ACCURACY ESTIMATES OF DIFFERENCE SCHEMES FOR QUASI-LINEAR PARABOLIC EQUATIONS TAKING INTO ACCOUNT THE INITIAL-BOUNDARY EFFECT

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Abstract — For difference schemes the initial-boundary problem for quasi-linear parabolic-type equations, 'a priori weight estimates' of the error have been found. These estimates show how much the accuracy of difference schemes near the boundary of a time rectangle is higher than in the middle of it. Sufficient conditions of smoothness of the coefficients and the right-hand side of the quasi-linear parabolic equation and the initial conditions have been found. These conditions ensure a correctness of these a priori estimates.

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1. Introduction

The subject of this paper is associated with the ideas of monograph [9]. These ideas were announced in [5] and referred to the obtaining of a special form of 'a priori weight estimates' of difference schemes for elliptic equations in $\mathbb{R}^1$, $\mathbb{R}^2$ with the Dirichlet conditions in canonical domains (segments, squares). These estimates have shown how the order of accuracy of the difference schemes increases near the boundary of the domain. This phenomenon is quite natural and qualitatively well-known. It is due to fact that the error of the difference scheme for Dirichlet conditions equals zero at the boundary of a canonical domain. Quantitative estimates of the boundary effect were first given in [5]. As a development of the results of [5], we put the question, whether or not the accuracy of difference schemes for parabolic equations increases near $t = 0$ and the boundary of the canonical domain, for which the initial and Dirichlet conditions are given. The basis for the affirmative answer to this question is the fact that the error of difference schemes equals zero at $t = 0$ and at the domain boundary. For the linear case, this is really so [6]. The main goal of this paper is to generalize the

results of [6] to quasi-linear parabolic equations by using the technique developed in [2–4], for both constructing of difference schemes and analyzing them.

The paper is organized as follows. In Section 2, we investigate a boundary effect of the accuracy of the difference schemes increasing close to the domain boundary, and Section 3 is devoted to the initial effect of the difference schemes accuracy increasing near $t = 0$ with Cauchy conditions.

2. A priori weight error estimates of the difference schemes taking into account the boundary effect

Let $Q_T = (0, 1) \times (0, T)$ denote the time rectangular with a boundary Γ_T , $\overline{Q_T} = Q_T \cup \Gamma_T$. We will use in this paper the following generally accepted notations for Banach spaces $\overset{0}{W}_2^1(0, 1)$, $H^{2+\beta}[0, 1]$, $W_2^{1,1}(Q_T)$, $H^{l, \frac{l}{2}}(\overline{Q_T})$ (see, e.g. [2]).

Let us introduce the grid $\omega_{Q_T} = \omega \times \omega_\tau$ in the rectangular Q_T , where

$$\begin{aligned}\omega &= \{x_i = ih : i = \overline{1, N-1}, h = 1/N\}, \\ \omega_\tau &= \{t_j = j\tau : j = \overline{1, M-1}, \tau = T/M\}, \\ \overline{\omega} &= \omega \cup \{0\} \cup \{1\}, \quad \omega^+ = \omega \cup \{1\}.\end{aligned}$$

For discrete functions defined on the grid ω_{Q_T} , we use the following notations [7]:

$$\begin{aligned}y(x, t) &= y(x, t_j) = y^j = y, \quad y^{(\pm 1)} = y(x \pm h, t), \\ y_t &= y_t^j = (y^{j+1} - y^j)/\tau = (\hat{y} - y)/\tau, \quad y_{\bar{x}} = (y - y^{(-1)})/h, \\ y_x &= (y^{(+1)} - y)/h, \quad y_{\bar{x}\bar{x}} = (y^{(+1)} - 2y + y^{(-1)})/h^2, \quad y^\pm = \frac{1}{2}(\hat{y} + y), \\ (y, v) &= (y(x, t), v(x, t)) = (y(\cdot, t), v(\cdot, t)) = \sum_{x \in \omega} y(x, t)v(x, t)h, \\ \|y\| &= \|y(x, t)\| = \|y(\cdot, t)\| = (y, y)^{1/2}, \\ (y, v) &= (y(x, t), v(x, t)) = (y(\cdot, t), v(\cdot, t)) = \sum_{x \in \omega^+} y(x, t)v(x, t), \\ \|y\| &= \|y(x, t)\| = \|y(\cdot, t)\| = (y, y)^{1/2}.\end{aligned}$$

Let $\overset{0}{H}_h$ denote the Hilbert space of discrete functions defined on the grid $\overline{\omega}$, vanishing at points $x = 0, 1$ with scalar product $(u, v) \forall u, v \in \overset{0}{H}_h$.

For functions $u(x, t)$ defined on Q_T , we introduce the following averaging operator of the exact finite difference schemes [9]

$$T^x(v(\cdot)) = \frac{1}{h^2} \int_{x-h}^{x+h} (h - |x - \xi|) v(\xi) d\xi, \quad x \in \omega.$$

Here the dot in the function argument shows that corresponding operator is acting on this argument. It is known that the operator T^x have the following properties [9]:

- $T^x \left(\frac{\partial^2 u(\cdot, t)}{\partial x^2} \right) = u_{\bar{x}x}, \quad x \in \omega;$
- $T^x(u(\cdot)) = u$ if u is a polynomial of the first degree.

The operator T^x is considered in the literature as iterated Steklov averaging operator T_+^x :

$$T^x = (T_+^x)^2, \quad \text{where } T_+^x(u(\cdot, t)) = \frac{1}{h} \int_{x-h/2}^{x+h/2} u(\xi, t) d\xi.$$

We consider the following initial-boundary problem for the quasi-linear parabolic equation:

$$\begin{aligned} \frac{\partial u(x, t)}{\partial t} &= \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t, u(x, t)), \quad (x, t) \in Q_T, \\ u(0, t) &= u(1, t) = 0, \quad t > 0, \\ u(x, 0) &= \varphi(x), \end{aligned} \quad (1)$$

where the function $f(x, t, u)$ satisfies the Lipschitz condition

$$|f(x, t, u_1) - f(x, t, u_2)| \leq L |u_1 - u_2| \quad \forall (x, t) \in \overline{Q_T}, \quad u_1, u_2 \in \mathbb{R}^1. \quad (2)$$

For the sake of simplicity we restrict ourselves to the consideration of the one-dimensional case.

We propose the following finite-difference approximation of problem (1):

$$\begin{aligned} y_t(x, t) &= \frac{1}{2} [\hat{y}(x, t) + y(x, t)]_{\bar{x}x} + T^x \left(f \left(\cdot, t + \frac{\tau}{2}, \frac{\hat{y}(x, t) + y(x, t)}{2} \right) \right), \quad (x, t) \in \omega_{Q_T}, \\ y(0, t) &= y(1, t) = 0, \quad t \in \omega_\tau, \quad y(x, 0) = \varphi(x), \quad x \in \omega, \end{aligned} \quad (3)$$

where $y(x, t)$ is an approximation of $u(x, t)$.

Let us introduce the function $z(x, t) = y(x, t) - u(x, t)$ characterizing the error of the difference method. Substituting $y = z + u$ into (3), we obtain the following finite-difference scheme for the error z :

$$\begin{aligned} z_t(x, t) - \frac{1}{2} [\hat{z}(x, t) + z(x, t)]_{\bar{x}x} &= T^x \left(f \left(\cdot, t + \frac{\tau}{2}, \frac{\hat{y}(x, t) + y(x, t)}{2} \right) \right. \\ &\quad \left. - f \left(\cdot, t + \frac{\tau}{2}, \frac{\hat{u}(x, t) + u(x, t)}{2} \right) \right) + \psi(x, t, u), \quad (x, t) \in \omega_{Q_T}, \\ z(0, t) &= z(1, t) = 0, \quad t \in \omega_\tau, \quad z(x, 0) = 0, \quad x \in \omega, \end{aligned} \quad (4)$$

where $\psi(x, t, u)$ is the local truncation error which has the following form:

$$\begin{aligned} \psi(x, t, u) &= T^x \left(f \left(\cdot, t + \frac{\tau}{2}, u^+(x, t) \right) - \frac{1}{2} \left(\hat{f}(\cdot, t, u(\cdot, t)) + f(\cdot, t, u(\cdot, t)) \right) \right. \\ &\quad \left. + \frac{d}{dt} T^x(u^+(\cdot, t)) \right) - u_t(x, t). \end{aligned} \quad (5)$$

To obtain the a priori estimate of the error, we scalar-multiply both sides of equation (5) by themselves and make use of condition (2). Then we have

$$\begin{aligned} & \|z_t(x, t)\|^2 + \|z^+(x, t)_{\bar{x}x}\|^2 - 2(z_t(x, t), z^+(x, t)_{\bar{x}x}) \\ &= \|T^x(f(\cdot, t + \tau/2, y^+(x, t)) - f(\cdot, t + \tau/2, u^+(x, t)))\|^2 + \|\psi(x, t, u)\|^2 \\ &+ 2(T^x(f(\cdot, t + \tau/2, y^+(x, t)) - f(\cdot, t + \tau/2, u^+(x, t))), \psi(x, t, u)) \\ &\leq 2L^2 \|z^+(x, t)\|^2 + 2\|\psi(x, t, u)\|^2. \end{aligned} \quad (6)$$

For further transformations of inequality (6) we need

Lemma 2.1. *For any function $v(x, t)$ which at each fixed $t \in \bar{\omega}_\tau$ belongs to the space H_h^0 and $v(x, 0) = 0 \quad \forall x \in \bar{\omega}$, the equality*

$$-2 \sum_{\eta=0}^{t-\tau} \tau (v_\eta(x, \eta), v^+(x, \eta)_{\bar{x}x}) = \|v_{\bar{x}}(x, t)\|^2$$

holds.

Proof. Using the formulas of summation by parts, we have

$$-2(v_\eta(x, \eta), v^+(x, \eta)_{\bar{x}x}) = \frac{1}{\tau} \|v_{\bar{x}}(x, \eta + \tau)\|^2 - \frac{1}{\tau} \|v_{\bar{x}}(x, \eta)\|^2 \quad \forall v(x, \eta) \in H_h^0.$$

Multiply both sides of this equality by τ , then sum both sides by η from 0 to $t - \tau$ and take into account the condition $v(x, 0) = 0 \quad \forall x \in \bar{\omega}$. As a result, we obtain the required identity. $\square$

Let us multiply both sides of inequality (2.6) by τ , sum by η from 0 to $t - \tau$, and take into account Lemma 2.1. Then we get

$$\begin{aligned} & \sum_{\eta=0}^{t-\tau} \tau \|z_\eta(x, \eta)\|^2 + \sum_{\eta=0}^{t-\eta} \tau \|z^+(x, \eta)_{\bar{x}x}\|^2 + 2\|z_{\bar{x}}(x, t)\|^2 \\ & \leq \sum_{\eta=0}^{t-\tau} \left[2L^2 \|z^+(x, \eta)\|^2 + 2\|\psi(x, \eta, u)\|^2 \right] \\ & \leq L^2 \sum_{\eta=0}^t \tau \|z_{\bar{x}}(x, \eta)\|^2 + 2 \sum_{\eta=0}^{t-\tau} \tau \|\psi(x, \eta, u)\|^2. \end{aligned} \quad (7)$$

The solution of inequality (7) can be obtained with the help of discrete analog of Gronwall lemma (see, e.g., [8, p. 171]). We will give here a more subtle result, but under stronger assumption than in [8, p. 171].

Lemma 2.2 [6]. *We will seek a nonnegative solution $v(t)$ of the following inequality:*

$$v(t) \leq c + \sum_{\eta=\tau}^t \tau g(\eta) v(\eta), \quad t = 0, \tau, \dots, \quad \tau > 0, \quad (8)$$

where $c > 0$, $g(\eta)$ at $\eta \geq 0$ is a nonnegative, nonincreasing function of the continuous argument and such that

$$\tau \max_{\eta \geq \tau} g(\eta) \leq \frac{1}{2}. \quad (9)$$

Then for the solution of inequality (8) the estimate

$$v(t) \leq c \exp \left\{ \ln 4 \int_0^t g(\eta) d\eta \right\}$$

holds.

Proof. It is easy to see that the solution of inequality (8) is majored by the solution of the difference equation

$$V(t) = c + \sum_{\eta=\tau}^t \tau g(\eta) V(\eta), \quad t = 0, \tau, \dots \quad (10)$$

From (10) the equality

$$V(t) - V(t - \tau) = \tau g(t) V(t)$$

or

$$V(t) = [1 - \tau g(t)]^{-1} V(t - \tau) \quad (11)$$

follows.

We use the inequality

$$(1 - \alpha)^{-1/\alpha} \leq 4, \quad \alpha \in [0, 1/2].$$

Then, taking into account condition (9), from (11) we consecutively get

$$V(t) \leq 4^{\tau g(t)} V(t - \tau) \leq \dots \leq 4^{[g(t) + g(t-\tau) + \dots + g(\tau)]} c. \quad (12)$$

Since the function $g(\eta)$ is nonincreasing at $\eta \geq 0$, the inequality

$$\sum_{\eta=\tau}^t \tau g(\eta) = \sum_{k=1}^{t/\tau} \tau g(k\tau) \leq \int_0^t g(\eta) d\eta$$

holds and together with (12) proves the lemma statement. $\square$

Returning to inequality (7) by the just proved lemma we get

$$\sum_{\eta=0}^{t-\tau} \tau \|z_\eta(x, \eta)\|^2 + \sum_{\eta=0}^{t-\tau} \tau \|z_{\bar{x}\bar{x}}^+(x, \eta)\|^2 + 2 \|z_{\bar{x}}(x, t)\|^2 \leq 2 \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2 4^{\tau L^2/2} \quad (13)$$

under the condition that

$$L^2 \tau \leq 1. \quad (14)$$

Then we rewrite the difference scheme (4) in the equivalent form

$$z^+(x, t) = \sum_{\xi \in \omega} hG(x, \xi) [\psi(\xi, t, u) - z_t(\xi, t) + T^\xi T_0 (f(\cdot, \cdot, y^+(\xi, t)) - f(\cdot, \cdot, u^+(\xi, t)))] , \quad (15)$$

where

$$G(x, \xi) = \begin{cases} x(1 - \xi), & 0 \leq x \leq \xi, \\ \xi(1 - x), & \xi \leq x \leq 1 \end{cases}$$

is the Green function of the operator $\Lambda v(x) = -v_{\bar{x}x}(x)$, $x \in \omega$, $v(x) \in \overset{0}{H}_h$. Hence

$$0 \leq G(x, \xi) \leq x(1 - x) \quad \forall \xi \in \bar{\omega}.$$

Making use of condition (2), from (15) we obtain

$$\left| \frac{z^+(x, t)}{x(1 - x)} \right| \leq \|\psi(\cdot, t, u)\| + \|z_t(\cdot, t)\| + L \|z^+(\cdot, t)\| ,$$

which together with (13) leads to the inequality

$$\begin{aligned} \sum_{\eta=0}^{t-\tau} \tau \left[\frac{z^+(x, \eta)}{x(1 - x)} \right]^2 &\leq 3 \sum_{\eta=0}^{t-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 + 6 \cdot 4^{tL^2/2} \sum_{\eta=0}^{t-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 \\ &\quad + 3L^2 \sum_{\eta=0}^{t-\tau} \tau \left\| \sum_{\xi=h}^{\cdot} h z^+(\xi, t) \right\|^2 \\ &\leq \left(3 + 6 \cdot 4^{tL^2/2} \right) \sum_{\eta=0}^{T-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 + \frac{3}{2} L^2 \cdot 2 \sum_{\eta=0}^{T-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 \sum_{\eta=0}^{T-\tau} \tau 4^{\eta L^2/2} \\ &= \left[3 + 6 \cdot 4^{tL^2/2} + 3L^2 \frac{4^{TL^2/2} - 1}{4^{\tau L^2/2} - 1} \tau \right] \sum_{\eta=0}^{T-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 \\ &\leq \left[3 + 6 \cdot 4^{TL^2/2} + \frac{6}{\ln 4} 4^{TL^2/2} \right] \sum_{\eta=0}^{T-\tau} \tau \|\psi(\cdot, \eta, u)\|^2. \end{aligned}$$

Thus, we obtain the following statement.

Lemma 2.3. *Assume that conditions (2) and (14) are satisfied. Then for the error of difference scheme (3) the following a priori weight estimate is valid:*

$$\left\{ \sum_{\eta=0}^{t-\tau} \tau \left[\frac{z^+(x, \eta)}{x(1 - x)} \right]^2 \right\}^{\frac{1}{2}} \leq \left[3 + 6 \left(1 + \frac{1}{\ln 4} \right) 4^{TL^2/2} \right]^{\frac{1}{2}} \left\{ \sum_{\eta=0}^{T-\tau} \tau \|\psi(\cdot, \eta, u)\|^2 \right\}^{\frac{1}{2}}. \quad (16)$$

To obtain the estimate of the error z from (16), we have to estimate the norm of the local truncation error $\psi(x, t, u)$.

Lemma 2.4. *Assume that there exists a constant $f_{0,0,2}$ and functions $f_{0,2,0}(x, t) \in L_2(Q_T)$, $f_{0,1,1}(x, t) \in L_4(Q_T)$ such that the function $f(x, t, u)$ satisfies the conditions*

$$\begin{aligned}
\left| \frac{\partial^2 f(x, t, u)}{\partial t^2} \right| &\leq f_{0,2,0}(x, t), \quad u \in \mathbb{R}^1, \\
\left| \frac{\partial^2 f(x, t, u)}{\partial u^2} \right| &\leq f_{0,0,2}, \quad (x, t) \in \bar{Q}_T, \quad u \in \mathbb{R}^1, \\
\left| \frac{\partial^2 f(x, t, u)}{\partial t \partial u} \right| &\leq f_{0,1,1}(x, t), \quad u \in \mathbb{R}^1.
\end{aligned} \tag{17}$$

Assume also that the solution of problem (1) is such that

$$\frac{\partial^3 u(x, t)}{\partial t^3} \in L_2(Q_T), \quad \frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \in L_2(Q_T).$$

Then the following estimate

$$\begin{aligned}
\sum_{t=0}^{T-\tau} \tau \|\psi(x, t, u)\|^2 &\leq \tau^4 \left[\frac{1}{2} \|f_{0,2,0}\|_{L_2(Q_T)}^2 + 4 \|f_{0,1,1}\|_{L_2(Q_T)}^2 + 8 \left(\frac{1}{4} + f_{0,0,2} \right)^2 \left\| \frac{\partial u}{\partial t} \right\|_{L_4(Q_T)}^2 \right. \\
&\quad \left. + 8 \left\| \frac{\partial^3 u}{\partial t^3} \right\|_{L_2(Q_T)}^2 \right] + 32h^4 \left\| \frac{\partial^3 u}{\partial x^2 \partial t} \right\|_{L_2(Q_T)}^2
\end{aligned}$$

holds.

Proof. From (5) we have $\psi(x, t, u) = \sum_{i=1}^3 \psi_i(x, t, u)$, where

$$\begin{aligned}
\psi_1(x, t, u) &= T^x \left(f\left(\cdot, t + \frac{\tau}{2}, u^+(x, t)\right) - \frac{1}{2} \left(f(\cdot, t + \tau, u^+(x, t)) + f(\cdot, t, u^+(x, t)) \right) \right), \\
\psi_2(x, t, u) &= T^x \left(\frac{1}{2} \left(f(\cdot, t + \tau, u^+(x, t)) + f(\cdot, t, u^+(x, t)) \right) \right. \\
&\quad \left. - \frac{1}{2} \left(\hat{f}(\cdot, t, u(\cdot, t)) + f(\cdot, t, u(\cdot, t)) \right) \right), \\
\psi_3(x, t, u) &= \frac{1}{2} \frac{d}{dt} T^x (\hat{u}(\cdot, t) + u(\cdot, t)) - u_t(x, t).
\end{aligned}$$

We estimate each term $\psi_i(x, t, u)$, $i = \overline{1, 3}$ separately. Thus, we have

$$\psi_1(x, t, u) = -\frac{1}{2} T^x \left(\int_{t+\frac{\tau}{2}}^{t+\tau} (t + \tau - \eta t) \frac{\partial^2 f(\cdot, \eta, u^+(x, t))}{\partial \eta^2} d\eta + \int_t^{t+\frac{\tau}{2}} (\eta - t) \frac{\partial^2 f(\cdot, \eta, u^+(x, t))}{\partial \eta^2} d\eta \right).$$

Hence

$$\begin{aligned}
 |\psi_1(x, t, u)| &\leq \frac{1}{2h^2} \int_{x-h}^{x+h} (h - |x - \xi|) \left[\int_{t+\frac{\tau}{2}}^{t+\tau} (t + \tau - \eta) f_{0,2,0}(\xi, \eta) d\eta \right. \\
 &\quad \left. + \int_t^{t+\frac{\tau}{2}} (\eta - t) f_{0,2,0}(\xi, \eta) d\eta \right] d\xi \leq \frac{\sqrt{h^3 \tau^3}}{12h^2} \left\{ \left[\int_{x-h}^{x+h} \int_{t+\frac{\tau}{2}}^{t+\tau} |f_{0,2,0}(\xi, \eta)|^2 d\eta d\xi \right]^{\frac{1}{2}} \right. \\
 &\quad \left. + \left[\int_{x-h}^{x+h} \int_t^{t+\frac{\tau}{2}} |f_{0,2,0}(\xi, \eta)|^2 d\eta d\xi \right]^{\frac{1}{2}} \right\} \leq \frac{\sqrt{2}}{12} \sqrt{\frac{\tau^3}{h}} \left[\int_{x-h}^{x+h} \int_t^{t+\tau} (f_{0,2,0}(\xi, \eta))^2 d\eta d\xi \right]^{\frac{1}{2}}.
 \end{aligned} \tag{18}$$

Then

$$\begin{aligned}
 \psi_2(x, t, u) &= -T^x \left(\frac{1}{2} \int_t^{t+\tau} \frac{\partial u(\cdot, \eta)}{\partial \eta} d\eta \int_t^{t+\tau} \frac{\partial^2 f(\cdot, \eta, u^+(\cdot, t))}{\partial \eta \partial u} d\eta \right. \\
 &\quad \left. + \int_{u^+(\cdot, t)}^{u(\cdot, t+\tau)} [u(\cdot, t+\tau) - v] \frac{\partial^2 f(\cdot, t+\tau, v)}{\partial v^2} dv + \int_{u(\cdot, t)}^{u^+(\cdot, t)} [v - u(\cdot, t)] \frac{\partial^2 f(\cdot, t, v)}{\partial v^2} dv \right).
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 |\psi_2(x, t, u)| &\leq \frac{\tau}{2h^2} \int_{x-h}^{x+h} (h - |x - \xi|) \left(\int_t^{t+\tau} \left| \frac{\partial u(\xi, \eta)}{\partial \eta} \right|^2 d\eta \right)^{\frac{1}{2}} \left(\int_t^{t+\tau} [f_{0,1,1}(\xi, \eta)]^2 d\eta \right)^{\frac{1}{2}} d\xi \\
 &\quad + \frac{1}{h^2} \int_{x-h}^{x+h} (h - |x - \xi|) [u^-(\xi, t)]^2 d\xi f_{0,0,2} \\
 &\leq \sqrt{\frac{\tau^3}{h}} \left\{ \left(\frac{1}{2\sqrt{2}} + \sqrt{2} f_{0,0,2} \right) \left[\int_{x-h}^{x+h} \int_t^{t+\tau} \left| \frac{\partial u(\xi, \eta)}{\partial \eta} \right|^4 d\eta d\xi \right]^{\frac{1}{2}} \right. \\
 &\quad \left. + \left[\int_{x-h}^{x+h} \int_t^{t+\tau} [f_{0,1,1}(\xi, \eta)]^4 d\eta d\xi \right]^{\frac{1}{2}} \right\}.
 \end{aligned} \tag{19}$$

Finally,

$$\begin{aligned}
 \psi_3(x, t, u) &= \frac{1}{2\tau} \int_t^{t+\tau} \frac{1}{h^2} \int_{x-h}^{x+h} (h - |x - \xi|) \left\{ \int_s^{t+\tau} \frac{1}{\tau} \int_t^{t+\tau} \int_{s_3}^{s_1} \frac{\partial^3 u(\xi, s_4)}{\partial s_4^3} ds_4 ds_3 ds_1 \right. \\
 &\quad \left. + \int_s^t \frac{1}{\tau} \int_t^{t+\tau} \int_{s_3}^{s_2} \frac{\partial^3 u(\xi, s_5)}{\partial s_5^3} ds_5 ds_3 ds_2 + 2 \int_x^\xi \frac{1}{2h} \int_{x-h}^{x+h} \int_{\xi_1}^{\xi_2} \frac{\partial^3 u(\xi_3, s)}{\partial \xi_3^2 \partial s} d\xi_3 d\xi_2 d\xi_1 \right\} d\xi ds,
 \end{aligned}$$

where upon corresponding transformations the operator property of exact difference schemes $T^x(\cdot - x) = 0$ and the equality

$$\int_t^{t+\tau} \left[\int_s^{t+\tau} 1 ds_1 + \int_s^t 1 ds_2 \right] ds = 0$$

were used.

Hence the estimate

$$\begin{aligned} |\psi_3(x, t, u)| &\leq \sqrt{\frac{2\tau^3}{h}} \left(\int_{x-h}^{x+h} \int_t^{t+\tau} \left| \frac{\partial^3 u(\xi, \eta)}{\partial \eta^3} \right|^2 d\eta d\xi \right)^{\frac{1}{2}} \\ &\quad + \sqrt{\frac{8h^3}{\tau}} \left(\int_t^{t+\tau} \int_{x-h}^{x+h} \left| \frac{\partial^3 u(\xi, \eta)}{\partial \xi^2 \partial \eta} \right|^2 d\xi d\eta \right)^{\frac{1}{2}} \end{aligned} \quad (20)$$

follows.

Now we can estimate the norm of local truncation error $\psi(x, t, u)$. Using (17) - (20), we have

$$\begin{aligned} \sum_{t=0}^{T-\tau} \tau \|\psi(x, t, u)\|^2 &\leq 3 \sum_{t=0}^{T-\tau} \tau \sum_{i=1}^3 \|\psi_i(x, t, u)\|^2 \leq \frac{\tau^4}{12} \int_0^T \int_0^1 (f_{0,2,0}(\xi, \eta))^2 d\xi d\eta \\ &\quad + \tau^4 \left\{ 8 \left(\frac{1}{4} + f_{0,0,2} \right)^2 \int_0^T \int_0^1 \left[\frac{\partial u(\xi, \eta)}{\partial \eta} \right]^4 d\xi d\eta \right. \\ &\quad \left. + 4 \int_0^T \int_0^1 [f_{0,1,1}(\xi, \eta)]^4 d\xi d\eta \right\} \\ &\quad + 8\tau^4 \int_0^T \int_0^1 \left[\frac{\partial^3 u(\xi, \eta)}{\partial \eta^3} \right]^2 d\xi d\eta + 32h^4 \int_0^T \int_0^1 \left[\frac{\partial^3 u(\xi, \eta)}{\partial \xi^2 \partial \eta} \right]^2 d\xi d\eta. \end{aligned} \quad (21)$$

The proof is completed. $\square$

Note that an estimate of the type of (17) can be obtained with the help of the Bremler-Hilbert lemma [9], but the right side of the corresponding inequality will include an indefinite constant.

Find sufficient conditions of smoothness of the functions $f(x, t, u)$, $\varphi(x)$, which will ensure the existence of finite norms in the right side of (17).

We first make use of the Theorem 5.2 [2, p. 645]. Let the following conditions be satisfied:

$$f(x, t, u)u \leq |u| \Phi(|u|), \quad \int_1^\infty \frac{d\tau}{\Phi(\tau)} = \infty \quad \forall (x, t) \in \bar{Q}_T; \quad (22)$$

$f(x, t, u)$ is a continuous function differentiable with respect to x and u and

$$|f(x, t, u)| \leq f_{0,0,0} \quad \forall (x, t) \in \bar{Q}_T, \quad |u| \leq M, \quad (23)$$

$$\left| \frac{\partial f(x, t, u)}{\partial x} \right| \leq f_{1,0,0} \quad \forall (x, t) \in \bar{Q}_T, \quad |u| \leq M, \quad (24)$$

$$\frac{\partial f(x, t, u)}{\partial u} \leq f_{0,0,1} = L \quad \forall (x, t) \in \bar{Q}_T, \quad |u| \leq M; \quad (25)$$

the function $f(x, t, u)$ is Hölder-continuous with respect to variable t with the exponent $\frac{\beta}{2}$, and with respect to variables x, u with the exponent β ($0 < \beta < 1$). Let the function $\varphi(x)$ be such that there exists a function $\mu(x, t)$

$$\begin{aligned} \mu(x, 0) &= \varphi(x), \quad \mu(0, t) = \mu(1, t) = 0, \quad \mu(x, t) \in H^{2+\beta, 1+\beta/2}(\bar{Q}_T), \\ [\mu_t(x, 0) - \mu_{xx}(x, 0) - f(x, 0, \mu(x, 0))]_{x=0,1} &= 0. \end{aligned} \quad (26)$$

Then problem (1) has a unique solution in $H^{2+\beta, 1+\beta/2}(\bar{Q}_T)$.

Note that if the function $f(x, t, u)$ satisfies the Lipschitz conditions (2) and bounded in the sense of (23), then the function $\Phi(s)$ can be considered as

$$\Phi(s) = \left(L + \frac{1}{2} \right) s + \frac{f_{0,0,0}^2}{2s}. \quad (27)$$

Indeed,

$$\begin{aligned} f(x, t, u)u &= [f(x, t, u) - f(x, t, 0)]u + f(x, t, 0)u \\ &\leq f_{0,0,1}u^2 + \frac{1}{2}u^2 + \frac{1}{2}f^2(x, t, 0) \leq \left(L + \frac{1}{2} \right) u^2 + \frac{f_{0,0,0}^2}{2}, \end{aligned}$$

from which it follows that the function $\Phi(s)$ can be chosen in the form of (27).

We give some of the properties of the function $\mu(x, t)$. If $\varphi(x) \in C[0, 1]$, $\varphi(0) = \varphi(1) = 0$, then function $\mu(x, t)$ taken in the form

$$\mu(x, t) = \cos t \cdot \varphi(x) \quad (28)$$

will belong to the class $C(\bar{Q}_T)$ and, therefore, the zero-order consistency condition (of the initial and boundary conditions) will be fulfilled (see [2, p. 509]). If, in addition, $\varphi(x) \in C^2[0, 1]$ and

$$[\varphi''(x) + f(x, 0, \varphi(x))]_{x=0,1} = 0, \quad (29)$$

then, the consistency condition (of the initial and boundary conditions and the equation) of the first order (see there again) will be fulfilled. If additionally $\varphi(x) \in H^{2+\beta}[0, 1]$, then function (28) will satisfy condition (26).

Thus, instead of condition (26), we can take the following condition:

$$\varphi(x) \in H^{2+\beta}[0, 1], \quad \varphi(0) = \varphi(1) = 0, \quad (30)$$

the function $\varphi(x)$ satisfies condition (29).

It can easily be seen that if conditions (22) - (25), (29), (30) hold true, the solution of problem (1) will satisfy the inequality

$$\begin{aligned} & \int_0^t \int_0^1 \left[\frac{\partial u(x, t_1)}{\partial t_1} \right]^2 dx dt_1 + \int_0^1 \left[\frac{\partial u(x, t)}{\partial x} \right]^2 dx + \int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t_1)}{\partial x^2} \right]^2 dx dt_1 \\ & \leq e^{L^2 t} \left\{ \int_0^1 [\varphi'(x)]^2 dx + 2 \int_0^T \int_0^1 [f(x, t, 0)]^2 dx dt_1 \right\}. \end{aligned} \quad (31)$$

Indeed, under our assumptions we can pass from problem (1) to the integral equation

$$\begin{aligned} & \int_0^t \int_0^1 \left[\frac{\partial u(x, t_1)}{\partial t_1} \right]^2 dx dt_1 + \int_0^1 \left[\frac{\partial u(x, t)}{\partial x} \right]^2 dx + \int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t_1)}{\partial x^2} \right]^2 dx dt_1 \\ & = \int_0^1 [\varphi'(x)]^2 dx + \int_0^t \int_0^1 [f(x, t_1, u(x, t_1))]^2 dx dt_1. \end{aligned}$$

From here we get

$$\begin{aligned} & \int_0^t \int_0^1 \left[\frac{\partial u(x, t_1)}{\partial t_1} \right]^2 dx dt_1 + \int_0^1 \left[\frac{\partial u(x, t)}{\partial x} \right]^2 dx + \int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t_1)}{\partial x^2} \right]^2 dx dt_1 \\ & \leq L^2 \int_0^t \int_0^1 \left[\frac{\partial u(x, t_1)}{\partial x} \right]^2 dx dt_1 + 2 \int_0^t \int_0^1 [f(x, t_1, 0)]^2 dx dt_1. \end{aligned}$$

Solving the last inequality with the help of the Gronwall lemma, we obtain (31). The following statement holds.

Lemma 2.5. Suppose conditions (22) - (25), (29), (30) and $\left| \frac{\partial f(x, t, u)}{\partial t} \right| \leq f_{0,1,0}(x, t) \in L_2(Q_T) \forall |u| \leq M$, $\psi_0(x) = \varphi''(x) + f(x, 0, \varphi(x)) \in W_2^1(0, 1)$, are satisfied. Then the solution $u(x, t)$ of problem (1) is such that

$$\frac{\partial u(x, t)}{\partial t} \in W_2^{1,1}(Q_T)$$

and the estimate

$$\begin{aligned} & \int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right]^2 dx dt + \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial x \partial t} \right]^2 dx \\ & \leq \int_0^1 \left[\varphi'''(x) + \frac{\partial f(x, 0, \varphi(x))}{\partial x} + \frac{\partial f(x, 0, \varphi(x))}{\partial u} \varphi'(x) \right]^2 dx \\ & + \int_0^T \int_0^1 [f_{0,1}(x, t)]^2 dx dt + L^2 \int_0^T \int_0^1 \left[\frac{\partial u(x, t)}{\partial t} \right]^2 dx dt, \\ & t \in [0, T], \end{aligned} \quad (32)$$

where $\|\frac{\partial u}{\partial t}\|_{L_2(Q_T)}^2$ is estimated with the help of inequality (31), holds.

Proof. On assumptions made

$$\mathcal{F}_1(x, t) \equiv \frac{d}{dt} f(x, t, u(x, t)) = \frac{\partial f(x, t, u(x, t))}{\partial t} + \frac{\partial f(x, t, u(x, t))}{\partial u} \frac{\partial u}{\partial t} \in L_2(Q_T),$$

$$\psi_0(x) = \varphi''(x) + f(x, 0, \varphi(x)) \in W_2^1(0, 1).$$

So, according to Theorem 6.1 from [2, p. 209], the solution of the problem

$$\frac{\partial w(x, t)}{\partial t} = \frac{\partial^2 w(x, t)}{\partial x^2} + \mathcal{F}_1(x, t),$$

$$w(0, t) = w(1, t) = 0, \quad w(x, 0) = \psi_0(x)$$
(33)

exists, is unique, and belongs to the space $W_2^{1,1}(Q_T)$. But, obviously, this solution coincides with $\frac{\partial u(x, t)}{\partial t}$ and problem (33) takes the form

$$\frac{\partial^2 u(x, t)}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial u(x, t)}{\partial t} + \frac{d}{dt} f(x, t, u(x, t)),$$

$$\frac{\partial u(0, t)}{\partial t} = \frac{\partial u(1, t)}{\partial t} = 0, \quad \frac{\partial u(x, 0)}{\partial t} = \varphi''(x) + f(x, 0, \varphi(x)).$$
(34)

Multiply both sides of equation (34) by $\frac{\partial^2 u(x, t)}{\partial t^2}$ and integrate over Q_t . Then, by simple transformations, we get (32). Lemma is proved. $\square$

Proceeding from the just proved lemma, the following statement is correct.

Lemma 2.6. *Let the conditions of Lemma 2.5 hold true and*

$$\left| \frac{\partial^2 f(x, t, u)}{\partial t^2} \right| \leq f_{0,2,0}(x, t) \in L_2(Q_T), \quad \left| \frac{\partial^2 f(x, t, u)}{\partial t \partial u} \right| \leq f_{0,1,1} \in L_2(Q_T),$$

$$\left| \frac{\partial^2 f(x, t, u)}{\partial u^2} \right| \leq f_{0,0,2}(x, t) \in L_2(Q_T), \quad |u| \leq M,$$

$$\psi_1(x) = \varphi'''(x) + \frac{\partial^2 f(x, 0, \varphi(x))}{\partial x^2} + 2 \frac{\partial f(x, 0, \varphi(x))}{\partial x \partial u} \varphi'(x)$$

$$+ \frac{\partial^2 f(x, 0, \varphi(x))}{\partial u^2} [\varphi'(x)]^2 + \frac{\partial f(x, 0, \varphi(x))}{\partial u} \varphi''(x) + \frac{\partial f(x, 0, \varphi(x))}{\partial t}$$

$$+ \frac{\partial f(x, 0, \varphi(x))}{\partial u} \psi_0(x) \in W_2^1(0, 1).$$

Then the solution of problem (2.1) $u(x, t)$ is such that

$$\frac{\partial^2 u(x, t)}{\partial t^2} \in W_2^{1,1}(Q_T)$$

and satisfies the inequalities

$$\int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right]^2 dx + 2 \int_0^t \int_0^1 \left[\frac{\partial^3 u(x, t)}{\partial t \partial x^2} \right]^2 dx dt \leq e^{2(2+L)t} \left\{ \int_0^1 \psi_1^2(x) dx + F^2 \right\}, \quad (35)$$

$$\begin{aligned} \int_0^t \int_0^1 \left[\frac{\partial^3 u(x, t)}{\partial t^3} \right]^2 dx dt + \int_0^1 \left[\frac{\partial^3 u(x, t)}{\partial x \partial t^2} \right]^2 dx &\leq \int_0^1 \psi_1^2(x) dx + 4F^2 + \\ &+ 4L^2 \int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right]^2 dx dt, \end{aligned} \quad (36)$$

where

$$\begin{aligned} F^2 = \int_0^T \int_0^1 [f_{0,2,0}(x, t)]^2 dx dt + 2 \int_0^T \int_0^1 [f_{0,1,1}(x, t)]^2 dx dt \max_{0 \leq t \leq T} \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial x \partial t} \right]^2 dx \\ + \int_0^T \int_0^1 [f_{0,0,2}(x, t)]^2 dx dt \max_{0 \leq t \leq T} \left\{ \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial x \partial t} \right]^2 dx \right\}^2. \end{aligned}$$

Here $\max_{0 \leq t \leq T} \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial x \partial t} \right]^2 dx$ and $\int_0^t \int_0^1 \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right]^2 dx dt$ are estimated by (32).

Proof. Under the assumptions made in the lemma the function

$$\begin{aligned} \mathcal{F}_2(x, t) &= \frac{d}{dt^2} f(x, t, u(x, t)) = \frac{\partial^2 f(x, t, u(x, t))}{\partial t^2} + 2 \frac{\partial^2 f(x, t, u(x, t))}{\partial t \partial u} \left[\frac{\partial u(x, t)}{\partial t} \right] \\ &+ \frac{\partial f(x, t, u(x, t))}{\partial u} \frac{\partial^2 u(x, t)}{\partial t^2} + \frac{\partial^2 f(x, t, u(x, t))}{\partial u^2} \left[\frac{\partial u(x, t)}{\partial t} \right]^2 \in L_2(Q_T), \\ \psi_1(t) &= \frac{\partial^2 u(x, 0)}{\partial t^2} = \varphi'''(x) + \frac{\partial^2 f(x, 0, \varphi(x))}{\partial x^2} + 2 \frac{\partial^2 f(x, 0, \varphi(x))}{\partial x \partial u} \varphi'(x) \\ &+ \frac{\partial^2 f(x, 0, \varphi(x))}{\partial u^2} [\varphi'(x)]^2 + \frac{\partial f(x, 0, \varphi(x))}{\partial u} \varphi''(x) + \frac{\partial f(x, 0, \varphi(x))}{\partial t} \\ &+ \frac{\partial f(x, 0, \varphi(x))}{\partial u} \psi_0(x) \in \overset{0}{W}_2^1(0, 1) \end{aligned}$$

will belong to the class $\overset{0}{W}_2^1(0, 1)$. Then, according to Theorem 6.1 [2, p. 209] a solution of the problem

$$\begin{aligned} \frac{\partial w(x, t)}{\partial t} &= \frac{\partial^2 w(x, t)}{\partial x^2} + \mathcal{F}_2(x, t), \\ w(0, t) &= w(1, t) = 0, \quad w(x, 0) = \psi_1(x) \end{aligned} \quad (37)$$

exists, is unique, and belongs to the space $W_2^{1,1}(Q_T)$. But, obviously, this solution coincides with $\frac{\partial^2 u(x, t)}{\partial t^2}$ and problem (37) takes the form

$$\begin{aligned} \frac{\partial}{\partial t} \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right] &= \frac{\partial^2}{\partial x^2} \left[\frac{\partial^2 u(x, t)}{\partial t^2} \right] + \frac{d^2}{dt^2} f(x, t, u(x, t)), \\ \frac{\partial^2 u(0, t)}{\partial t^2} &= \frac{\partial^2 u(1, t)}{\partial t^2} = 0, \quad \frac{\partial^2 u(x, 0)}{\partial t^2} = \psi_1(x). \end{aligned} \quad (38)$$

Multiply both sides of the equation from (38) by $\frac{\partial^2 u(x,t)}{\partial t^2}$ and integrate over Q_t . Then we get

$$\begin{aligned} \frac{1}{2} \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial t^2} \right]^2 dx + \int_0^t \int_0^1 \left[\frac{\partial^3 u(x,t)}{\partial t \partial x^2} \right]^2 dx dt &\leq \frac{1}{2} \int_0^1 \psi_1^2(x) dx \\ &+ (2+L) \int_0^t \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial t^2} \right]^2 dx dt + \frac{1}{2} \int_0^t \int_0^1 [f_{0,2,0}(x,t)]^2 dx dt \\ &+ \int_0^t \int_0^1 [f_{0,1,1}(x,t)]^2 dx dt \max_{0 \leq t \leq T} \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial x \partial t} \right]^2 dx \\ &+ \frac{1}{2} \int_0^t \int_0^1 [f_{0,0,2}(x,t)]^2 dx dt \max_{0 \leq t \leq T} \left\{ \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial x \partial t} \right]^2 dx \right\}^2. \end{aligned}$$

Solving the last inequality with the help of the Gronwall Lemma, we obtain (35).

If we multiply equation (38) by $\frac{\partial^3 u}{\partial t^3}$ and integrate it over Q_T , then we get

$$\begin{aligned} \int_0^t \int_0^1 \left[\frac{\partial^3 u(x,t)}{\partial t^3} \right]^2 dx dt + \frac{1}{2} \int_0^t \int_0^1 \frac{\partial}{\partial t} \left(\frac{\partial^3 u(x,t)}{\partial x \partial t^2} \right)^2 dx dt \\ = \int_0^t \int_0^1 \left[\frac{d^2}{dt^2} f(x,t,u(x,t)) \right] \frac{\partial^3 u(x,t)}{\partial t^3} dx dt \end{aligned}$$

hence

$$\begin{aligned} \int_0^t \int_0^1 \left[\frac{\partial^3 u(x,t)}{\partial t^3} \right]^2 dx dt + \int_0^1 \left(\frac{\partial^3 u(x,t)}{\partial x \partial t^2} \right)^2 dx &\leq \int_0^1 \left(\frac{\partial^3 u(x,0)}{\partial x \partial t^2} \right)^2 dx \\ &+ \int_0^t \int_0^1 \left[\frac{d^2}{dt^2} f(x,t,u(x,t)) \right]^2 dx dt \leq \int_0^1 [\psi_1(x)]^2 dx + 4 \int_0^t \int_0^1 [f_{0,2,0}(x,t)]^2 dx dt \\ &+ 8 \int_0^t \int_0^1 [f_{0,1,1}(x,t)]^2 dx dt \max_{0 \leq t \leq T} \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial x \partial t} \right]^2 dx + 4L^2 \int_0^t \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial t^2} \right]^2 dx dt \\ &+ 4 \int_0^t \int_0^1 [f_{0,0,2}(x,t)]^2 dx dt \max_{0 \leq t \leq T} \left\{ \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial x \partial t} \right]^2 dx \right\}^2, \end{aligned}$$

moreover, $\max_{0 \leq t \leq T} \int_0^1 \left[\frac{\partial^2 u}{\partial x \partial t} \right]^2 dx$ and $\int_0^t \int_0^1 \left[\frac{\partial^2 u(x,t)}{\partial t^2} \right]^2 dx dt$ are estimated by (24). Lemma is proved. $\square$

From Lemma 2.3 and Lemma 2.6 the following theorem takes place.

Theorem 2.1. Suppose the conditions of Lemmas 2.3 and 2.6 are satisfied. Then the difference scheme (3) has a weight a priori estimate of accuracy of the following form:

$$\left\{ \sum_{\eta=0}^{t-\tau} \tau \left[\frac{z^+(x, \eta)}{x(1-x)} \right]^2 \right\}^{\frac{1}{2}} \leq M_1 (\tau^2 + h^2),$$

where the constant M_1 is defined in explicit form by the input data with the help of estimates (21), (32), (35), (36).

Remark 2.1. Conditions (17) for the function $f(x, t, u)$, which were used in proving Lemma 2.5, are stronger than required for $\frac{\partial^3 u(x, t)}{\partial t^3} \in L_2(Q_T)$, $\frac{\partial^3 u(x, t)}{\partial x^2 \partial t} \in L_2(Q_T)$ (see Lemma 2.6).

Remark 2.2. For the solution of the initial problem (1), the a priori estimate taking into account the boundary effect can also be obtained. Let us demonstrate this. If conditions (22) - (25), (29), (30) are satisfied, problem (1) can be transformed to the equivalent form

$$u(x, t) = \int_0^1 G(x, \xi) \left[f(\xi, t, u(\xi, t)) - \frac{\partial u(\xi, t)}{\partial t} \right] d\xi,$$

from which we obtain the estimate

$$\begin{aligned} \left| \frac{u(x, t)}{x(1-x)} \right| &\leq 2L \int_0^1 \left| \int_0^x \frac{\partial u(\xi, t)}{\partial \xi} d\xi \right| dx + \int_0^1 \left| \frac{\partial u(\xi, t)}{\partial t} \right| d\xi \\ &\leq L \left\{ \int_0^1 \left[\frac{\partial u(\xi, t)}{\partial \xi} \right]^2 d\xi \right\}^{\frac{1}{2}} + \left\{ \int_0^1 \left[\frac{\partial u(\xi, t)}{\partial t} \right]^2 d\xi \right\}^{\frac{1}{2}}. \end{aligned}$$

Let us square both sides of the last inequality, integrate them with respect to t from 0 to t , and use (31), then we get

$$\int_0^t \left[\frac{u(x, t_1)}{x(1-x)} \right]^2 dt_1 \leq \left\{ \int_0^1 [\varphi'(\xi)]^2 d\xi + 2 \int_0^T \int_0^1 [f(\xi, t_1, 0)]^2 d\xi dt_1 \right\} 4e^{L^2 t}.$$

3. Weighted a priori error estimate taking into account the boundary effect

To obtain the desired a priori estimate, we scalar-multiply both sides of equation (4) by $\tau z^+(x, t)$ and then sum over t from 0 to $t - \tau$. We get

$$\begin{aligned} \frac{1}{2} \|z(x, t)\|^2 + \sum_{\eta=0}^{t-\tau} \tau \|z_{\bar{x}}^+(x, \eta)\|^2 &= \sum_{\eta=0}^{t-\tau} \left(T^x \left(f\left(\cdot, \eta + \frac{\tau}{2}, y^+(x, \eta)\right) \right. \right. \\ &\quad \left. \left. - f\left(\cdot, \eta + \frac{\tau}{2}, u^+(x, \eta)\right) \right), z^+(x, \eta) \right) \tau + \sum_{\eta=0}^{t-\tau} \tau (\psi(x, \eta, u), z^+(x, \eta)). \end{aligned}$$

Thus, we come to the inequality

$$\frac{1}{2} \|z(x, t)\|^2 \leq \frac{1}{2} \sum_{\eta=\tau}^t \tau (\|\psi(x, \eta - \tau, u)\| + \|\psi(x, \eta, u)\|) \|z(x, \eta)\| + L \sum_{\eta=\tau}^t \tau \|z(x, \eta)\|^2. \quad (39)$$

Divide both sides of inequality (39) by the function $(2 + \ln^2 t) t$. Then, by simple transformations, we obtain the relation

$$\frac{1}{2} \frac{\|z(x, t)\|^2}{(2 + \ln^2 t) t} \leq 2 \sum_{\eta=0}^t \tau \|\psi(x, \eta, u)\|^2 + \sum_{\eta=\tau}^t \tau \left[L + \frac{1}{(2 + \ln^2 \eta) \eta} \right] \frac{\|z(x, \eta)\|^2}{(2 + \ln^2 \eta) \eta},$$

which, using the notation

$$\frac{1}{2} \frac{\|z(x, t)\|^2}{(2 + \ln^2 t) t} = v(t), \quad 2 \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2 = c, \quad 2 \left[L + \frac{1}{(2 + \ln^2 t) t} \right] = g(\eta),$$

transforms to the form

$$v(t) \leq c + \sum_{\eta=\tau}^t \tau g(\eta) v(\eta). \quad (40)$$

Inequality (40) satisfies the conditions of the Lemma 2.2 at

$$\tau \max_{\eta \geq \tau} g(\eta) = 2\tau L + \frac{2}{2 + \ln^2 \tau} \leq 1/2 \quad (41)$$

and its solution is

$$\begin{aligned} v(t) = \frac{\|z(x, t)\|^2}{2(2 + \ln^2 t) t} &\leq 2 \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2 \exp \left\{ \ln 4 \int_0^t 2 \left[L + \frac{1}{(2 + \ln^2 \eta) \eta} \right] d\eta \right\} \\ &= 2 \exp \left\{ 2 \ln 4 \left[Lt + \frac{1}{\sqrt{2}} \left(\arctg \left(\frac{\ln t}{\sqrt{2}} \right) + \frac{\pi}{2} \right) \right] \right\} \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2 \\ &\leq 2 \exp \left\{ 2 \ln 4 \left(Lt + \frac{\pi}{\sqrt{2}} \right) \right\} \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2. \end{aligned}$$

Thus, we have proved the following statement.

Lemma 3.1. *Suppose conditions (2), (41) hold true. Then for the error of the difference scheme (3) the following weight a priori estimate is valid:*

$$\frac{\|z(x, t)\|^2}{2(2 + \ln^2 t) t} \leq 2 \exp \left\{ 4 \ln 2 \left(Lt + \frac{\pi}{\sqrt{2}} \right) \right\} \sum_{\eta=0}^{T-\tau} \tau \|\psi(x, \eta, u)\|^2. \quad (42)$$

From this lemma and Lemma 2.6 the next theorem follows.

Theorem 3.1. Suppose the conditions of Lemma 2.6 and Lemma 3.1 hold true. Then the difference scheme (3) has the weight a priori estimate of accuracy in the following form:

$$\frac{\|z(x, t)\|}{\sqrt{(2 + \ln^2 t) t}} \leq M_2 (\tau^2 + h^2),$$

where the constant M_2 in explicit form is estimated by the input data with the help of (21), (32), (35), (36).

Remark 3.1. Analogously to (42), we can obtain weight a priori estimate taking into account the initial effect for the solution of the difference problem (1). This estimate is obtained in a similar manner by the Gronwall lemma [1] and has the form

$$\frac{\left\{ \int_0^1 [u(x, t) - \varphi(x)]^2 dx \right\}^{\frac{1}{2}}}{\sqrt{(2 + \ln^2 t) t}} \leq f_{0,0,0} \sqrt{T} \exp\left(\frac{\pi}{2\sqrt{2}}\right).$$

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