manner one obtains a weak form of IBVP (1)

$$\int_0^T \int_S \frac{\partial u}{\partial t} v \, dS dt + \int_0^T \int_{\Omega} \nabla u \cdot \nabla v \, dx dt = \int_0^T \int_{\Omega} f v \, dx dt + \int_0^T \int_S g v \, dS dt \quad \forall v \in \overset{\circ}{W}_2^{1,0}(Q),$$

where $\nabla u \cdot \nabla v = \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_1} + \frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_2}$ and $\overset{\circ}{W}_2^{1,0}(Q) = \{v \in W_2^{1,0}(Q) : v = 0 \text{ on } \Gamma \times (0, T)\}$.

The same weak form corresponds to the following IBVP:

$$\begin{aligned} \delta_S \frac{\partial u}{\partial t} - \Delta u &= \varphi(x, t), & (x, t) &\in Q, \\ u(x, t) &= 0, & (x, t) &\in \Gamma \times (0, T), \\ u(x, 0) &= u_0(x), & x &\in S, \end{aligned} \quad (2)$$

where $\delta_S(x) = \delta(x_2 - x_2^0)$ is the Dirac delta function concentrated on S , $\varphi(x, t) = f(x, t) + \delta_S(x) g(x, t)$, where the equality is considered in the sense of the theory of distributions [22]. In this sense, IBVPs (1) and (2) are equivalent. Further, problem (2) will be called a **weakly parabolic problem** since its parabolic nature appears only on $S \times (0, T)$.

The following abstract Cauchy problem correspond to the IBVP (2):

$$Bu' + Au = \varphi(t), \quad t \in (0, T); \quad Bu(0) = Bu_0, \quad (3)$$

where u_0 is a given element of the Hilbert space H , $\varphi(t)$ is given and $u(t)$ is the unknown function with values in H , A is unbounded a self-adjoint positively definite linear operator in H , and B is a self-adjoint nonnegative linear operator in H . We assume that the domain $D(A)$ of the operator A is dense in H and $D(A) \subset D(B) \subset H$. Also the operator inequality $A \geq B$ is supposed. The inner product and the norm in H are denoted by (u, v) and $\|u\|$, respectively. Problems of this type have been extensively investigated (see e.g. [4, 18, 24] and the references therein).

In a standard fashion we introduce the inner products $(u, v)_A = (Au, v)$, $(u, v)_{A^{-1}} = (A^{-1}u, v)$ and $(u, v)_B = (Bu, v)$ and the corresponding energy spaces H_A , $H_{A^{-1}}$ and H_B . We also introduce the Sobolev space $W_2^k(0, T; H)$ of functions which map $(0, T)$ into H [17]. Specially, by $\widetilde{W}_2^{1/2}(0, T; H)$ we denote the space of functions $u : (0, T) \rightarrow H$ with the norm

$$\|u\|_{\widetilde{W}_2^{1/2}(0, T; H)}^2 = \int_0^T \int_0^T \frac{\|u(t) - u(t')\|^2}{|t - t'|^2} dt dt' + \int_0^T \left(\frac{1}{t} + \frac{1}{T-t} \right) \|u(t)\|^2 dt.$$

The following assertion holds [12]:

Lemma 1. *For the solution of (3) the following a priori estimates hold:*

$$\int_0^T \|u(t)\|_A^2 dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|_B^2}{|t - t'|^2} dt dt' + \max_{t \in [0, T]} \|u(t)\|_B^2 \leq C \left(\|u_0\|_B^2 + \int_0^T \|\varphi(t)\|_{A^{-1}}^2 dt \right) \quad (4)$$

and

$$\int_0^T \|u(t)\|_B^2 dt + \max_{t \in [0, T]} \|Bu(t)\|_{A^{-1}}^2 \leq C \left(\|Bu_0\|_{A^{-1}}^2 + \int_0^T \|A^{-1} \varphi(t)\|_B^2 dt \right). \quad (5)$$

In the case where $\varphi(t) = B\psi'(t)$, we have

$$\begin{aligned} & \int_0^T \|u(t)\|_A^2 dt + \int_0^T \int_0^T \frac{\|u(t) - u(t')\|_B^2}{|t - t'|^2} dt dt' + \max_{t \in [0, T]} \|u(t)\|_B^2 \\ & \leq C \left\{ \|u_0\|_B^2 + \int_0^T \int_0^T \frac{\|\psi(t) - \psi(t')\|_B^2}{|t - t'|^2} dt dt' + \int_0^T \left(\frac{1}{t} + \frac{1}{T-t} \right) \|\psi(t)\|_B^2 dt \right\} \end{aligned} \quad (6)$$

and

$$\int_0^T \|u(t)\|_B^2 dt + \max_{t \in [0, T]} \|Bu(t)\|_{A^{-1}}^2 \leq C \left(\|Bu_0\|_{A^{-1}}^2 + \max_{t \in [0, T]} \|B\psi(t)\|_{A^{-1}}^2 + \int_0^T \|\psi(t)\|_B^2 dt \right).$$

In problem (2) we have: $H = L_2(\Omega)$, $Au = -\Delta u$, $Bu = \delta_S u$, $H_A = \overset{\circ}{W}_2^1(\Omega)$, $H_B = L_2(S)$ and

$$\begin{aligned} \|u\|_A^2 &= \int_{\Omega} \left[\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 \right] dx \asymp \|u\|_{\overset{\circ}{W}_2^1(\Omega)}^2, \quad u \in \overset{\circ}{W}_2^1(\Omega), \\ \|u\|_B^2 &= \int_S u^2 dS = \|u\|_{L_2(S)}^2. \end{aligned}$$

Therefore, $H_{A^{-1}} = W_2^{-1}(\Omega)$ and $\|u\|_{A^{-1}} \asymp \|u\|_{W_2^{-1}(\Omega)}$. By $\widetilde{W}_2^{1/2}(S)$ we denote the space of functions $u(x_1, x_2^0)$ defined on S , with the norm

$$\|u\|_{\widetilde{W}_2^{1/2}(S)}^2 = \int_0^{l_1} \int_0^{l_1} \left[\frac{u(x_1, x_2^0) - u(x'_1, x_2^0)}{x_1 - x'_1} \right]^2 dx_1 dx'_1 + \int_0^{l_1} \left(\frac{1}{x_1} + \frac{1}{l_1 - x_1} \right) u^2(x_1, x_2^0) dx_1$$

and by $\widetilde{W}_2^{-1/2}(S)$ – its dual space. As is well known, the inner product $(u, v)_{L_2(S)}$ can be continuously extended to the map $(u, v)_{\widetilde{W}_2^{-1/2}(S) \times \widetilde{W}_2^{1/2}(S)}$ from $\widetilde{W}_2^{-1/2}(S) \times \widetilde{W}_2^{1/2}(S)$ into R , and the norm in $\widetilde{W}_2^{-1/2}(S)$ is defined as follows:

$$\|u\|_{\widetilde{W}_2^{-1/2}(S)} = \sup_{v \in \widetilde{W}_2^{1/2}(S)} \frac{|(u, v)_{\widetilde{W}_2^{-1/2}(S) \times \widetilde{W}_2^{1/2}(S)}|}{\|v\|_{\widetilde{W}_2^{1/2}(S)}}.$$

From the results of [1] it follows that

$$\|u\|_{\widetilde{W}_2^{1/2}(S)} \asymp \inf_{v \in \overset{\circ}{W}_2^1(\Omega), v|_S = u} \|v\|_{W_2^1(\Omega)}.$$

Also, it is easy to prove that

$$\|Bu\|_{A^{-1}} \asymp \|u\|_{\widetilde{W}_2^{-1/2}(S)}.$$

In this way, the a priori estimates (4) and (5) in the case of the IBVP (2), take the form

$$\begin{aligned} &\int_0^T \|u(\cdot, t)\|_{W_2^1(\Omega)}^2 dt + \int_0^T \int_0^T \frac{\|u(\cdot, t) - u(\cdot, t')\|_{L_2(S)}^2}{|t - t'|^2} dt dt' + \max_{t \in [0, T]} \|u(\cdot, t)\|_{L_2(S)}^2 \\ &\leq C \left(\|u_0\|_{L_2(S)}^2 + \int_0^T \|f(\cdot, t)\|_{W_2^{-1}(\Omega)}^2 dt + \int_0^T \|g(\cdot, t)\|_{\widetilde{W}_2^{-1/2}(S)}^2 dt \right) \end{aligned} \quad (7)$$

and

$$\int_0^T \|u(\cdot, t)\|_{L_2(S)}^2 dt + \max_{t \in [0, T]} \|u(\cdot, t)\|_{\widetilde{W}_2^{-1/2}(S)}^2 \leq C \left(\|u_0\|_{\widetilde{W}_2^{-1/2}(S)}^2 + \int_0^T \|A^{-1} \varphi(\cdot, t)\|_{L_2(S)}^2 dt \right).$$

Remark 1. The proof of Lemma 1 can be found in [8] (for $B = I = \text{identity operator}$) and [12] (for $B > 0$). The Fourier expansion is used to obtain estimates (4) and (6). Estimates of

these types for the parabolic PDEs obtained using the Fourier transform are given in [14]. In our case, the operator B is only non-negative and (4) leads to the nonstandard a priori estimate (7). Note that the functional space containing the generalized solution of the IBVP (1) is determined by (7): if $u_0 \in L_2(S)$, $f \in L_2(0, T; W_2^{-1}(\Omega))$ and $g \in L_2(0, T; \widetilde{W}_2^{-1/2}(S))$, then $u \in L_2(0, T; W_2^1(\Omega)) \cap W_2^{1/2}(0, T; L_2(S))$.

Using Lemma 1 and equation (1), we get the following assertion:

Lemma 2. *If $f \in L_2(Q)$, $g \in L_2(0, T; \widetilde{W}_2^{-1/2}(S))$ and $u_0 \in \overset{\circ}{W}_2^1(S)$, then the solution of the IBVP (2) belongs to the space $L_2(0, T; W_2^2(\Omega^-)) \cap L_2(0, T; W_2^2(\Omega^+)) \cap W_2^1(0, T; L_2(S))$ and the following a priori estimate is fulfilled:*

$$\begin{aligned} & \int_0^T \left(\left\| \frac{\partial^2 u}{\partial x_1^2}(\cdot, t) \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2}(\cdot, t) \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial^2 u}{\partial x_2^2}(\cdot, t) \right\|_{L_2(\Omega^-)}^2 \right. \\ & \quad \left. + \left\| \frac{\partial^2 u}{\partial x_2^2}(\cdot, t) \right\|_{L_2(\Omega^+)}^2 + \left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|_{L_2(S)}^2 \right) dt + \max_{t \in [0, T]} \|u(\cdot, t)\|_{W_2^1(S)}^2 \\ & \leq C \left(\|u_0\|_{W_2^1(S)}^2 + \int_0^T \|f(\cdot, t)\|_{L_2(\Omega)}^2 dt + \int_0^T \|g(\cdot, t)\|_{\widetilde{W}_2^{-1/2}(S)}^2 dt \right). \end{aligned}$$

If $f \in L_2(0, T; W_2^1(\Omega))$, $f = 0$ at $x_1 = 0$ and $x_1 = l_1$, $g \in L_2(0, T; W_2^{3/2}(S))$ and $u_0 \in W_2^2(S) \cap W_2^1(S)$, then the solution of the IBVP (1) belongs to the space $L_2(0, T; W_2^3(\Omega^-)) \cap L_2(0, T; W_2^3(\Omega^+))$, $\frac{\partial u}{\partial x_1} \in W_2^1(0, T; L_2(S)) \cap W_2^{1/2}(0, T; W_2^1(S))$ and the following estimate holds:

$$\begin{aligned} & \int_0^T \left(\left\| \frac{\partial^3 u}{\partial x_1^3}(\cdot, t) \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2}(\cdot, t) \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2}(\cdot, t) \right\|_{L_2(\Omega^-)}^2 + \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2}(\cdot, t) \right\|_{L_2(\Omega^+)}^2 \right. \\ & \quad \left. + \left\| \frac{\partial^3 u}{\partial x_2^3}(\cdot, t) \right\|_{L_2(\Omega^-)}^2 + \left\| \frac{\partial^3 u}{\partial x_2^3}(\cdot, t) \right\|_{L_2(\Omega^+)}^2 + \left\| \frac{\partial^2 u}{\partial t \partial x_1}(\cdot, t) \right\|_{L_2(S)}^2 \right) dt \\ & \quad + \int_0^T \int_0^T \frac{\left\| \frac{\partial^2 u}{\partial x_1^2}(\cdot, t) - \frac{\partial^2 u}{\partial x_1^2}(\cdot, t') \right\|_{L_2(S)}^2}{|t - t'|^2} dt dt' + \max_{t \in [0, T]} \|u(\cdot, t)\|_{W_2^2(S)}^2 \\ & \leq C \left(\|u_0\|_{W_2^2(S)}^2 + \int_0^T \|f(\cdot, t)\|_{W_2^1(\Omega)}^2 dt + \int_0^T \|g(\cdot, t)\|_{W_2^{3/2}(S)}^2 dt \right). \end{aligned}$$

If, in addition, $\frac{\partial f}{\partial t} \in L_2(0, T; W_2^{-1}(\Omega))$, $f(\cdot, 0) \in L_2(\Omega)$, $g \in \widetilde{W}_2^{-1/2}(0, T; L_2(S))$ and $g(\cdot, 0) =$

0, then $\frac{\partial u}{\partial t} \in L_2(0, T; W_2^1(\Omega)) \cap W_2^{1/2}(0, T; L_2(S))$ and the following estimate holds:

$$\begin{aligned} & \int_0^T \left(\left\| \frac{\partial^2 u}{\partial t \partial x_1}(\cdot, t) \right\|_{L_2(\Omega)}^2 + \left\| \frac{\partial^2 u}{\partial t \partial x_2}(\cdot, t) \right\|_{L_2(\Omega)}^2 \right) dt + \int_0^T \int_0^T \frac{\left\| \frac{\partial u}{\partial t}(\cdot, t) - \frac{\partial u}{\partial t}(\cdot, t') \right\|_{L_2(S)}^2}{|t - t'|^2} dt dt' \\ & + \max_{t \in [0, T]} \left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|_{L_2(S)}^2 \leq C \left[\|u_0\|_{W_2^2(S)}^2 + \|f(\cdot, 0)\|_{L_2(\Omega)}^2 + \int_0^T \left\| \frac{\partial f}{\partial t}(\cdot, t) \right\|_{W_2^{-1}(\Omega)}^2 dt \right. \\ & \left. + \int_0^T \int_0^T \frac{\|g(\cdot, t) - g(\cdot, t')\|_{L_2(S)}^2}{|t - t'|^2} dt dt' + \int_0^T \left(\frac{1}{t} + \frac{1}{T-t} \right) \|g(\cdot, t)\|_{L_2(S)}^2 dt \right]. \end{aligned}$$

3. Difference approximation

Let ω_i be a uniform mesh with a step $h_i = l_i/N_i$ in $(0, l_i)$, $i = 1, 2$. For simplicity, we suppose that x_2^0 is a mesh point of ω_2 . Suppose that $\omega_i^- = \omega_i \cup \{0\}$, $\bar{\omega}_i = \omega_i \cup \{0, l_i\}$, $\omega_h = \omega_1 \times \omega_2$, $\bar{\omega}_h = \bar{\omega}_1 \times \bar{\omega}_2$, $\gamma_h = \bar{\omega}_h \cap \Gamma$, $\sigma_h = \omega_h \cap S$ and $\sigma_h^- = \sigma_h \cup \{(0, x_2^0)\}$. Let ω_τ be a uniform mesh with a step $\tau = T/N$ in $(0, T)$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$ and $\bar{\omega}_\tau = \omega_\tau \cup \{0, T\}$.

Further, we shall use the standard notations [19]

$$\begin{aligned} v_{x_i} &= \frac{v(x + h_i e_i, t) - v(x, t)}{h_i}, \quad v_{\bar{x}_i} = \frac{v(x, t) - v(x - h_i e_i, t)}{h_i}, \quad e_1 = (1, 0), \quad e_2 = (0, 1), \\ v_t &= \frac{v(x, t + \tau) - v(x, t)}{\tau}, \quad v_{\bar{t}} = \frac{v(x, t) - v(x, t - \tau)}{\tau}. \end{aligned}$$

We shall approximate the IBVP (2) on the mesh $\bar{Q}_{h\tau} = \bar{\omega}_1 \times \bar{\omega}_2 \times \bar{\omega}_\tau$ by the implicit difference scheme with the averaged right-hand side

$$\begin{aligned} \delta_{\sigma_h} v_{\bar{t}} - \Delta_h v &= T_1^2 T_2^2 T_t^- \varphi, \quad (x, t) \in \omega_h \times \omega_\tau^+, \\ v(x, t) &= 0, \quad (x, t) \in \gamma_h \times \omega_\tau^+, \\ v(x, 0) &= u_0(x), \quad x \in \sigma_h, \end{aligned} \tag{8}$$

where $\Delta_h v = v_{\bar{x}_1 x_1} + v_{\bar{x}_2 x_2}$ is the difference Laplace operator,

$$\delta_{\sigma_h}(x) = \delta_{h_2}(x_2 - x_2^0) = \begin{cases} 0, & x \in \omega_h \setminus \sigma_h, \\ 1/h_2, & x \in \sigma_h \end{cases}$$

is the difference Dirac delta function, and T_i , T_t^- are the Steklov averaging operators [15]:

$$\begin{aligned} T_1 f(x_1, x_2, t) &= T_1^- f(x_1 + h_1/2, x_2, t) = T_1^+ f(x_1 - h_1/2, x_2, t) = \frac{1}{h_1} \int_{x_1 - h_1/2}^{x_1 + h_1/2} f(x'_1, x_2, t) dx'_1, \\ T_2 f(x_1, x_2, t) &= T_2^- f(x_1, x_2 + h_2/2, t) = T_2^+ f(x_1, x_2 - h_2/2, t) = \frac{1}{h_2} \int_{x_2 - h_2/2}^{x_2 + h_2/2} f(x_1, x'_2, t) dx'_2, \\ T_t^- f(x_1, x_2, t) &= T_t^+ f(x_1, x_2, t - \tau) = \frac{1}{\tau} \int_{t-\tau}^t f(x_1, x_2, t') dt'. \end{aligned}$$

Note that these operators are self-comutative and transform the derivatives to difference relations, for example,

$$T_i^- \frac{\partial u}{\partial x_i} = u_{\bar{x}_i}, \quad T_i^2 \frac{\partial^2 u}{\partial x_i^2} = u_{x_i \bar{x}_i}, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}}. \quad (9)$$

Let us consider the operator-difference scheme of the form

$$\begin{aligned} B_h v_{\bar{t}} + A_h v &= \varphi(t), \quad t \in \omega_\tau^+, \\ B_h v(0) &= B_h v_0, \end{aligned} \quad (10)$$

which corresponds to problem (8). Here v_0 is a given element in the finite-dimensional Hilbert space H_h , $\varphi(t)$ is a given function and $v(t)$ is the unknown function with values in H_h , A_h is linear operator in H_h , B_h is a non-negative self-adjoint linear operator in H_h , and $A_h \geq B_h$. The inner product and the norm in H_h are denoted by $(v, w)_h$ and $\|v\|_h$.

The following analog of Lemma 1 holds [12]:

Lemma 3. *For the solution of the problem (10) the following a priori estimates hold*

$$\begin{aligned} & \tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{A_h}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(t) - v(t')\|_{B_h}^2}{|t - t'|^2} + \max_{t \in \omega_\tau^+} \|v(t)\|_{B_h}^2 \\ & \leq C \left(\|v_0\|_{B_h}^2 + \tau \|v_0\|_{A_h}^2 + \tau \sum_{t \in \omega_\tau^+} \|\varphi(t)\|_{A_h^{-1}}^2 \right), \\ & \tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{B_h}^2 + \max_{t \in \omega_\tau^+} \|B_h v(t)\|_{A_h^{-1}}^2 \leq C \left(\|B_h v_0\|_{A_h^{-1}}^2 + \tau \|v_0\|_{B_h}^2 + \tau \sum_{t \in \omega_\tau^+} \|A_h^{-1} \varphi(t)\|_{B_h}^2 \right). \end{aligned}$$

In the case of $\varphi = B_h \psi_{\bar{t}}$ the following a priori estimates hold

$$\begin{aligned} & \tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{A_h}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|v(t) - v(t')\|_{B_h}^2}{|t - t'|^2} + \max_{t \in \omega_\tau^+} \|v(t)\|_{B_h}^2 \\ & \leq C \left\{ \|v_0\|_{B_h}^2 + \tau \|v_0\|_{A_h}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|\psi(t) - \psi(t')\|_{B_h}^2}{|t - t'|^2} + \tau \sum_{t \in \omega_\tau} \left(\frac{1}{t} + \frac{1}{T - t} \right) \|\psi(t)\|_{B_h}^2 \right\}, \\ & \tau \sum_{t \in \omega_\tau^+} \|v(t)\|_{B_h}^2 + \max_{t \in \omega_\tau^+} \|B_h v(t)\|_{A_h^{-1}}^2 \leq C \left(\|B_h v_0\|_{A_h^{-1}}^2 + \max_{t \in \bar{\omega}_\tau} \|B_h \psi(t)\|_{A_h^{-1}}^2 + \tau \sum_{t \in \bar{\omega}_\tau} \|\psi(t)\|_{B_h}^2 \right). \end{aligned}$$

Let H_h be the space of mesh functions defined on ω_h , equal to zero on γ_h , endowed with the inner product and the norm

$$(v, w)_h = h_1 h_2 \sum_{x \in \omega_h} v(x) w(x), \quad \|v\|_h = \|v\|_{L_2(\omega_h)} = (v, v)_h^{1/2}.$$

We also denote

$$(v, w)_{i,h} = h_1 h_2 \sum_{x_i \in \omega_i^-, x_{3-i} \in \omega_{3-i}} v(x) w(x), \quad \|v\|_{i,h} = (v, v)_{i,h}^{1/2},$$

$$\|v\|_{W_2^1(\omega_h)}^2 = \|v_{x_1}\|_{1,h}^2 + \|v_{x_2}\|_{2,h}^2 + \|v\|_h^2$$

and

$$(v, w)_{\sigma_h} = h_1 \sum_{x \in \sigma_h} v(x) w(x), \quad \|v\|_{\sigma_h} = \|v\|_{L_2(\sigma_h)} = (v, v)_{\sigma_h}^{1/2}.$$

Problem (8) can be reduced to form (10) if one sets $A_h v = -\Delta_h v$ and $B_h v = \delta_{\sigma_h} v$. Then

$$\begin{aligned} \|v\|_{A_h}^2 &= \|v_{x_1}\|_{1,h}^2 + \|v_{x_2}\|_{2,h}^2 \asymp \|v\|_{W_2^1(\omega_h)}^2, \quad v \in H_h, \\ \|v\|_{B_h} &= \|v\|_{L_2(\sigma_h)}. \end{aligned}$$

Further we shall make use of the assertion:

Lemma 4. *Let $v \in H_h$ and w be a mesh function defined on σ_h^- . Then*

$$|(w_{\bar{x}_1}, v)_{\sigma_h}| \leq C \|v\|_{W_2^1(\omega_h)} \left\{ h_1^2 \sum_{x \in \sigma_h^-} \sum_{x' \in \sigma_h^-, x' \neq x} \frac{|w(x) - w(x')|^2}{|x_1 - x'_1|^2} \right\}^{1/2}.$$

The proof is analogous to the proof of Lemma 2 in [10].

4. Convergence of the difference approximation

Let u be the solution of the IBVP (2) and v – the solution of the difference scheme (8). The error $z = u - v$ satisfies the difference problem

$$\begin{aligned} \delta_{\sigma_h} z_{\bar{t}} - \Delta_h z &= \phi, \quad (x, t) \in \omega_h \times \omega_{\tau}^+, \\ z(x, t) &= 0, \quad (x, t) \in \gamma_h \times \omega_{\tau}^+, \\ z(x, 0) &= 0, \quad x \in \sigma_h, \end{aligned} \tag{11}$$

where

$$\phi = \delta_{\sigma_h} \xi_{\bar{t}} + \eta_{1, \bar{x}_1 x_1} + \eta_{2, \bar{x}_2 x_2}, \quad \xi = u - T_1^2 u, \quad \eta_i = T_{3-i}^2 T_t^- u - u.$$

Let us set

$$\zeta = \frac{h_2^2}{6} \left[T_t^- \frac{\partial u}{\partial x_2} \right]_S, \quad x \in \sigma_h; \quad \tilde{\eta}_1 = \eta_1 - \delta_{\sigma_h} \zeta.$$

We rewrite the difference scheme (11) as the operator difference scheme

$$\begin{aligned} B_h z_{\bar{t}} + A_h z &= \phi(t), \quad t \in \omega_{\tau}^+, \\ B_h z(0) &= 0, \end{aligned}$$

where $A_h z = -\Delta_h z$, $B_h z = \delta_{\sigma_h} z$, $\phi = \delta_{\sigma_h} \xi_{\bar{t}} + \tilde{\eta}_{1, \bar{x}_1 x_1} + \eta_{2, \bar{x}_2 x_2} + \delta_{\sigma_h} \zeta_{\bar{x}_1 x_1}$.

Using Lemmas 3 and 4, we get the a priori estimate

$$\begin{aligned} &\tau \sum_{t \in \omega_{\tau}^+} \|z(\cdot, t)\|_{W_2^1(\omega_h)}^2 + \tau^2 \sum_{t \in \bar{\omega}_{\tau}} \sum_{t' \in \bar{\omega}_{\tau}, t' \neq t} \frac{\|z(\cdot, t) - z(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2} + \max_{t \in \omega_{\tau}^+} \|z(\cdot, t)\|_{L_2(\sigma_h)}^2 \\ &\leq C \left\{ \tau \sum_{t \in \omega_{\tau}^+} \left(\|\tilde{\eta}_{1, x_1}(\cdot, t)\|_{1,h}^2 + \|\eta_{2, x_2}(\cdot, t)\|_{2,h}^2 \right) \right. \\ &\quad + \tau \sum_{t \in \omega_{\tau}^+} h_1^2 \sum_{x \in \sigma_h^-} \sum_{x' \in \sigma_h^-, x' \neq x} \frac{|\zeta_{x_1}(x, t) - \zeta_{x_1}(x', t)|^2}{|x_1 - x'_1|^2} \\ &\quad \left. + \tau^2 \sum_{t \in \bar{\omega}_{\tau}} \sum_{t' \in \bar{\omega}_{\tau}, t' \neq t} \frac{\|\xi(\cdot, t) - \xi(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2} + \tau \sum_{t \in \omega_{\tau}} \left(\frac{1}{t} + \frac{1}{T-t} \right) \|\xi(\cdot, t)\|_{L_2(\sigma_h)}^2 \right\}. \end{aligned} \tag{12}$$

Therefore, in order to estimate the rate of convergence of the difference scheme (8), it is sufficient to estimate the norms of ξ , $\tilde{\eta}_1$, η_2 and ζ on the right-hand side of (12).

We begin with η_2 . Letting

$$\eta_2 = \eta_{21} + \eta_{22} + \eta_{23} = (T_1^2 T_t^- u - T_t^- u) + (T_1^2 T_t^- u - T_1^2 u) + (T_t^- u - u - T_1^2 T_t^- u + T_1^2 u)$$

and using (9), it is easy to find the integral representations

$$\eta_{21,x_2} = \frac{1}{h_1 h_2 \tau} \int_{x_1-h_1}^{x_1+h_1} \int_{x_1}^{x_1'} \int_{x_1}^{x_1''} \int_{x_2}^{x_2+h_2} \int_{t-\tau}^t \left(1 - \frac{|x_1' - x_1|}{h_1}\right) \frac{\partial^3 u}{\partial x_1^2 \partial x_2} (x_1''', x_2', t') dt' dx_2' dx_1''' dx_1'' dx_1'$$

and

$$\eta_{22,x_2} = \frac{1}{h_1 h_2 \tau} \int_{x_1-h_1}^{x_1+h_1} \int_{x_2}^{x_2+h_2} \int_{t-\tau}^t \int_t^{t'} \left(1 - \frac{|x_1' - x_1|}{h_1}\right) \frac{\partial^2 u}{\partial t \partial x_2} (x_1', x_2', t'') dt'' dt' dx_2' dx_1'.$$

Hence

$$\left(\tau \sum_{t \in \omega_\tau^+} \|\eta_{21,x_2}\|_{2,h}^2 \right)^{1/2} \leq C h_1^2 \left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L_2(Q)}, \quad \left(\tau \sum_{t \in \omega_\tau^+} \|\eta_{22,x_2}\|_{2,h}^2 \right)^{1/2} \leq C \tau \left\| \frac{\partial^3 u}{\partial t \partial x_2} \right\|_{L_2(Q)}.$$

Using the technique from [9], we get

$$\begin{aligned} \left(\tau \sum_{t \in \omega_\tau^+} \|\eta_{23,x_2}\|_{2,h}^2 \right)^{1/2} &\leq C (h_1^2 + \tau) \left(h_2 \sum_{x_2 \in \omega_2^-} \left\| T_2^+ \frac{\partial u}{\partial x_2} (\cdot, x_2, \cdot) \right\|_{W_2^{2,1}((0,l_1) \times (0,T))}^2 \right)^{1/2} \\ &\leq C (h_1^2 + \tau) \left(\left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial^2 u}{\partial t \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_2} \right\|_{L_2(Q)} \right). \end{aligned}$$

The last three estimates give

$$\left(\tau \sum_{t \in \omega_\tau^+} \|\eta_{2,x_2}\|_{2,h}^2 \right)^{1/2} \leq C (h_1^2 + \tau) \left(\left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial^2 u}{\partial t \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_2} \right\|_{L_2(Q)} \right). \quad (13)$$

In a similar way, we let $\tilde{\eta}_1 = \tilde{\eta}_{11} + \tilde{\eta}_{12} + \tilde{\eta}_{13}$, where

$$\begin{aligned} \tilde{\eta}_{11} &= \begin{cases} T_2^2 T_t^- u - T_t^- u, & x \in \omega_h \setminus \sigma_h, \\ 0.5(\tilde{\eta}_{11}^+ + \tilde{\eta}_{11}^-), & x \in \sigma_h, \end{cases} & \tilde{\eta}_{11}^\pm &= T_{2\pm}^2 T_t^- u - T_t^- u \mp \frac{h_2}{3} T_t^- \frac{\partial u}{\partial x_2} \Big|_{x_2=x_2^0 \pm 0}, \\ \tilde{\eta}_{12} &= T_2^2 T_t^- u - T_2^2 u, & x &\in \omega_h, \\ \tilde{\eta}_{13} &= \begin{cases} T_t^- u - u - T_2^2 T_t^- u + T_2^2 u, & x \in \omega_h \setminus \sigma_h, \\ 0.5(\tilde{\eta}_{13}^+ + \tilde{\eta}_{13}^-), & x \in \sigma_h, \end{cases} & \tilde{\eta}_{13}^\pm &= T_t^- u - u - T_{2\pm}^2 T_t^- u + T_{2\pm}^2 u \end{aligned}$$

and

$$T_{2-}^2 u = \frac{2}{h_2} \int_{x_2^0-h_2}^{x_2^0} \left(1 + \frac{x_2' - x_2^0}{h_2}\right) u(x_1, x_2', t) dx_2', \quad T_{2+}^2 u = \frac{2}{h_2} \int_{x_2^0}^{x_2^0+h_2} \left(1 - \frac{x_2' - x_2^0}{h_2}\right) u(x_1, x_2', t) dx_2'.$$

When $x \in \omega_h \setminus \sigma_h$ the expressions $\tilde{\eta}_{1j}$ can be estimated in a similar way as their corresponding η_{2j} ($j = 1, 2, 3$). When $x \in \sigma_h$ the expression $\tilde{\eta}_{13}^\pm$ can be estimated as $\tilde{\eta}_{13}$ when $x \in \omega_h \setminus \sigma_h$. Further,

$$\tilde{\eta}_{11,x_1}^+ = \frac{1}{h_1 h_2 \tau} \int_{x_1}^{x_1+h_1} \int_{x_2^0}^{x_2^0+h_2} \int_{x_2^0}^{x_2^0} \int_{x_2^0}^{x_2^0} \int_{t-\tau}^t \left(1 - \frac{x_2' - x_2}{h_2}\right) \frac{\partial^3 u}{\partial x_1 \partial x_2^2}(x_1', x_2''', t') dt' dx_2''' dx_2'' dx_2' dx_1'$$

and an analogous representation holds for $\tilde{\eta}_{11,x_1}^-$. Reasoning as in the last case, we get the estimate

$$\begin{aligned} \left(\tau \sum_{t \in \omega_\tau^+} \|\tilde{\eta}_{11,x_1}\|_2^2 \right)^{1/2} &\leq C (h_2^2 + \tau) \left(\left\| \frac{\partial^3 u}{\partial x_2^2 \partial x_1} \right\|_{L_2(Q^-)} + \left\| \frac{\partial^3 u}{\partial x_2^2 \partial x_1} \right\|_{L_2(Q^+)} \right. \\ &\quad \left. + \left\| \frac{\partial^2 u}{\partial t \partial x_1} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L_2(Q)} \right). \end{aligned} \quad (14)$$

The integral representation

$$\zeta_{x_1} = \frac{h_2^2}{6 h_1 \tau} \int_{t-\tau}^t \int_{x_1}^{x_1+h_1} \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1'', x_2^0 + 0, t) - \frac{\partial^2 u}{\partial x_1 \partial x_2}(x_1'', x_2^0 - 0, t) \right) dx_1'' dt'$$

and the imbedding theorem imply

$$\begin{aligned} \left\{ \tau \sum_{t \in \omega_\tau^+} h_1^2 \sum_{x \in \sigma_h^-} \sum_{x' \in \sigma_h^-, x' \neq x} \frac{|\zeta_{x_1}(x, t) - \zeta_{x_1}(x', t)|^2}{|x_1 - x_1'|^2} \right\}^{1/2} &\leq \frac{h_2^2}{6} \sqrt{\frac{7}{6}} \left\| \left[\frac{\partial^2 u}{\partial x_1 \partial x_2} \right]_S \right\|_{L_2(0,T;W_2^{1/2}(S))} \\ &\leq C h_2^2 \left(\left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L_2(0,T;W_2^1(\Omega^-))} + \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L_2(0,T;W_2^1(\Omega^+))} \right) \\ &\leq C h_2^2 \left(\left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L_2(Q^-)} + \left\| \frac{\partial^3 u}{\partial x_1 \partial x_2^2} \right\|_{L_2(Q^+)} + \left\| \frac{\partial^2 u}{\partial x_1 \partial x_2} \right\|_{L_2(Q)} \right). \end{aligned} \quad (15)$$

The terms containing the expression ξ can be estimated as in [12]. Therefore, we get

$$\left\{ \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|\xi(\cdot, t) - \xi(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2} \right\}^{1/2} \leq C (h_1^2 + \tau) \|u\|_{W_2^{3,3/2}(S \times (0,T))} \quad (16)$$

and

$$\left\{ \tau \sum_{t \in \omega_\tau} \left(\frac{1}{t} + \frac{1}{T-t} \right) \|\xi(\cdot, t)\|_{L_2(\sigma_h)}^2 \right\}^{1/2} \leq C h_1^2 \sqrt{\ln 1/\tau} \|u\|_{W_2^{3,3/2}(S \times (0,T))}. \quad (17)$$

Now, from (12)–(17) we obtain the desired estimate for the convergence rate of the difference scheme (8):

$$\begin{aligned} &\left\{ \tau \sum_{t \in \omega_\tau^+} \|z(\cdot, t)\|_{W_2^1(\omega_h)}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|z(\cdot, t) - z(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2} + \max_{t \in \omega_\tau^+} \|z(\cdot, t)\|_{L_2(\sigma_h)}^2 \right\}^{1/2} \\ &\leq C (h_1^2 + h_2^2 + \tau) \sqrt{\ln 1/\tau} \left(\left\| \frac{\partial^3 u}{\partial x_1^2 \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial^3 u}{\partial x_2^2 \partial x_1} \right\|_{L_2(Q^-)} + \left\| \frac{\partial^3 u}{\partial x_2^2 \partial x_1} \right\|_{L_2(Q^+)} \right. \\ &\quad \left. + \left\| \frac{\partial^2 u}{\partial t \partial x_1} \right\|_{L_2(Q)} + \left\| \frac{\partial^2 u}{\partial t \partial x_2} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_1} \right\|_{L_2(Q)} + \left\| \frac{\partial u}{\partial x_2} \right\|_{L_2(Q)} + \|u\|_{W_2^{3,3/2}(S \times (0,T))} \right). \end{aligned}$$

Thus, in view of Lemma 2, we have proved the main result of the paper.

Theorem 1. *Let $z = u - v$ be the error and the conditions of Lemma 2 are fulfilled. Then the difference scheme (8) converges and the following estimate holds:*

$$\left\{ \tau \sum_{t \in \omega_\tau^+} \|z(\cdot, t)\|_{W_2^1(\omega_h)}^2 + \tau^2 \sum_{t \in \bar{\omega}_\tau} \sum_{t' \in \bar{\omega}_\tau, t' \neq t} \frac{\|z(\cdot, t) - z(\cdot, t')\|_{L_2(\sigma_h)}^2}{|t - t'|^2} + \max_{t \in \omega_\tau^+} \|z(\cdot, t)\|_{L_2(\sigma_h)}^2 \right\}^{1/2} \\ \leq C (h_1^2 + h_2^2 + \tau) \sqrt{\ln 1/\tau} \left(\|u_0\|_{W_2^2(S)} + \|f(\cdot, 0)\|_{L_2(\Omega)} + \|f\|_{L_2(0, T; W_2^1(\Omega))} \right. \\ \left. + \|f\|_{W_2^1(0, T; W_2^{-1}(\Omega))} + \|g\|_{L_2(0, T; W_2^{3/2}(S))} + \|g\|_{\widetilde{W}_2^{1/2}(0, T; L_2(S))} \right).$$

Remark 2. An analogous result holds for the weighted scheme

$$\begin{aligned} \delta_{\sigma_h} v_{\bar{t}} - \Delta_h v^{(\kappa)} &= T_1^2 T_2^2 T_t^- \varphi, & (x, t) &\in \omega_h \times \omega_\tau^+, \\ v(x, t) &= 0, & (x, t) &\in \gamma_h \times \omega_\tau^+, \\ v(x, 0) &= u_0(x), & x &\in \sigma_h, \end{aligned}$$

where $v^{(\kappa)} = \kappa v(x, t) + (1 - \kappa) v(x, t - \tau)$. The condition $\kappa \geq 1/2$ is assumed.

Remark 3. In the case of a lower smoothness of the input data ($u_0 \in \overset{\circ}{W}_2^1(S)$, $f \in L_2(Q)$, $g \in L_2(0, T; \widetilde{W}_2^{1/2}(S))$), one can prove the convergence of the difference scheme (8) in the weaker norm

$$\left\{ \tau \sum_{t \in \omega_\tau^+} \|z(t)\|_{B_h}^2 + \max_{t \in \omega_\tau^+} \|B_h z(t)\|_{A_h^{-1}}^2 \right\}^{1/2}.$$

Acknowledgement: The research of the first author was supported by the MSTD of the Republic of Serbia under grant 1840 and the research of the second author was supported by the University of Rouse under grant 2002-PF-04.

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Received 10 Jan. 2003

Revised 24 Feb. 2003