THE SDFEM FOR A CONVECTION-DIFFUSION PROBLEM WITH TWO SMALL PARAMETERS

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Dedicated to John J.H. Miller on the occasion of his 65th birthday.

Abstract — A singularly perturbed convection-diffusion problem with two small parameters is considered. The problem is solved using the streamline-diffusion finite element method on a Shishkin mesh. We prove that the method is convergent independently of the perturbation parameters. Numerical experiments support these theoretical results.

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1. Introduction

We consider the singularly perturbed convection-diffusion problem

$$Lu := -\varepsilon_1 u'' + \varepsilon_2 b(x)u' + c(x)u = f(x) \quad x \in I = (0, 1), \quad u(0) = 0, \quad u(1) = 0, \quad (1.1)$$

where b, c, f are smooth functions, $b(x) \geq b_0 > 0$, $c(x) \geq c_0 > 0$ and $0 < \varepsilon_{1,2} \ll 1$ are both small perturbation parameters. Although physical problems frequently involve the solution of boundary-value problems with many small parameters, not much is known about robust numerical methods for solving two parameter problems (see O'Malley [8] for some physical problems involving two small parameters and for details concerning asymptotic expansions).

Shishkin and Titov [12] proved, for an exponentially fitted difference scheme on an equidistant mesh, for some $\nu < 2/5$, that

$$|u(x_i) - u_i| \leq Ch^\nu,$$

where C is independent of ε and h . Instead of using exponential fitting, the use of layer-adapted meshes becomes more and more popular. Vulcanovic [14] considers Bakhvalov and

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Shishkin meshes but assumes $\varepsilon_2 = \varepsilon_1^{\frac{1}{2}+p}$ with $p > 0$. Then the problem (1.1) behaves similarly as the standard reaction-diffusion problem with $\varepsilon_2 = 0$. Two-parameter problems are considered also in Li [2] and in Madden and Stynes [7], but are of different type than the one considered here.

Recently, simple upwinding on a properly chosen Shishkin mesh was analyzed independently in [5,9]. Both papers are based on similar estimates for a priori bounds for derivatives. These estimates are also the basis of our analysis, and because [5] and the present paper were written almost in parallel, we present the estimates in Section 2 (it turns out that Lemma 2.1 and Lemma 2.2 are almost identical with Lemma 3.1 in [9]).

Simple upwinding results in a first order scheme. It is well known that for a convection-diffusion problem with $\varepsilon_2 = 1$ the streamline diffusion finite element method (SDFEM) can generate a second order scheme [13]. Therefore in the present paper we shall analyze the SDFEM on a properly chosen Shishkin mesh for the two-parameter problem and prove almost second-order pointwise convergence uniformly with respect to the parameters $\varepsilon_1, \varepsilon_2$.

Our analysis is absolutely different from the analysis in [13]. While Stynes and Tobiska use the consistency error of the difference scheme generated by the SDFEM, our approach uses mainly finite element tools. This technique was already presented in [10] for a convection-diffusion problem with one small parameter and a point source.

In the literature so far convection-diffusion problems ($\varepsilon_2 = 1$) and reaction-diffusion problems ($\varepsilon_2 = 0$) are handled separately. Our approach offers a unified treatment of problem (1.1) for all possible classes of subproblems.

Throughout the paper C will represent a constant independent of $\varepsilon_1, \varepsilon_2$, and of the mesh. For a continuous function g , the value at a point x_i will be denoted by g_i . Furthermore, $\|\cdot\|_\infty$ denotes the maximum norm, while $\|\cdot\|_{L_1}$ denotes the L_1 norm.

2. Some properties of the continuous problem

The solution of the reduced problem $cu_0 = f$, in general, does not satisfy the boundary conditions, so there exist boundary layers both at $x = 0$ and at $x = 1$. To describe the layers let us introduce the characteristic equation

$$-\varepsilon_1 \lambda^2(x) + \varepsilon_2 b(x) \lambda(x) + c(x) = 0.$$

It has two real solutions $\lambda_0(x) < 0$ and $\lambda_1(x) > 0$. Because $|\lambda_0| < \lambda_1$, the layer at $x = 1$ is stronger than the layer at $x = 0$. Let

$$\mu_0 = -\max_{x \in [0,1]} \lambda_0(x) \quad \text{and} \quad \mu_1 = \min_{x \in [0,1]} \lambda_1(x).$$

Lemma 2.1. *For any $0 < p < 1$ we have up to the a certain order q that depends on the smoothness of the data*

$$|u^{(k)}(x)| \leq C \{1 + \mu_0^k e^{-p\mu_0 x} + \mu_1^k e^{-p\mu_1(1-x)}\} \quad \text{for } 0 \leq k \leq q. \quad (2.1)$$

Proof. We shall prove (2.1) by induction with respect to k . For $k = 0$ the estimate follows immediately from the maximum principle.

Since

$$(Lu)' = Lu' + \varepsilon_2 b'u' + c'u,$$

then by differentiating the original equation k times, we get

$$L_k u^{(k)} := -\varepsilon_1 u^{(k+2)} + \varepsilon_2 b u^{(k+1)} + c_k u^{(k)} = g_k, \quad \text{for } k = 1, \dots \quad (2.2)$$

where

$$c_0 = c, \quad g_0 = f$$

and

$$c_k = c + k\varepsilon_2 b', \quad g_k = g'_{k-1} - c'_{k-1} u^{(k-1)}, \quad \text{for } k = 1, \dots$$

The $(k+1)^{th}$ derivative satisfies the equation

$$L_{k+1} u^{(k+1)} = g_{k+1}$$

with

$$|g_{k+1}| \leq C\{1 + \mu_0^k e^{-p\mu_0 x} + \mu_1^k e^{-p\mu_1(1-x)}\}.$$

In the first step of the proof we assume that we have already proved the inequalities

$$|u^{(k+1)}(0)| \leq C\mu_0^{k+1} \quad \text{and} \quad |u^{(k+1)}(1)| \leq C\mu_1^{k+1}. \quad (2.3)$$

We shall bound $u^{(k+1)}$ using (2.2), (2.3) and the following barrier function

$$\Phi = C(1 + \mu_0^{k+1} w_{0,p}(x) + \mu_1^{k+1} w_{1,p}(x)),$$

with

$$w_{0,p}(x) = e^{-p\mu_0 x} \quad \text{and} \quad w_{1,p}(x) = e^{-p\mu_1(1-x)}.$$

It is sufficient to show that $L_{k+1} w_{0,p} \geq \alpha_0 w_{0,p}$ and $L_{k+1} w_{1,p} \geq \alpha_1 w_{1,p}$ with some $\alpha_0, \alpha_1 > 0$. Since

$$\begin{aligned} L_{k+1} w_{0,p} &= \{-\varepsilon_1 p^2 \mu_0^2 - \varepsilon_2 p \mu_0 b + c + \mathcal{O}(\varepsilon_2)\} w_{0,p} \\ &\geq \{p(-\varepsilon_1 \mu_0^2 - \varepsilon_2 b \mu_0) + c + \mathcal{O}(\varepsilon_2)\} w_{0,p} \\ &\geq \{p(-\varepsilon_1 \lambda_0^2(x) + \varepsilon_2 b(x) \lambda_0(x)) + c(x) + \mathcal{O}(\varepsilon_2)\} w_{0,p} \\ &= \{c(x)(1-p) + \mathcal{O}(\varepsilon_2)\} w_{0,p}, \end{aligned}$$

for any $0 < p < 1$, we have the desired property if ε_2 is small enough. Similarly, we estimate $L_{k+1} w_{1,p}$.

In the next step we shall prove (2.3) starting from

$$L_k u^{(k)} = g_k \quad \text{and} \quad |u^{(k)}(0)| \leq C\mu_0^k, \quad |u^{(k)}(1)| \leq C\mu_1^k.$$

Setting

$$u^{(k)} = y + y_0 \quad \text{with} \quad y_0 = u^{(k)}(0)(1-x),$$

we obtain for y the partially homogeneous problem

$$\begin{aligned} L_k y &= g_k - \varepsilon_2 b y'_0 - c_k y_0, \\ y(0) &= 0, \quad y(1) = u^{(k)}(1). \end{aligned}$$

Now we bound y by a barrier function of the type

$$\Phi = C_1(1 + e^{-p\mu_0 x} - 2e^{-\theta\mu_0 x})\mu_0^k + C_2(e^{-p\mu_1(1-x)} - e^{-p\mu_1})\mu_1^k,$$

where $\theta > 0$ is large enough. It is easy to check that $\Phi(0) = 0$ and $\Phi(1) \geq y(1)$ if C_2 is sufficiently large. Since $L_k e^{-\theta \mu_0 x} \leq 0$ and $L_k e^{-p \mu_0 x} \geq 0, L_k e^{-p \mu_1 (1-x)} \geq 0$, if θ and p are chosen adequately, Φ majorizes y if C_1 is sufficiently large.

Then the definition of the derivative gives

$$|y'(0)| \leq C \mu_0^{k+1} + C \mu_1^{k+1} e^{-p \mu_1} \leq C \mu_0^{k+1}.$$

Similarly we prove the second estimate in (2.3). $\square$

From (2.1) we obtain immediately the existence of a Shishkin decomposition [4, p. 892]:

Lemma 2.2. *The solution u has the representation $u = S + E_0 + E_1$, where*

$$|S^{(k)}(x)| \leq C \quad \text{for } 0 \leq k \leq g$$

and

$$\begin{aligned} |E_0^{(k)}(x)| &\leq C \mu_0^k e^{-p \mu_0 x} & \text{for } 0 \leq k \leq g, \\ |E_1^{(k)}(x)| &\leq C \mu_1^k e^{-p \mu_1 (1-x)} & \text{for } 0 \leq k \leq g. \end{aligned}$$

Let us for a moment assume b and c to be constant. Then the two extremal layer situations are characterized by

$$(i) \quad \varepsilon_2^2 \ll \varepsilon_1$$

which implies $\mu_0 \approx \mu_1 \approx \sqrt{c/\varepsilon_1}$, and we have layers similar to the reaction-diffusion case $\varepsilon_2 = 0$;

$$(ii) \quad \varepsilon_1 \ll \varepsilon_2^2$$

which implies $\mu_0 \approx c/(b\varepsilon_2)$, but $\mu_1 \approx (b\varepsilon_2)/\varepsilon_1$ is much larger than μ_0 , so the layer at $x = 1$ is much stronger than the layer at $x = 0$.

In the intermediate case, when neither (i) nor (ii) hold, μ_0 and μ_1 move in the interval given by the values from (i) and (ii).

Fig. 1 below presents the exact solutions, for various ε_1 and ε_2 , for the problem considered in [3]:

$$-\varepsilon_1 u'' + \varepsilon_2 u' + u = \cos \pi x \quad x \in (0, 1), \quad u(0) = u(1) = 0.$$

Figure 1. The exact solution for $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.01$ and $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.1$

The exact solution is

$$u(x) = a \cos \pi x + b \sin \pi x + A e^{\lambda_0 x} + B e^{-\lambda_1(1-x)},$$

where

$$\begin{aligned} a &= \frac{\varepsilon_1 \pi^2 + 1}{\varepsilon_2^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2}, & b &= \frac{\varepsilon_2 \pi}{\varepsilon_2^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2}, & A &= -a \frac{1 + e^{-\lambda_1}}{1 - e^{\lambda_0 - \lambda_1}}, \\ B &= a \frac{1 + e^{\lambda_0}}{1 - e^{\lambda_0 - \lambda_1}}, & \lambda_{0,1} &= \frac{\varepsilon_2 \mp \sqrt{\varepsilon_2^2 + 4\varepsilon_1}}{2\varepsilon_1}. \end{aligned}$$

3. The SDFEM with linear elements

3.1. The scheme

For problem (1.1), the standard weak formulation is: find $u \in V$ such that

$$a_G(u, v) := (\varepsilon_1 u', v') + (\varepsilon_2 b u' + c u, v) = (f, v) \quad \forall v \in V,$$

where $V = H_0^1(\Omega)$ denotes the usual Sobolev space and $(\cdot, \cdot)$ is the scalar product on $L_2(\Omega)$.

On a given mesh

$$0 = x_0 < x_1 < \dots < x_{N-1} < x_N = 1,$$

with the mesh step $h_i = x_i - x_{i-1}$, $i = 1, \dots, N$, we set $\bar{h}_i = (h_i + h_{i+1})/2$, $i = 1, \dots, N-1$, and for a given mesh function $\{u_i\}$ we define the forward and backward difference operators D^+ and D^- by

$$D^+ u_i := \frac{u_{i+1} - u_i}{h_{i+1}} \quad \text{and} \quad D^- u_i := \frac{u_i - u_{i-1}}{h_i}.$$

Let $V_h \subset V$ be the space of continuous linear finite elements with the basis functions

$$\phi_i = \begin{cases} \frac{x - x_{i-1}}{h_i} & \text{for } x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h_{i+1}} & \text{for } x \in [x_i, x_{i+1}], \\ 0 & \text{for } x \notin [x_{i-1}, x_{i+1}]. \end{cases}$$

The Galerkin approach is characterized by: find $u_h \in V_h$ such that

$$a_G(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

Based on the decomposition $u = S + E_0 + E_1$ given in Lemma 2.2, we take $p = 1/2$ and introduce a corresponding S-mesh. Namely, the mesh is equidistant on $\Omega_0 := [0, \sigma_0]$, $\Omega_g := [\sigma_0, 1 - \sigma_1]$ and $\Omega_1 := [1 - \sigma_1, 1]$. We choose

$$\sigma_0 = \frac{2\tau}{\mu_0} \ln N, \quad \sigma_1 = \frac{2\tau}{\mu_1} \ln N$$

(such that $e^{-\mu_0 x/2}|_{x=\sigma_0} = N^{-\tau}$, for example). Since μ_0 and μ_1 are large and $\mu_1 \geq \mu_0$, the mesh is fine on Ω_0 and Ω_1 and coarse on Ω_g , with mesh step sizes

$$h_0 = \mathcal{O}\left(\frac{N^{-1} \ln N}{\mu_0}\right), \quad h_1 = \mathcal{O}\left(\frac{N^{-1} \ln N}{\mu_1}\right) \quad \text{and} \quad H = h_g = \mathcal{O}(N^{-1}),$$

respectively.

If u^I denotes the linear interpolant of u in V_h , simple modifications of the well-known techniques of [6] result in:

Lemma 3.1. *The interpolation error satisfies*

$$\|u - u^I\|_{\infty, \Omega_g} \leq CN^{-2}, \quad \|u - u^I\|_{\infty, \Omega \setminus \Omega_g} \leq C(N^{-1} \ln N)^2.$$

The Galerkin method generates a difference scheme which can be written in the form

$$\begin{aligned} L_G^N u_i &:= -\varepsilon_1 \frac{D^+ u_i - D^- u_i}{\bar{h}_i} + \varepsilon_2 \left(\beta_{i+1} \frac{u_{i+1} - u_i}{\bar{h}_i} - \beta_{i-1} \frac{u_i - u_{i-1}}{\bar{h}_i} \right) + \gamma_{i-1} u_{i-1} + \gamma_i u_i + \gamma_{i+1} u_{i+1} \\ &= \frac{1}{\bar{h}_i} \int_{x_{i-1}}^{x_{i+1}} f \phi_i, \end{aligned}$$

where ϕ_i denotes again the basis function of V_h with $\phi_i(x_j) = \delta_{ij}$, and

$$\begin{aligned} \beta_{i-1} &= \int_{x_{i-1}}^{x_i} b \phi'_{i-1} \phi_i < 0, \quad \beta_{i+1} = \int_{x_i}^{x_{i+1}} b \phi'_{i+1} \phi_i > 0, \\ \gamma_{i-1} &= \frac{1}{\bar{h}_i} \int_{x_{i-1}}^{x_i} c \phi_{i-1} \phi_i, \quad \gamma_i = \frac{1}{\bar{h}_i} \int_{x_{i-1}}^{x_{i+1}} c \phi_i \phi_i, \quad \gamma_{i+1} = \frac{1}{\bar{h}_i} \int_{x_i}^{x_{i+1}} c \phi_{i+1} \phi_i > 0. \end{aligned}$$

When $c = \text{const}$ we have $\gamma_{i-1} = h_i/(6\bar{h}_i)$, $\gamma_i = 2/3$, $\gamma_{i+1} = h_{i+1}/(6\bar{h}_i)$.

If we wish to have an M -matrix, we have the conditions

$$\varepsilon_2 \beta_{i+1} + \bar{h}_i \gamma_{i+1} \leq \frac{\varepsilon_1}{h_{i+1}} \quad \text{and} \quad \gamma_{i-1} \bar{h}_i \leq \varepsilon_2 (-\beta_{i-1}) + \frac{\varepsilon_1}{h_i},$$

which for a lumped scheme ($\gamma_{i-1} = \gamma_{i+1} = 0$ and $\gamma_i^{\text{new}} = \gamma_{i-1} + \gamma_i + \gamma_{i+1}$) reduce to

$$\varepsilon_2 \beta_{i+1} \leq \frac{\varepsilon_1}{h_{i+1}}.$$

Because these conditions are too restrictive, one has to stabilize the FEM to generate M -matrices.

For the SDFEM we modify the Galerkin bilinear operator a_G by introducing lumping for the c -term:

$$a_{GL}(u, v) := \varepsilon_1(u', v') + \varepsilon_2(bu', v) + \sum_{k=1}^{N-1} \bar{h}_k c_k u_k v_k.$$

The streamline-diffusion method is characterized by the stabilization

$$a_{ST}(u, v) := \sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_k (-\varepsilon_1 u'' + \varepsilon_2 bu' + cu) v'$$

with some elementwise-defined parameter δ_k which will be determined later.

Now we again choose $V_h \subset V$ to be the space of piecewise linear elements on the given grid. Introducing $a(u, v) = a_{GL}(u, v) + a_{ST}(u, v)$, we define the SDFEM discretization: find $u_h \in V_h$ such that

$$a(u_h, v_h) = f_h(v_h) \quad \forall v_h \in V_h$$

with

$$f_h(v_h) := (f, v_h) + \sum_{k=1}^N \int_{x_{k-1}}^{x_k} \delta_k f v'_h.$$

The SDFEM generates a finite difference scheme, which can be written in the form

$$L^N u_i := -\varepsilon_1 (D^+ u_i - D^- u_i) + \alpha_i D^+ u_i + \beta_i D^- u_i + \gamma_i u_i = f_h(\varphi_i), \quad (3.1)$$

where

$$\begin{aligned} \alpha_i &= h_{i+1} \int_{x_i}^{x_{i+1}} (\varepsilon_2 b \varphi'_{i+1} \varphi_i + \delta_{i+1} \varepsilon_2 b \varphi'_{i+1} \varphi'_i + \delta_{i+1} c \varphi_{i+1} \varphi'_i), \\ \beta_i &= h_i \int_{x_{i-1}}^{x_i} (\varepsilon_2 b \varphi'_i \varphi_i + \delta_i \varepsilon_2 b \varphi'_i \varphi'_i + \delta_i c \varphi_{i-1} \varphi'_{i-1}), \\ \gamma_i &= \bar{h}_i c_i + \delta_i \int_{x_{i-1}}^{x_i} c \varphi'_i + \delta_{i+1} \int_{x_i}^{x_{i+1}} c \varphi'_i. \end{aligned}$$

The choice of the parameter δ_i is determined by the structure of the coefficient matrix of scheme (3.1), as we wish to have an M -matrix. Namely, if the local mesh size is small enough, standard Galerkin can be applied, i.e., it is possible to choose $\delta_i = 0$, $i = 1, \dots, N$. This leads to the condition

$$\alpha_{i-1}|_{\delta_i=0} \leq \varepsilon_1 \quad \text{i.e.,} \quad \varepsilon_2 h_i \hat{b}_i \leq \varepsilon_1 \quad \text{for } i = 1, \dots, N,$$

where $\hat{b}_i = (1/h_i) \int_{x_{i-1}}^{x_i} b \varphi_{i-1}$. Otherwise, we choose δ_i from the condition $\alpha_{i-1} = 0$, $i = 1, \dots, N$. With the notations

$$\hat{c}_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} c \varphi_i, \quad \tilde{b}_i = \frac{1}{h_i} \int_{x_{i-1}}^{x_i} b,$$

this condition is equivalent to

$$\delta_i \left(\frac{\varepsilon_2 \tilde{b}_i}{h_i} + \hat{c}_i \right) = \varepsilon_2 \hat{b}_i.$$

Finally we obtain

$$\delta_i = \begin{cases} 0 & \text{if } \varepsilon_2 h_i \hat{b}_i \leq \varepsilon_1, \\ \frac{\varepsilon_2 \hat{b}_i h_i}{\varepsilon_2 \tilde{b}_i + \hat{c}_i h_i} & \text{otherwise.} \end{cases} \quad (3.2)$$

We can see that

$$\begin{aligned} \delta_i &= \mathcal{O}(h_i) \quad \text{if } h_i \text{ is smaller than } \varepsilon_2, \\ \delta_i &= \mathcal{O}(\varepsilon_2) \quad \text{if } \varepsilon_2 \text{ is smaller than } h_i. \end{aligned}$$

In the case of $\varepsilon_2 = 1$ and $\varepsilon_1 = \varepsilon$, we choose $\delta_i = 0$ on the fine mesh. If we check the condition $\varepsilon_2 h_i \hat{b}_i \leq \varepsilon_1$, we see that it is satisfied on Ω_1 because $\mu_1 \geq (\varepsilon_2 b_0)/\varepsilon_1$, i.e., $\varepsilon_1 \geq \varepsilon_2 b_0/\mu_1$. On Ω_0 , however, the situation is different.

Summarizing, we have
on Ω_1 :

$$\delta_i = 0;$$

on Ω_g :

$$\delta_i = \begin{cases} 0 & \text{if } \varepsilon_2 H \hat{b}_i \leq \varepsilon_1, \\ \frac{\varepsilon_2 H \hat{b}_i}{\varepsilon_2 \tilde{b}_i + \hat{c}_i H} & \text{otherwise;} \end{cases}$$

on Ω_0 :

$$\delta_i = \begin{cases} 0 & \text{if } \varepsilon_2 h_0 \hat{b}_i \leq \varepsilon_1, \\ \frac{\varepsilon_2 h_0 \hat{b}_i}{\varepsilon_2 \tilde{b}_i + \hat{c}_i h_0} & \text{otherwise.} \end{cases}$$

Since $h_0 = \mathcal{O}(N^{-1} \ln N / \mu_0)$ and $1/\mu_0 = \mathcal{O}(\varepsilon_2 + \varepsilon_1^{1/2})$, the stabilization on Ω_0 is necessary if ε_1 is extremely small, roughly speaking, in comparison with $\varepsilon_2^2 N^{-1} \ln N$. The corresponding stabilization parameter is of order $\mathcal{O}(h_0)$.

3.2. Pointwise error analysis

For estimating the pointwise error we use an approach similar to [10, 11] based on a discrete Green's function. Let us define a function $G^j \in V_h$ by

$$a(\varphi_i, G^j) = \delta_{ij} \quad \text{for all } i = 1, \dots, N-1.$$

In our estimates we need bounds for the L_∞ and $W^{1,1}$ norm of G^j (the L_1 norm of its derivative). Since G^j is piecewise linear, the continuous and the discrete versions of these norms are equivalent. For the discrete norms Andreev [1] proved estimates for an upwind finite difference operator in the case of $\varepsilon_2 = 1$. In the Appendix we shall prove similar estimates for the two-parameter problem.

It is a well known fact that the scheme generated by SDFEM can be written as a generalized upwind scheme (see [13] and especially [11], Section 5). We do not want to repeat the details from [11], but it is not difficult to see that scheme (3.1) with (3.2) is equivalent to

$$L_h u_h = f_h$$

with

$$(L_h u)_i := -\frac{\varepsilon_1}{\bar{h}_i} (p_{i+1} D^+ u_i - p_i D^- u_i) + \varepsilon_2 b_i^* \frac{u_i - u_{i-1}}{\bar{h}_i} + c_i^* u_i.$$

Here all coefficients are the result of rewriting (3.1) with (3.2) in this form, especially

$$c_i^* = c_i + \frac{\delta_i}{\bar{h}_i} \hat{c}_i - \frac{\delta_{i+1}}{\bar{h}_i} \hat{c}_{i+1}.$$

Do we have $c_i^* \geq c_0^* > 0$ if $c_i \geq c_0 > 0$? It is not difficult to see that we are in trouble only if $\delta_i = 0$ and $\delta_{i+1} \neq 0$, but this happens only in the first subinterval of Ω_g with follows the transition point σ_0 . On that interval we have

$$\delta_{i+1} = \frac{\varepsilon_2 H \hat{b}_i}{\varepsilon_2 \tilde{b}_i + \hat{c}_i H} \quad \text{if } \varepsilon_1 \leq \varepsilon_2 H \hat{b}_i.$$

Do we need $c_i^* \geq c_0^* > 0$? The proof in the Appendix shows that the strong positivity is only necessary if $\varepsilon_2^2 \leq \kappa \varepsilon_1$. Let us assume for a moment that $b = \text{const.}$ and $c = \text{const.}$ Then

$$c_i^* \geq c \left(1 - \frac{2}{H} \frac{\varepsilon_2(H/2)b}{(\varepsilon_2 b + cH/2)}\right) = c \frac{H/2}{\varepsilon_2 b + cH/2}.$$

Because $\varepsilon_2^2/\kappa \leq \varepsilon_1 \leq \varepsilon_2 H b$ implies $\varepsilon_2 \leq \kappa b H$, it is true that

$$\frac{H/2}{\varepsilon_2 b + cH/2} \geq \frac{1/2}{\kappa b^2 + c/2} > 0.$$

The nonconstant-coefficient case can be handled similarly.

The discrete Green's function G^j now satisfies (see the Appendix for some details and estimates for the Green's function)

$$L_h G = \delta_h.$$

Using the linear operator $P : V \rightarrow V_h$ with

$$Pv := \sum_{i=1}^{N-1} a(v, G^i) \varphi_i$$

we have the following representation for the error at the mesh point x_j :

$$(u - u_h)(x_j) = P(u^I - u)|_{x=x_j} + (Pu - u_h)|_{x=x_j}. \quad (3.3)$$

Furthermore,

$$P(u^I - u)|_{x=x_j} = a(u - u^I, G^j)$$

and the consistency error $Pu - u_h$ vanishes for a consistent finite element method.

Based on the error representation (3.3) we want to estimate the pointwise error of our SDFEM on a S-mesh. Since the stabilization itself is consistent, the consistency error $(Pu - u_h)|_{x=x_j}$ reduces to the error induced by lumping of the c -term. That error part can be estimated as in [11] and it is of order $(N^{-1} \ln N)^2$.

To estimate the other error part we start from

$$\begin{aligned} a(u - u^I, G^j) &= \varepsilon_1((u - u^I)', (G^j)') + \sum_{k=1}^{N-1} \bar{h}_k c_k (u - u^I)(x_k) G^j(x_k) \\ &\quad + \varepsilon_2(b(u - u^I)', G^j) + \sum_{k=1}^N \delta_k (-\varepsilon_1 u'' + \varepsilon_2 b(u - u^I)' + c(u - u^I), (G^j)'). \end{aligned}$$

The first two terms are equal to zero. To estimate the remaining terms we want to apply the results concerning the discrete Green's function from the Appendix.

Now we estimate several error contributions. First, using integration by parts

$$|\varepsilon_2(b(u - u^I)', G^j)| \leq C \|u - u^I\|_\infty \|G^j\|_{L_1} + C |\varepsilon_2(u - u^I, (G^j)')|.$$

Furthermore,

$$|\varepsilon_2(u - u^I, (G^j)')| \leq C \varepsilon_2 \|u - u^I\|_\infty \|(G^j)'\|_{L_1} \leq C \frac{\varepsilon_2}{\sqrt{\varepsilon_2^2 + \varepsilon_1}} (N^{-1} \ln N)^2.$$

Next we estimate the contribution of the stabilization on Ω_g . Let us recall that

$$\delta_g = \mathcal{O}\left(\frac{\varepsilon_2 H}{\varepsilon_2 + H}\right) \quad \text{if } \varepsilon_1 \leq C\varepsilon_2 H.$$

$$(i) \quad \Sigma \varepsilon_1 \delta_k(u'', (G^j)').$$

For the smooth part we get

$$C\varepsilon_1 \delta_g \frac{1}{\sqrt{\varepsilon_2^2 + \varepsilon_1}} \leq CH^2, \quad \left(\varepsilon_1 \leq C\varepsilon_2 H, \quad \delta_g = \mathcal{O}(H), \quad \frac{\varepsilon_2}{\sqrt{\varepsilon_2^2 + \varepsilon_1}} \leq 1 \right).$$

For the layer part we get (only for E_1 because that layer is stronger)

$$C\varepsilon_1 \delta_g \|E_1''\|_{L_1} \|(G^j)'\|_\infty \leq C\varepsilon_1 \delta_g \mu_1 H^\tau H^{-1} \|G^j\|_\infty.$$

This error contribution is small enough because $\varepsilon_1 \mu_1 / \sqrt{\varepsilon_2^2 + \varepsilon_1}$ is bounded.

$$(ii) \quad \Sigma \varepsilon_2 \delta_k(b(u - u^I)', (G^j)').$$

Since $(G^j)'' = 0$, $(u - u^I)_{x_j} = 0$, integration by parts yields

$$|\Sigma \varepsilon_2 \delta_k(b(u - u^I)', (G^j)')| \leq C\varepsilon_2 \delta_g \|u - u^I\|_\infty \|(G^j)'\|_{L_1} \leq C\delta_g N^{-2}.$$

$$(iii) \quad \Sigma \delta_k(c(u - u^I), (G^j)').$$

Using $\delta_g = \mathcal{O}(\varepsilon_2)$, we get

$$|\Sigma \delta_k(c(u - u^I), (G^j)')| \leq C\varepsilon_2 \frac{1}{\sqrt{\varepsilon_2^2 + \varepsilon_1}} \|u - u^I\|_\infty \leq CN^{-2}.$$

Finally, we consider the contributions of the stabilization on Ω_0 . Let us recall that

$$\delta_0 = \mathcal{O}(h_0) \quad \text{if } \varepsilon_1 \leq C\varepsilon_2 h_0,$$

where

$$h_0 = \mathcal{O}\left(\frac{N^{-1} \ln N}{\mu_0}\right), \quad \frac{1}{\mu_0} = \mathcal{O}\left(\varepsilon_2 + \sqrt{\varepsilon_2^2 + \varepsilon_1}\right).$$

$$(i') \quad \Sigma \varepsilon_1 \delta_0(u'', (G^j)').$$

For the smooth part there is no problem. For the layer part E_2 (E_1 is extremely small on Ω_0) using $\varepsilon_1 \leq C\varepsilon_2 h_0$, $h_0 = \mathcal{O}(N^{-1} \ln N) / \mu_0$, we get

$$|\Sigma \varepsilon_1 \delta_0(u'', (G^j)')| \leq \varepsilon_1 \delta_0 \mu_0^2 \|(G^j)'\|_{L_1} \leq C\varepsilon_2 (h_0 \mu_0)^2 \frac{1}{\sqrt{\varepsilon_2^2 + \varepsilon_1}} \leq C(N^{-1} \ln N)^2.$$

The two other error terms on Ω_0 , which correspond to (ii) and (iii), can be estimated as on Ω_g .

Theorem 3.1. *The pointwise error of the SDFEM on a S-mesh satisfies the following estimate:*

$$\|u - u_h\|_\infty \leq C(N^{-1} \ln N)^2.$$

4. Appendix: The discrete Green's function

Denoting the upwind difference operator on an arbitrary mesh by L_h :

$$(L_h u_i) := -\frac{\varepsilon_1}{\bar{h}_i} (D^+ u_i - D^- u_i) + b_i \varepsilon_2 \frac{u_i - u_{i-1}}{\bar{h}_i} + c_i u_i,$$

(assuming $b_i > 0, c_i > 0$ as above), we define the discrete Green's function $G(x_i, \xi_j)$ for L_h , for fixed ξ_j , by

$$L_h G(x_i, \xi_j) = \delta_h(x_i, \xi_j) \quad G(0, \xi_j) = G(1, \xi_j) = 0,$$

$$\delta_h(x_i, \xi_j) = \begin{cases} \frac{1}{\bar{h}_i} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Setting $W = \sqrt{\varepsilon_2^2 + \varepsilon_1}$, we shall prove

$$\|G(x_i, \xi_j)\|_\infty \leq \frac{C}{W}. \quad (4.1)$$

To do this, let us introduce the barrier function w defined by

$$w_j = \frac{C}{W} \begin{cases} \prod_{k=i}^j \left(1 - \frac{\mu_0}{2} h_k\right)^{-1} & \text{for } j > i, \\ \prod_{k=j}^i \left(1 + \frac{\mu_1}{2} h_k\right)^{-1} & \text{for } j < i. \end{cases} \quad (4.2)$$

From [3] we know already that $L_h w \geq 0$ at the mesh points $\{x_1, \dots, x_{i-1}\}$ and $\{x_{i+1}, \dots, x_{N-1}\}$. It remains to verify

$$(L_h w)_i \geq \frac{1}{\bar{h}_i}, \quad (4.3)$$

if the constant C in (4.2) is chosen adequately.

We use the abbreviations

$$\bar{\mu}_0 = \frac{\mu_0}{2} < 0, \quad \bar{\mu}_1 = \frac{\mu_1}{2} > 0$$

and introduce

$$\tilde{w} = \begin{pmatrix} \tilde{w}_{i+1} \\ \tilde{w}_i \\ \tilde{w}_{i-1} \end{pmatrix} = \begin{pmatrix} (1 - \bar{\mu}_0 h_{i+1})^{-1} \\ 1 \\ (1 + \bar{\mu}_1 h_i)^{-1} \end{pmatrix}.$$

Then

$$D^+ \tilde{w} = \frac{\bar{\mu}_0}{1 - \bar{\mu}_0 h_{i+1}}, \quad D^- \tilde{w} = \frac{\bar{\mu}_1}{1 + \bar{\mu}_1 h_i},$$

and we obtain

$$(L_h w)_i = \frac{C}{W} \left(-\frac{\varepsilon_1}{\bar{h}_i} \left(\frac{\bar{\mu}_0}{1 - \bar{\mu}_0 h_{i+1}} - \frac{\bar{\mu}_1}{1 + \bar{\mu}_1 h_i} \right) + \varepsilon_2 b_i h_i \frac{\bar{\mu}_1}{\bar{h}_i (1 + \bar{\mu}_1 h_i)} + c_i \right)$$

Case A: $\varepsilon_2^2 \leq \kappa \varepsilon_1$.

In this case we neglect the b -term and use the existence of a constant C_1 such that

$$\mu_1 \leq 2C_1 \frac{\varepsilon_2 + W}{\varepsilon_1} = 2\mu_1^* \quad \text{as well as} \quad -\mu_0 \leq 2\mu_1^*.$$

Then

$$\frac{1}{1 - \bar{\mu}_0 h_{i+1}} \geq \frac{1}{1 + \mu_1^* \bar{h}_i}, \quad \frac{1}{1 + \bar{\mu}_1 h_i} \geq \frac{1}{1 + \mu_1^* \bar{h}_i},$$

and we obtain

$$(L_h w)_i \geq \frac{C}{\bar{h}_i} \left(\frac{\varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0)}{W(1 + \mu_1^* \bar{h}_i)} + c_i \frac{\bar{h}_i}{W} \right).$$

If $\bar{h}_i \geq \rho W$ for some $\rho > 0$, we have the desired estimate. Otherwise, we have

$$\frac{\varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0)}{W(1 + \mu_1^* \bar{h}_i)} \geq \frac{\varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0)}{W} \frac{1}{1 + \mu_1^* \rho \bar{h}_i}.$$

Since $\varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0)/W$ is bounded away from zero and $\mu_1^* W$ is bounded if $\varepsilon_2^2 \leq \kappa \varepsilon_1$, it follows that (4.3) holds true if C is chosen adequately.

Case B: $\varepsilon_2^2 \geq \kappa \varepsilon_1$

In this case we neglect the c -term and get

$$\begin{aligned} (L_h w)_i &\geq \frac{C}{W} \frac{1}{\bar{h}_i(1 + \bar{\mu}_1 h_i)} \left\{ \varepsilon_1 \bar{\mu}_1 - \varepsilon_1 \bar{\mu}_0 \frac{1 + \bar{\mu}_1 h_i}{1 - \bar{\mu}_0 h_{i+1}} + \varepsilon_2 b_i \bar{\mu}_1 h_i \right\} \\ &= \frac{C}{W} \frac{1}{\bar{h}_i(1 + \bar{\mu}_1 h_i)} \left\{ \varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0) - \varepsilon_1 \bar{\mu}_0 \frac{\bar{\mu}_1 h_i + \bar{\mu}_0 h_{i+1}}{1 - \bar{\mu}_0 h_{i+1}} + \varepsilon_2 b_i \bar{\mu}_1 h_i \right\}. \end{aligned}$$

Using

$$\bar{\mu}_1 h_i + \bar{\mu}_0 h_{i+1} = \bar{\mu}_1 h_i(1 - \bar{\mu}_0 h_{i+1}) + \bar{\mu}_0 h_{i+1}(1 + \bar{\mu}_1 h_i)$$

we obtain

$$\begin{aligned} (L_h w)_i &\geq \frac{C}{W} \frac{1}{\bar{h}_i(1 + \bar{\mu}_1 h_i)} \left\{ \varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0) + (\varepsilon_2 b_i - \varepsilon_1 \bar{\mu}_0) \bar{\mu}_1 h_i - \frac{\varepsilon_1 \bar{\mu}_0 \bar{\mu}_0 h_{i+1}(1 + \bar{\mu}_1 h_i)}{1 - \bar{\mu}_0 h_{i+1}} \right\} \\ &= \frac{C}{\bar{h}_i} \left\{ \frac{\varepsilon_1(\bar{\mu}_1 - \bar{\mu}_0)}{W} + \frac{\varepsilon_2 b_i - \varepsilon_1 \bar{\mu}_0}{W} \bar{\mu}_1 h_i - \frac{\varepsilon_1 \bar{\mu}_0 \bar{\mu}_0 h_{i+1}}{W(1 - \bar{\mu}_0 h_{i+1})} \right\}. \end{aligned}$$

This expression has the structure

$$(L_h w)_i \geq \frac{C}{W} \left\{ \frac{D + E \bar{\mu}_1 h_i}{1 + \bar{\mu}_1 h_i} - F \right\}$$

with $|F| < E$, $|F| < D$ and $|F| < -\varepsilon_1 \bar{\mu}_0 / W$.

Since $D - F$ is bounded away from zero, it remains to show that $E - F$ is bounded away from zero also. We have the necessary bound since $\varepsilon_2^2 \geq \kappa \varepsilon_1$ and $E - F > \varepsilon_2 b_i / W$.

Remark 4.1. It is a simple exercise to replace the difference operator by the more general operator

$$(L_h u)_i := -\frac{\varepsilon_1}{h_i} (p_{i+1} D^+ u_i - p_i D^- u_i) + \varepsilon_2 b_i^* \frac{u_i - u_{i-1}}{h_i} + c_i^* u_i,$$

assuming $p_i \geq p > 0$, and again $b_i^* > 0, c_i^* > 0$. (SDFEM is related to a difference operator of that form)

Remark 4.2. The estimate remains true if we consider the standard upwind difference operator with the difference quotient $(u_i - u_{i-1})/h_i$.

Let us summarize the properties of the discrete Green's function for the two parameter problem:

Lemma 4.1. *The discrete Green's function G satisfies*

$$\|G\|_{L_1} \leq C; \quad \|G\|_{\infty} \leq \frac{C}{\sqrt{\varepsilon_2^2 + \varepsilon_1}}; \quad \|G'\|_{L_1} \leq \frac{C}{\sqrt{\varepsilon_2^2 + \varepsilon_1}}.$$

The L_1 estimate can be obtained by summing up the equations for G at all meshpoints. The $W^{1,1}$ bound follows from the L_{∞} bound from the same arguments used by Andreev [1, p. 925].

5. Numerical experiments

In this section we present some numerical results for scheme (3.1), applied to the following example.

Example 5.1.

$$-\varepsilon_1 u'' + \varepsilon_2 u' + u = \cos \pi x \quad x \in (0, 1), \quad u(0) = u(1) = 0.$$

The exact solution is

$$u(x) = a \cos \pi x + b \sin \pi x + A e^{\lambda_0 x} + B e^{-\lambda_1(1-x)},$$

where

$$a = \frac{\varepsilon_1 \pi^2 + 1}{\varepsilon_2^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2}, \quad b = \frac{\varepsilon_2 \pi}{\varepsilon_2^2 \pi^2 + (\varepsilon_1 \pi^2 + 1)^2},$$

$$A = -a \frac{1 + e^{-\lambda_1}}{1 - e^{\lambda_0 - \lambda_1}}, \quad B = a \frac{1 + e^{\lambda_0}}{1 - e^{\lambda_0 - \lambda_1}}, \quad \lambda_{0,1} = \frac{\varepsilon_2 \mp \sqrt{\varepsilon_2^2 + 4\varepsilon_1}}{2\varepsilon_1}.$$

Table 1. Errors E^N at the mesh points and convergence rates Ord^N for $N = 2^5$

ε_2	ε_1							
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	
1	2.272e-05	2.124e-04	1.513e-03	6.686e-02	9.962e-02	1.031e-01	1.024e-01	E^N
	2.004	2.002	2.028	3.782	-0.033	-0.013	-0.096	Ord^N
10^{-1}	1.189e-05	1.316e-04	8.723e-04	1.173e-02	1.585e-01	2.067e-01	2.215e-01	E^N
	2.004	2.003	2.006	2.130	0.411	0.0897	0.094	Ord^N
10^{-2}	6.416e-06	7.698e-05	4.598e-04	4.171e-03	2.822e-01	3.223e-01	3.384e-01	E^N
	2.002	2.002	2.002	2.028	3.843	0.097	0.088	Ord^N
10^{-3}	3.850e-06	4.882e-05	3.673e-04	1.124e-03	2.330e-02	2.968e-01	2.981e-01	E^N
	2.002	2.001	1.998	1.996	2.136	0.028	0.100	Ord^N
10^{-4}	2.682e-06	3.538e-05	3.068e-04	1.268e-03	3.597e-03	1.772e-01	1.882e-01	E^N
	2.002	2.000	2.000	1.996	1.973	3.195	-0.147	Ord^N
10^{-5}	2.147e-06	2.900e-05	2.691e-04	1.260e-03	4.243e-03	1.392e-02	1.352e-01	E^N
	2.001	2.001	1.999	1.995	1.980	2.029	0.497	Ord^N
10^{-6}	1.887e-06	2.588e-05	2.486e-04	1.221e-03	4.255e-03	1.467e-02	4.869e-02	E^N
	1.996	1.999	1.999	1.995	1.955	1.878	1.948	Ord^N
10^{-7}	1.770e-06	2.441e-05	2.373e-04	1.189e-03	4.216e-03	1.524e-02	4.609e-02	E^N
	2.001	2.000	1.995	1.991	1.974	1.912	1.670	Ord^N
10^{-8}	1.711e-06	2.367e-05	2.316e-04	1.169e-03	4.154e-03	1.505e-02	4.266e-02	E^N
	2.001	2.000	1.994	1.988	1.978	1.912	1.506	Ord^N
10^{-9}	1.682e-06	2.329e-05	2.290e-04	1.158e-03	4.112e-03	1.480e-02	3.760e-02	E^N
	2.000	2.000	1.995	1.987	1.978	1.912	1.344	Ord^N
10^{-10}	1.667e-06	2.311e-05	2.277e-04	1.153e-03	4.088e-03	1.463e-02	3.494e-02	E^N
	2.000	2.000	1.996	1.986	1.978	1.913	1.264	Ord^N
10^{-11}	1.656e-06	2.296e-05	2.267e-04	1.150e-03	4.068e-03	1.449e-02	3.290e-02	E^N
	2.000	2.000	1.996	1.987	1.978	1.913	1.204	Ord^N
10^{-12}	1.653e-06	2.293e-05	2.265e-04	1.149e-03	4.063e-03	1.445e-02	3.238e-02	E^N
	2.000	2.000	1.996	1.987	1.978	1.913	1.189	Ord^N
10^{-13}	1.652e-06	2.292e-05	2.264e-04	1.148e-03	4.062e-03	1.444e-02	3.221e-02	E^N
	2.000	2.000	1.996	1.987	1.978	1.913	1.184	Ord^N
10^{-14}	1.652e-06	2.292e-05	2.264e-04	1.148e-03	4.061e-03	1.444e-02	3.221e-02	E^N
	2.000	2.000	1.996	1.987	1.978	1.913	1.184	Ord^N

In Table 1 and Table 2 we present maximum pointwise errors E^N for $N = 2^5$ and $N = 2^{10}$, respectively, which are given by

$$E^N = \max_{0 \leq i \leq N} |u(x_i) - u_i|.$$

Assuming the convergence order $(N^{-1})^r$, for some r , we use the double-mesh method to compute the experimental rates of convergence Ord^N . For every fixed ε_1 and ε_2 , we computed the rate of convergence from

$$Ord^N = \frac{\ln E^N - \ln E^{2N}}{\ln 2}.$$

The results clearly show robust uniform convergence of order two.

Table 2. Errors E^N at the mesh points and convergence rates Ord^N for $N = 2^{10}$

ε_2	ε_1							
	1	10^{-2}	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	
1	6.768e-08	7.393e-07	5.712e-06	6.753e-05	1.502e-04	1.407e-01	1.063e-01	E^N
	4.996	2.577	2.037	2.003	1.700	11.58	-0.002	Ord^N
10^{-1}	3.532e-08	4.236e-07	3.283e-06	4.074e-05	3.732e-04	3.813e-04	1.969e-01	E^N
	4.674	3.404	2.145	2.005	1.701	1.686	-0.443	Ord^N
10^{-2}	1.982e-08	2.504e-07	1.595e-06	1.595e-05	2.843e-04	7.012e-04	3.467e-01	E^N
	3.566	3.223	2.674	2.019	2.000	1.693	10.58	Ord^N
10^{-3}	1.268e-08	1.680e-07	1.338e-06	4.249e-06	8.206e-05	8.452e-04	9.594e-04	E^N
	2.905	2.683	2.351	2.167	2.002	1.698	1.691	Ord^N
10^{-4}	9.414e-09	1.281e-07	1.156e-06	4.907e-06	1.433e-05	3.720e-04	9.376e-04	E^N
	2.499	2.372	2.171	2.059	2.015	2.000	1.699	Ord^N
10^{-5}	7.889e-09	1.086e-07	1.033e-06	4.916e-06	1.693e-05	5.448e-05	6.448e-04	E^N
	2.264	2.196	2.086	2.026	2.005	2.001	1.706	Ord^N
10^{-6}	7.158e-09	9.900e-08	9.630e-07	4.782e-06	1.724e-05	6.425e-05	2.149e-04	E^N
	2.136	2.101	2.044	2.013	2.002	2.000	1.999	Ord^N
10^{-7}	6.801e-09	9.426e-08	9.258e-07	4.672e-06	1.693e-05	6.529e-05	2.530e-04	E^N
	2.069	2.052	2.022	2.006	2.001	1.999	1.998	Ord^N
10^{-8}	6.625e-09	9.190e-08	9.067e-07	4.606e-06	1.663e-05	6.404e-05	2.570e-04	E^N
	2.035	2.026	2.011	2.003	2.000	1.999	1.999	Ord_{u^n}
10^{-9}	6.538e-09	9.072e-08	8.971e-07	4.570e-06	1.644e-05	6.285e-05	2.516e-04	E^N
	2.018	2.013	2.006	2.002	2.000	2.000	1.997	Ord^N
10^{-10}	6.494e-09	9.013e-08	8.922e-07	4.551e-06	1.634e-05	6.207e-05	2.470e-04	E^N
	2.009	2.007	2.003	2.001	2.000	1.999	1.998	Ord^N
10^{-11}	6.462e-09	8.969e-08	8.885e-07	4.537e-06	1.625e-05	6.144e-05	2.423e-04	E^N
	2.002	2.002	2.001	2.000	2.000	1.999	1.998	Ord^N
10^{-12}	6.454e-09	8.958e-08	8.876e-07	4.533e-06	1.623e-05	6.127e-05	2.410e-04	E^N
	2.001	2.000	2.000	2.000	2.000	1.999	1.998	Ord_{u^n}
10^{-13}	6.451e-09	8.955e-08	8.873e-07	4.532e-06	1.623e-05	6.122e-05	2.406e-04	E^N
	2.000	2.000	2.000	2.000	2.000	1.999	1.998	Ord^N
10^{-14}	6.451e-09	8.955e-08	8.873e-07	4.532e-06	1.623e-05	6.121e-05	2.406e-04	E^N
	2.000	2.000	2.000	2.000	2.000	1.999	1.998	Ord^N

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