D-GAP FUNCTIONS FOR A CLASS OF EQUILIBRIUM PROBLEMS IN BANACH SPACES ¹

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Abstract — We consider rather a general class of equilibrium problems in a real Banach space, which involve nonsmooth convex functions. We apply the *D*-gap function approach to these problems and show that, under certain additional assumptions, they can be converted into a problem of finding a stationary point of a differentiable function. Based on this property, we suggest a descent type algorithm to find a solution to the initial problem. An example of applications to nonlinearly constrained equilibrium problems is also given.

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1. Introduction

Let U be a nonempty closed and convex subset of a real Banach space E and $\Phi : U \times U \rightarrow R$ an equilibrium bifunction, i.e., $\Phi(u, u) = 0$ for every $u \in U$. Then one can define the *equilibrium problem* (EP), that is, to find an element $u^* \in U$ such that

$$\Phi(u^*, v) \geq 0 \quad \forall v \in U. \quad (1)$$

It is well known that equilibrium problems represent rather a general and suitable format for the formulation and investigation of various complex problems arising in economics, mathematical physics, transportation, operations research, and other fields. Moreover, they are closely related to other general problems of nonlinear analysis such as fixed point, optimization, complementarity and variational inequality ones. For this reason, various aspects of EPs were investigated by many researchers (see, e.g., [3, 5–7] and references therein).

One of the most popular approaches to solve various problems of nonlinear analysis is to convert them into a suitable optimization problem with the help of so-called gap functions (see, e.g., [14]). In particular, Peng [15] suggested a *D*-gap function for the usual variational inequality problem (VI) and showed that it enables one to convert a VI into an unconstrained differentiable optimization problem without local minima if the cost mapping of a VI is strongly monotone and differentiable. Konnov [8] extended this approach to mixed VIs involving nonsmooth convex functions, afterwards Konnov and Kum [9] extended these results to Hilbert spaces.

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The first main goal of this paper is to apply the D -gap functions approach to rather a broad class of EPs, which also contain nonsmooth convex functions. In other words, we will show that such EPs can be reduced to a nonlinear equation under certain additional assumptions – an equation representing a stationarity condition for an unconstrained optimization problem. The second goal is to extend the results to a real Banach space setting thus providing very general fields of applications. On the basis of the above results we present a descent method, which strongly converges to a solution under rather natural assumptions, and give an additional example of applications from the theory of nonlinearly constrained EPs handled in terms of exact penalty functions.

2. Gap functions

Let U be a nonempty closed and convex subset of a real reflexive Banach space E , $h : E \times E \rightarrow R$ an equilibrium bifunction, i.e., $h(u, u) = 0$ for every $u \in U$, $f : E \rightarrow R$ a convex continuous but not necessarily differentiable function. We consider an EP of the form: find a point $u^* \in U$ such that

$$h(u^*, v) + f(v) - f(u^*) \geq 0 \quad \forall v \in U. \quad (2)$$

Clearly, this problem coincides with EP(1) if we set $\Phi(u, v) = h(u, v) + f(u) - f(v)$. Some existence and uniqueness results for just this problem were presented in [3, 6].

Many real problems arising in various fields can be formulated as an EP of the form of (2). It suffices to recall the obstacle and dam problems in mathematical physics (see, e.g., [13]), economic equilibrium problems with perfect and non-perfect competition (see, e.g., [12]).

We denote by U^* the set of solutions of EP(2). In addition, we assume throughout that h is differentiable and that $h(u, \cdot)$ is convex for each $u \in E$.

Together with EP(2) we shall consider the perturbed EP, which is to find an element $\bar{u} \in U$ such that

$$h(\bar{u}, v) + f(v) - f(\bar{u}) + 0.5\alpha\|v - \bar{u}\|^2 \geq 0 \quad \forall v \in U, \quad (3)$$

where $\alpha > 0$ is a fixed number. We denote by U^α the set of solutions of EP(3).

We now give an optimality condition for EP(2) in the form of a mixed variational inequality.

Proposition 2.1. *An element u^* solves EP(2) if and only if $u^* \in U$ and*

$$\langle \nabla_v h(u^*, u^*), v - u^* \rangle + f(v) - f(u^*) \geq 0 \quad \forall v \in U. \quad (4)$$

Proof. Suppose $u^* \in U^*$. Then u^* solves the convex minimization problem

$$\min_{v \in U} \rightarrow \{h(u^*, v) + f(v)\}.$$

Writing the optimality condition for this problem, we have

$$\langle \nabla_v h(u^*, u^*) + g^*, v - u^* \rangle \geq 0 \quad \forall v \in U,$$

where g^* is a subgradient of f at u^* . Since $f(v) - f(u^*) \geq \langle g^*, v - u^* \rangle$, this means that u^* solves (4).

Conversely, suppose u^* solves (4). By convexity of $h(u^*, \cdot)$, we have

$$\langle \nabla_v h(u^*, u^*), v - u^* \rangle \leq h(u^*, v) - h(u^*, u^*) \quad \forall v \in U,$$

hence,

$$h(u^*, v) + f(v) - f(u^*) \geq 0 \quad \forall v \in U,$$

i.e., $u^* \in U^*$, as desired. $\square$

Moreover, we can obtain the equivalence result between the usual and perturbed EP.

Proposition 2.2. $U^* = U^\alpha$ holds.

Proof. The result follows from Proposition 2.1, since $\nabla_v h(u, u) = \nabla_v \bar{h}(u, u)$, where $\bar{h}(u, v) = h(u, v) + 0.5\alpha\|v - u\|^2$. $\square$

Let

$$\Phi_\alpha(u, v) = -h(u, v) - f(v) + f(u) - 0.5\alpha\|v - u\|^2$$

and

$$\mu_\alpha(u) = \sup_{v \in U} \Phi_\alpha(u, v) = \Phi_\alpha(u, v_\alpha(u)). \quad (5)$$

The function μ_α can serve as a gap function for EP(2). Note that the inner maximization problem in (5) always has a unique solution $v_\alpha(u)$, since $\Phi_\alpha(u, \cdot)$ is strongly concave and continuous. The optimality condition for this problem can be formulated as follows.

Proposition 2.3. For all $u' \in U$

$$\langle \nabla_v h(u', v_\alpha(u')) + \alpha(v_\alpha(u') - u'), v - v_\alpha(u') \rangle + f(v) - f(v_\alpha(u')) \geq 0 \quad \forall v \in U \quad (6)$$

holds.

Proof. Fix an arbitrary $v \in U$. For brevity, set $v' = v_\alpha(u')$. Writing the usual optimality condition for the inner problem in (5), we have

$$\langle -\nabla_v h(u', v') - \alpha(v' - u') - g, v' - v \rangle \geq 0$$

(where g is a subgradient of f at v'). By the definition of the subgradient, we have $f(v) - f(v') \geq \langle g, v - v' \rangle$, and the result follows. $\square$

We are now ready to derive the basic properties of the gap function μ_α .

Proposition 2.4. (i) It is true that $\mu_\alpha(u) \geq 0$ for every $u \in U$.

(ii) If $\mu_\alpha(\bar{u}) = 0$ and $\bar{u} \in U$, then $\bar{u} \in U^*$.

(iii) The following properties are equivalent: (a) $u^* \in U^*$; (b) $u^* = v_\alpha(u^*)$; (c) $\mu_\alpha(u^*) = 0$ and $u^* \in U$.

Proof. Part (i) follows from the fact that $\Phi_\alpha(u, u) = 0$. Next, if $\bar{u} \in U$ and $\mu_\alpha(\bar{u}) = 0$, then

$$-h(\bar{u}, v) - f(v) + f(\bar{u}) - 0.5\alpha\|v - \bar{u}\|^2 \leq 0 \quad \forall v \in U,$$

i.e., $\bar{u} \in U^\alpha$ and $\bar{u} \in U^*$ because of Propositions 2.1 and 2.2. So, (ii) is true. Let us prove (iii). Note that the implication (c) $\implies$ (a) has been proven in (ii). Suppose that (a) holds, then (by Proposition 2.2), $u \in U^\alpha$ and, in view of Proposition 2.1,

$$\langle \nabla_v h(u^*, u^*), v^* - u^* \rangle + f(v^*) - f(u^*) \geq 0$$

with $v^* = v_\alpha(u^*)$. At the same time, by Proposition 2.3,

$$\langle \nabla_v h(u^*, v^*), u^* - v^* \rangle + f(u^*) - f(v^*) - \alpha\|u^* - v^*\|^2 \geq 0.$$

Adding both inequalities yields

$$-\alpha\|u^* - v^*\|^2 \geq \langle \nabla_v h(u^*, v^*) - \nabla_v h(u^*, u^*), v^* - u^* \rangle \geq 0,$$

since $\nabla_v h(u^*, \cdot)$ is monotone. It follows that $u^* = v^*$, i.e. (a) $\implies$ (b). If we suppose that $u^* = v_\alpha(u^*)$, then, by definition,

$$\mu_\alpha(u^*) = \Phi_\alpha(u^*, u^*) = -h(u^*, u^*) - f(u^*) + f(u^*) - 0.5\alpha\|u^* - u^*\| = 0,$$

hence, (b) $\implies$ (c), and (iii) is also true. □

However the gap function μ_α is nonsmooth in general and this fact may create additional difficulties in developing a suitable descent method. In order to obtain a smooth problem, we turn to the so-called *D-gap functions*.

3. D-gap functions

Let us now consider the function

$$\psi_{\alpha\beta}(u) = \mu_\alpha(u) - \mu_\beta(u),$$

where $0 < \alpha < \beta$. To obtain the basic properties of $\psi_{\alpha\beta}$, we need the following auxiliary assertion.

Proposition 3.1. *For every $u \in E$, we have*

$$\|u - v_\beta(u)\|^2 \leq 2\psi_{\alpha\beta}(u)/(\beta - \alpha) \leq \|u - v_\alpha(u)\|^2. \quad (7)$$

Proof. By definition,

$$\begin{aligned} \psi_{\alpha\beta}(u) &= \Phi_\alpha(u, v_\alpha(u)) - \Phi_\beta(u, v_\beta(u)) \geq \Phi_\alpha(u, v_\beta(u)) - \Phi_\beta(u, v_\beta(u)) \\ &\geq (\beta - \alpha)\|u - v_\beta(u)\|^2/2, \end{aligned}$$

i.e., the left inequality in (7) holds. Similarly, we obtain the right inequality in (7). □

The next proposition says that the *D-gap function* $\psi_{\alpha\beta}$ possesses the gap properties over the whole space rather than the feasible set U .

Proposition 3.2. (i) It is true that $\psi_{\alpha\beta}(u) \geq 0$ for every $u \in E$.

(ii) It is true that $\psi_{\alpha\beta}(u^*) = 0 \iff u^* \in U^*$.

Proof. Part (i) clearly follows from (7). Part (ii) follows from (7) and Proposition 2.4 (iii). $\square$

Thus, the initial problem (2) is equivalent to the unconstrained minimization problem

$$\min_{u \in E} \psi_{\alpha\beta}(u). \quad (8)$$

However, this problem can, in principle, have local minima differing from the global ones. For this reason, it is more suitable to replace EP(2) with the problem of finding a stationary point of problem (8), that is,

$$\nabla \psi_{\alpha\beta}(u^*) = 0. \quad (9)$$

This result needs certain additional assumptions and considerations, which will be given in the next sections.

4. Continuity and differentiability

We shall use the following additional assumptions.

(A1) For each $v \in E$, the map $\nabla_v h(\cdot, v)$ is Lipschitz continuous on every bounded subset of E .

(A2) The map $\nabla_u h(\cdot, \cdot)$ is continuous.

Lemma 4.1. If (A1) holds, then the map $u \mapsto v_\alpha(u)$ is locally Lipschitz continuous. If, in addition, $\nabla_v h(\cdot, v)$ is Lipschitz continuous with constant L_h for each $v \in E$, then the map $u \mapsto v_\alpha(u)$ is also Lipschitz continuous.

Proof. Fix $u', u'' \in E$ and set $v' = v_\alpha(u')$, $v'' = v_\alpha(u'')$. Then, according to Proposition 2.3, we have

$$\begin{aligned} \langle \nabla_v h(u', v') + \alpha(v' - u'), v'' - v' \rangle + f(v'') - f(v') &\geq 0, \\ \langle \nabla_v h(u'', v'') + \alpha(v'' - u''), v' - v'' \rangle + f(v') - f(v'') &\geq 0. \end{aligned}$$

Adding these inequalities gives

$$\langle \nabla_v h(u', v') - \nabla_v h(u'', v''), v'' - v' \rangle + \alpha \langle (v' - v'') + (u'' - u'), v'' - v' \rangle \geq 0.$$

It follows that

$$\begin{aligned} \langle \nabla_v h(u', v') - \nabla_v h(u'', v'), v'' - v' \rangle + \alpha \langle u'' - u', v'' - v' \rangle \\ \geq \langle \nabla_v h(u'', v') - \nabla_v h(u'', v''), v' - v'' \rangle + \alpha \|v'' - v'\|^2 \\ \geq \alpha \|v'' - v'\|^2 \end{aligned}$$

since $\nabla_v h(u'', \cdot)$ is monotone and, using (A1), we obtain

$$(L_h + \alpha) \|u' - u''\| \|v' - v''\| \geq \alpha \|v' - v''\|^2,$$

where L_h is the corresponding Lipschitz constant. Hence,

$$\|v' - v''\| \leq (1 + L_h/\alpha) \|u' - u''\|, \quad (10)$$

as desired. $\square$

We also need an additional fact, which originally stems from [16, Theorem 3.3] (see also [2, Theorem 1.7] and [9, Lemma 3]).

Lemma 4.2. *Let $\varphi : E \times U \rightarrow R$ be a bifunction such that $\nabla_u \varphi(u, v)$ exists and is continuous on $E \times U$. Define two functions as follows:*

$$\Psi(u) = \min_{v \in U} \varphi(u, v) \quad \forall u \in E,$$

$$M(u) = \{v \in U \mid \varphi(u, v) = \Psi(u)\}.$$

Assume that $M(u)$ is a singleton for every $u \in E$ and that M is continuous on E . Then Ψ is continuously differentiable and

$$\nabla \Psi(u) = \nabla_u \varphi(u, M(u)).$$

Now we are ready to establish the differentiability of $\psi_{\alpha\beta}$.

Theorem 4.1. *Let (A1) and (A2) hold. Then the function $\psi_{\alpha\beta}$ is continuously differentiable and*

$$\nabla \psi_{\alpha\beta}(u) = \nabla_u h(u, v_\beta(u)) - \nabla_u h(u, v_\alpha(u)) + \beta(u - v_\beta(u)) - \alpha(u - v_\alpha(u)). \quad (11)$$

Proof. Observe that

$$\mu_\alpha(u) = - \min_{v \in U} \{h(u, v) + 0.5\alpha\|v - u\|^2 + f(v)\} + f(u).$$

To apply Lemma 4.2, set

$$\varphi(u, v) = h(u, v) + 0.5\alpha\|v - u\|^2 + f(v).$$

Then $M(u) = v_\alpha(u)$ and $\Psi(u) = \min_{v \in U} \varphi(u, v)$. From Lemma 4.1 we have that M is continuous, hence

$$\nabla \Psi(u) = \nabla_u h(u, v_\alpha(u)) + \alpha(u - v_\alpha(u)).$$

Using this formula, we can compute the directional derivative of $\psi_{\alpha\beta}$ at any point $u \in E$ for each direction $d \in E$ as follows:

$$\begin{aligned} \psi'_{\alpha\beta}(u; d) &= \mu'_\alpha(u; d) - \mu'_\beta(u; d) \\ &= \langle -\nabla_u h(u, v_\alpha(u)) - \alpha(u - v_\alpha(u)), d \rangle + f'(u; d) \\ &\quad + \langle \nabla_u h(u, v_\beta(u)) + \beta(u - v_\beta(u)), d \rangle - f'(u; d) \\ &= \langle \nabla_u h(u, v_\beta(u)) - \nabla_u h(u, v_\alpha(u)) \\ &\quad + \beta(u - v_\beta(u)) - \alpha(u - v_\alpha(u)), d \rangle \\ &= \langle \nabla \psi_{\alpha\beta}(u), d \rangle. \end{aligned}$$

Hence, $\psi_{\alpha\beta}$ is Gâteaux differentiable at u . But $\nabla \psi_{\alpha\beta}$ is continuous due to (A2) and Lemma 4.1. Thus, $\psi_{\alpha\beta}$ is continuously (Fréchet) differentiable, as desired. $\square$

This smoothing property of D -gap functions for mixed variational inequalities was noted [8].

5. Stationarity properties

In this section, we establish the sufficient optimality condition for EP(2) in the form of (9).

Let

$$r(u) = v_\alpha(u) - v_\beta(u), \quad s(u) = \alpha[u - v_\alpha(u)] - \beta[u - v_\beta(u)].$$

Lemma 5.1. *It is true that $\langle r(u), s(u) \rangle \geq 0$ for every $u \in E$.*

Proof. Using Proposition 2.3 for $v_\alpha(u)$ and $v_\beta(u)$, we have

$$\langle \nabla_v h(u, v_\alpha(u)) + \alpha[v_\alpha(u) - u], v_\beta(u) - v_\alpha(u) \rangle + f(v_\beta(u)) - f(v_\alpha(u)) \geq 0$$

and

$$\langle \nabla_v h(u, v_\beta(u)) + \beta[v_\beta(u) - u], v_\alpha(u) - v_\beta(u) \rangle + f(v_\alpha(u)) - f(v_\beta(u)) \geq 0.$$

Adding the above inequalities yields

$$\langle \nabla_v h(u, v_\alpha(u)) - \nabla_v h(u, v_\beta(u)), v_\beta(u) - v_\alpha(u) \rangle + \langle s(u), r(u) \rangle \geq 0.$$

Since $\nabla_v h(u, \cdot)$ is monotone, we see that $\langle s(u), r(u) \rangle \geq 0$. □

We now need an additional assumption of the partial derivative of h .

(B1) *The map $\nabla_u h(u, \cdot)$ is strongly monotone with constant κ for each fixed $u \in E$.*

Theorem 5.1. *Let (A1), (A2) and (B1) hold. Then $\nabla \psi_{\alpha\beta}(u^*) = 0$ implies that $u^* \in U^*$.*

Proof. If $\nabla \psi_{\alpha\beta}(u^*) = 0$, then

$$\begin{aligned} 0 = \langle \nabla \psi_{\alpha\beta}(u^*), -r(u^*) \rangle &= \langle \nabla_u h(u^*, v_\beta(u^*)) - \nabla_u h(u^*, v_\alpha(u^*)), -r(u^*) \rangle + \langle s(u^*), r(u^*) \rangle \\ &\geq \kappa \|r(u^*)\|^2. \end{aligned}$$

due to Lemma 5.1 and (B1), hence $v_\alpha(u^*) = v_\beta(u^*)$. It follows that

$$0 = \nabla_u h(u^*, v_\alpha(u^*)) - \nabla_u h(u^*, v_\alpha(u^*)) + (\beta - \alpha)[u^* - v_\alpha(u^*)] = (\beta - \alpha)[u^* - v_\alpha(u^*)],$$

i.e., $u^* = v_\alpha(u^*)$. Taking into account Proposition 2.4 (iii), we conclude that $u^* \in U^*$. □

Thus, the initial nonsmooth and constrained problem (2) can be replaced with the problem of finding a stationary point of a continuously differentiable function $\psi_{\alpha\beta}$. This problem can be solved with the help of the usual unconstrained optimization methods. We intend to describe such a method in the next section.

6. Solution method

For each $u \in U$, we set $H(u) = \nabla_v h(u, v)|_{v=u}$.

Let us introduce the following additional assumptions.

(B2) *The map $H : E \rightarrow E^*$ is strongly monotone with constant τ .*

(B3) *The map $\nabla_u h(\cdot, \cdot)$ is uniformly Lipschitz continuous on each bounded subset of $E \times E$.*

(B4) *The map $\nabla_v h(\cdot, \cdot)$ is uniformly Lipschitz continuous with constant L'_h .*

First we recall the existence and uniqueness result for the initial EP(2).

Proposition 6.1. *If (B2) holds, then EP(2) has at most one solution. If in addition (B4) holds, then EP (2) has a unique solution.*

Proof. From Proposition 2.1 we have that EP(2) can be replaced with the equivalent mixed variational inequality (4). If (B2) holds, its solution is clearly unique (see e.g. [3, Theorem 10.2]). Moreover, (B2) yields the coercivity condition. Hence, (B2) and (B4) provide the existence of a solution (see, e.g., [3, Theorem 10.5] and [6]). $\square$

Lemma 6.1. *Let (B2)–(B4) hold. Then there exists a constant $\hat{\gamma} > 0$ such that*

$$\|u' - u^*\| \leq \hat{\gamma} \|u' - v_\beta(u')\| \quad \forall u' \in E, \quad (12)$$

where u^* solves EP (2).

Proof. Take an arbitrary point $u' \in E$. For brevity, set $v' = v_\beta(u')$. Adding (4) with $v = v'$ and (6) with $\alpha = \beta$, $v = u^*$ gives

$$\begin{aligned} 0 &\leq \langle \nabla_v h(u^*, u^*) - \nabla_v h(u', v'), v' - u^* \rangle + \beta \langle v' - u', u^* - v' \rangle \\ &= \langle \nabla_v h(u^*, u^*) - \nabla_v h(u', u'), v' - u^* \rangle \\ &\quad + \langle \nabla_v h(u', u') - \nabla_v h(u', v'), v' - u^* \rangle + \beta \langle v' - u', u^* - v' \rangle. \end{aligned}$$

Applying (B2), we have

$$\begin{aligned} \tau \|u^* - u'\|^2 &\leq \langle \nabla_v h(u^*, u^*) - \nabla_v h(u', u'), u^* - u' \rangle \\ &\leq \langle \nabla_v h(u^*, u^*) - \nabla_v h(u', u'), v' - u' \rangle + \langle \nabla_v h(u', u') - \nabla_v h(u', v'), u' - u^* \rangle \\ &\quad + \langle \nabla_v h(u', u') - \nabla_v h(u', v'), v' - u' \rangle + \beta \langle v' - u', u^* - u' \rangle + \beta \langle v' - u', u' - v' \rangle. \end{aligned}$$

Since $\nabla_v h(u', \cdot)$ is monotone, it follows that

$$\langle \nabla_v h(u', u') - \nabla_v h(u', v'), v' - u' \rangle \leq 0.$$

Besides, it is clear that $\beta \langle v' - u', u' - v' \rangle \leq 0$. Taking into account both inequalities and (B3), (B4), we obtain

$$\tau \|u^* - u'\|^2 \leq L'_h \|u^* - u'\| \|v' - u'\| + L'_h \|u^* - u'\| \|v' - u'\| + \beta \|u^* - u'\| \|v' - u'\|,$$

i.e., (12) holds with $\hat{\gamma} = (\beta + 2L'_h)/\tau$. $\square$

From this proposition, we can get the following global error result for EP.

Theorem 6.1. *Let (B2)–(B4) hold. Then there exists a constant $\tilde{\gamma} > 0$ such that*

$$\tilde{\gamma} \|u - u^*\|^2 \leq \psi_{\alpha\beta}(u) \quad \forall u \in U,$$

where u^* solves EP(2).

Proof. Combining Proposition 3.1 and Lemma 6.1, we have

$$\|u - u^*\|^2 \leq 2\hat{\gamma}/(\beta - \alpha) \psi_{\alpha\beta}(u) \quad \forall u \in E,$$

as desired. $\square$

For each $u \in U$, set

$$S(u) = \{v \in E \mid \psi_{\alpha\beta}(v) \leq \psi_{\alpha\beta}(u)\}.$$

Corollary 6.1. *Let all conditions of Theorem 6.1 hold. Then the level sets $S(u)$ are bounded.*

Based on the argument similar to that in [17, Lemma 5.1], we give a descent property of a direction associated with the gradient of the D -gap function.

Lemma 6.2. *Suppose that (B1)–(B4) hold. Then there exists a constant $\tilde{\rho} > 0$ such that for every $u \in S(u^0)$ and for every $\rho \in (0, \tilde{\rho})$, it is true that*

$$\langle \nabla \psi_{\alpha\beta}(u), r(u) + \rho s(u) \rangle \leq -(\kappa/2)(\|r(u)\| + \rho\|s(u)\|)^2,$$

where u^0 is an arbitrary point of E .

Proof. Set $d = r(u) + \rho s(u)$. From Theorem 4.1 we have

$$\begin{aligned} \langle d, \nabla \psi_{\alpha\beta}(u) \rangle &= \langle d, \nabla_u h(u, v_\beta(u)) - \nabla_u h(u, v_\alpha(u)) \rangle + \langle d, \beta(u - v_\beta(u)) - \alpha(u - v_\alpha(u)) \rangle \\ &= -\langle v_\alpha(u) - v_\beta(u), \nabla_u h(u, v_\alpha(u)) - \nabla_u h(u, v_\beta(u)) \rangle \\ &\quad - \rho \langle \alpha(u - v_\alpha(u)) - \beta(u - v_\beta(u)), \nabla_u h(u, v_\alpha(u)) - \nabla_u h(u, v_\beta(u)) \rangle \\ &\quad - \langle \alpha(u - v_\alpha(u)) - \beta(u - v_\beta(u)), v_\alpha(u) - v_\beta(u) \rangle \\ &\quad - \rho \|\alpha(u - v_\alpha(u)) - \beta(u - v_\beta(u))\|^2. \end{aligned} \tag{13}$$

Using (B1) gives

$$\begin{aligned} \langle v_\alpha(u) - v_\beta(u), \nabla_u h(u, v_\alpha(u)) - \nabla_u h(u, v_\beta(u)) \rangle &\geq \kappa \|v_\alpha(u) - v_\beta(u)\|^2 \\ &= \kappa \|r(u)\|^2. \end{aligned} \tag{14}$$

Moreover, since u is in the bounded set $S(u^0)$, $v_\alpha(u)$ and $v_\beta(u)$ are also in bounded sets due to Lemma 4.1, and it follows from (B3) that

$$\begin{aligned} -\langle \nabla_u h(u, v_\alpha(u)) - \nabla_u h(u, v_\beta(u)), s(u) \rangle &\leq \|\nabla_u h(u, v_\alpha(u)) - \nabla_u h(u, v_\beta(u))\| \|s(u)\| \\ &\leq L_h'' \|r(u)\| \|s(u)\|, \end{aligned} \tag{15}$$

where L_h'' is the corresponding Lipschitz constant. Besides, by Lemma 5.1, we have

$$\langle r(u), s(u) \rangle \geq 0. \tag{16}$$

Thus, it follows from (13)–(16) that

$$\langle d, \nabla \psi_{\alpha\beta}(u) \rangle \leq -\kappa \|r(u)\|^2 + L_h'' \rho \|r(u)\| \|s(u)\| - \rho \|s(u)\|^2. \tag{17}$$

We can rewrite the right-hand side of (17) as

$$\begin{aligned} &-\kappa \|r(u)\|^2 + L_h'' \rho \|r(u)\| \|s(u)\| - \rho \|s(u)\|^2 \\ &= -(\kappa/2)(\|r(u)\| + \rho\|s(u)\|)^2 - (1/2)(\sqrt{\kappa}\|r(u)\| - \sqrt{\rho}\|s(u)\|)^2 \\ &\quad + (1/2)(\kappa\rho - 1)\rho\|s(u)\|^2 + \sqrt{\rho}(\kappa\sqrt{\rho} + L_h''\sqrt{\rho} - \sqrt{\kappa})\|r(u)\| \|s(u)\|. \end{aligned} \tag{18}$$

Then, by choosing ρ such that

$$\rho \leq \tilde{\rho} = \min\{1/\kappa, \kappa/(\kappa + L_h'')^2\},$$

we obtain

$$\kappa\rho - 1 \leq 0, \quad \mu\sqrt{\rho} + L_h''\sqrt{\rho} - \sqrt{\mu} \leq 0. \quad (19)$$

It follows from (17)–(19) that

$$\langle d, \nabla\psi_{\alpha\beta}(u) \rangle \leq -(\kappa/2)(\|r(u)\| + \rho\|s(u)\|)^2,$$

as desired. □

Now we state an algorithm for EP, which can be viewed as an extension of the algorithms from [8, 9, 17].

Algorithm.

Step 0: Select an initial point $u^0 \in E$ and parameters $\rho > 0, \theta \in (0, 1), \gamma > 0$. Set $k = 0$.

Step 1: If $\psi_{\alpha\beta}(u^k) = 0$ then stop, u^k is a solution of EP.

Step 2: Set $d^k = r(u^k) + \rho s(u^k)$.

Step 3: Compute m as the smallest nonnegative integer such that

$$\psi_{\alpha\beta}(u^k + \theta^m d^k) - \psi_{\alpha\beta}(u^k) \leq -\theta^m \gamma (\|r(u^k)\| + \rho\|s(u^k)\|)^2.$$

Step 4: Set $\lambda_k = \theta^m, u^{k+1} = u^k + \lambda_k d^k, k = k + 1$ and go to Step 1.

Based on the previous result, we are now ready to establish a convergence theorem for this algorithm.

Theorem 6.2. *Let (B1)–(B4) hold. Then, the sequence $\{u^k\}$ generated by the algorithm with $\rho \in (0, \tilde{\rho})$ and $\gamma \leq \kappa/2$ converges strongly to a unique solution of EP(2).*

Proof. According to the descent property of the algorithm, the sequence $\{u^k\}$ is contained in the level set $S(u^0)$. In view of Corollary 6.1, $S(u^0)$ is bounded, hence, so are $\{u^k\}, \{v_\alpha(u^k)\}$ and $\{v_\beta(u^k)\}$ due to Lemma 4.1. Hence, there exists a closed bounded convex subset G of E which contains sequences $\{u^k\}$ and $\{u^k + d^k\}$. In view of (B3), (10) and (11), $\nabla\psi_{\alpha\beta}$ is Lipschitz continuous on G . Hence, we have, for all $\lambda \geq 0$,

$$\psi_{\alpha\beta}(u^k + \lambda d^k) \leq \psi_{\alpha\beta}(u^k) + \lambda \langle \nabla\psi_{\alpha\beta}(u^k), d^k \rangle + L\lambda^2 \|d^k\|^2/2,$$

where L is a Lipschitz constant for $\nabla\psi_{\alpha\beta}$ on G . Applying Lemma 6.2 to this inequality gives

$$\psi_{\alpha\beta}(u^k + \lambda d^k) - \psi_{\alpha\beta}(u^k) \leq -(\kappa - L\lambda)\lambda(\|r(u^k)\| + \rho\|s(u^k)\|)^2/2.$$

Thus

$$\psi_{\alpha\beta}(u^k + \lambda d^k) - \psi_{\alpha\beta}(u^k) \leq -\gamma\lambda(\|r(u^k)\| + \rho\|s(u^k)\|)^2$$

holds for all λ satisfying $0 \leq \lambda \leq (\kappa - 2\gamma)/L$ provided that $\gamma < \kappa/2$. In the manner determining the step size λ_k , we obtain

$$\lambda_k \geq \theta(\kappa - 2\gamma)/L = \tilde{\lambda} > 0 \quad \forall k.$$

It follows that

$$\psi_{\alpha\beta}(u^k) - \psi_{\alpha\beta}(u^{k+1}) \geq \tilde{\lambda}\gamma(\|r(u^k)\| + \rho\|s(u^k)\|)^2.$$

Taking the limit in the above inequality gives

$$\lim_{k \rightarrow \infty} [\psi_{\alpha\beta}(u^k) - \psi_{\alpha\beta}(u^{k+1})] \geq \lim_{k \rightarrow \infty} \tilde{\lambda}\gamma(\|r(u^k)\| + \rho\|s(u^k)\|)^2. \quad (20)$$

Clearly, the left-hand side of (20) is 0 because the sequence $\{\psi_{\alpha\beta}(u^k)\}$ is nonnegative and decreases monotonically. Therefore, $\lim_{k \rightarrow \infty} (\|r(u^k)\| + \rho\|s(u^k)\|) = 0$, so

$$\lim_{k \rightarrow \infty} \|r(u^k)\| = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \|s(u^k)\| = 0. \quad (21)$$

Since

$$\langle s(u^k), v_\beta(u^k) - u^k \rangle = (\beta - \alpha)\|u^k - v_\beta(u^k)\|^2 - \alpha\langle u^k - v_\beta(u^k), v_\beta(u^k) - v_\alpha(u^k) \rangle,$$

we have

$$(\beta - \alpha)\|u^k - v_\beta(u^k)\|^2 \leq \|s(u^k)\|\|u^k - v_\beta(u^k)\| + \alpha\|r(u^k)\|\|u^k - v_\beta(u^k)\|.$$

It follows that

$$\|u^k - v_\beta(u^k)\| \leq [\|s(u^k)\| + \alpha\|r(u^k)\|]/(\beta - \alpha).$$

From (12) and (21) we now obtain $\lim_{k \rightarrow \infty} \|u^k - u^*\| = 0$, as desired. $\square$

7. Exact penalties for nonlinearly constrained EPs

In this section, we intend to give an additional example of EP(2) via application of the known exact penalty approach to constrained equilibrium problems. Note that the exact penalty approach was adjusted for variational inequalities in [10]. So, we consider EP(1), where the feasible set U is defined as follows:

$$U = \{u \in D \mid f_i(u) \leq 0 \quad i = 1, \dots, m\}, \quad (22)$$

where D is a nonempty convex and closed subset of E , $f_i : E \rightarrow R, i = 1, \dots, m$ are continuous convex functions. Let us consider the following Slater constraint qualification.

(C) *There exists a point $\tilde{z} \in D$ such that $f_i(\tilde{z}) < 0$ for $i = 1, \dots, m$.*

We now recall that the bifunction Φ is said to be monotone if for all $u, v \in E$ we have

$$\Phi(u, v) + \Phi(v, u) \leq 0,$$

see, e.g., [6]. Set $f(v) = (f_1(v), \dots, f_m(v))$. The following result is an extension of the Kuhn-Tucker saddle point theorem for EPs.

Proposition 7.1. *(i) If a point $w^* = (u^*, y^*) \in D \times R_+^m$ is a solution to the system*

$$\Phi(u^*, v) + \langle y^*, f(v) - f(u^*) \rangle \leq 0 \quad \forall v \in D, \quad (23)$$

$$\langle f(u^*), y^* - y \rangle \geq 0 \quad \forall y \in R_+^m, \quad (24)$$

then u^ solves EP(1),(22).*

(ii) If u^ solves EP (1),(22) and (C) holds, then there exists a point $y \in R_+^m$ such that $w^* = (u^*, y^*)$ solves problem (23),(24).*

Proof. Part (i) follows, e.g., from Proposition 4.1 in [11]. In case (ii), if u^* solves EP (1), (22), then it solves the problem of minimizing $h(u^*, \cdot) + f(\cdot)$ over U . Applying the well-known in optimization Kuhn-Tucker theorem (e.g., see [1, Chapter 4, Theorem 5.3.1] or [4, Section 5.2]) to this problem, we conclude that assertion (ii) is also true. $\square$

Let us consider the auxiliary functions

$$F(u) = \max\{f_1(u), \dots, f_m(u)\}$$

and

$$H(u) = \max\{0, F(u)\}.$$

It is clear that both functions F and H are convex and continuous if $f_i, i = 1, \dots, m$ are the same. At the same time, F and H do not have to be differentiable, even if all f_i are differentiable. Nevertheless, we see that

$$U = \{u \in D \mid F(u) \leq 0\}, \quad (25)$$

and that (C) is equivalent to $F(\tilde{z}) < 0$ for some $\tilde{z} \in D$. Thus, we can specialize the above result for EP (1), (25).

Proposition 7.2. (i) If $(u^*, y^*) \in D \times R_+$ is a solution to the system

$$\Phi(u^*, v) + y^*(F(v) - F(u^*)) \geq 0 \quad \forall v \in U, \quad (26)$$

$$F(u^*)(y^* - y) \geq 0 \quad \forall y \in R_+, \quad (27)$$

then u^* solves EP(1), (25).

(ii) If u^* solves EP (1), (25) and (C) holds, then there exists $y^* \geq 0$ such that (u^*, y^*) solves problem (26), (27).

We are now in a position to present an analogue of the exact penalty function result for EP (1), (22).

Theorem 7.1. Suppose Φ is monotone. If $\lambda > y^*$, where y^* is defined in (26), (27), then each solution $u_\lambda^* \in D$ to the problem

$$\Phi(u_\lambda^*, v) + \lambda[H(v) - H(u_\lambda^*)] \geq 0 \quad \forall v \in D \quad (28)$$

solves EP (1), (22) (or, equivalently, EP (1), (25)).

Proof. First we note that $u_\lambda^* \in U$ implies $H(u_\lambda^*) = 0$ and then, for all $u \in U$, from (28) we must have (1). Hence, it suffices to show that $u_\lambda^* \in U$ whenever $\lambda > y^*$. Fix $u^* \in D$ and $y^* \geq 0$ satisfying (26), (27), then (27) implies $y^*F(u^*) = 0$ and $u^* \in U$. Moreover, if u_λ^* solves (28) for some $\lambda > y^*$, then

$$\Phi(u_\lambda^*, u^*) - \lambda H(u_\lambda^*) \geq 0,$$

but (26) now gives

$$\Phi(u^*, u_\lambda^*) + y^* H(u_\lambda^*) \geq \Phi(u^*, u_\lambda^*) + y^* F(u_\lambda^*) \geq 0.$$

Adding these inequalities yields

$$(y^* - \lambda)H(u_\lambda^*) \geq -\Phi(u_\lambda^*, u^*) - \Phi(u^*, u_\lambda^*) \geq 0$$

since Φ is monotone. But $y^* < \lambda$, hence we must have $H(u_\lambda^*) = 0$, i.e., $u_\lambda^* \in U$, and the result follows. $\square$

Thus, Proposition 7.2 and Theorem 7.1 give a sufficient condition for the penalty method to be exact in the case of EP. In fact, from Proposition 7.2 it follows that the number y^* in (26), (27) exists if EP(1), (22) is solvable, $\Phi(u, \cdot)$ is convex for every $u \in E$, and (C) holds.

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