BOUNDS OF INFORMATION EXPENSES IN CONSTRUCTING PROJECTION METHODS FOR SOLVING ILL-POSED PROBLEMS

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Abstract — An approach to constructing regularized projection methods to solve ill-posed problems is proposed. This approach is based on a modification of the Galerkin discretization scheme. It has been established that such a modification leads to a significant reduction of information expenses compared to other known methods.

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1. Introduction

In a Hilbert space X with inner product $(\cdot, \cdot)$ and norm $\|x\| = \sqrt{(x, x)}$ we shall consider the operator equation of the first kind

$$Ax = f. \quad (1)$$

Suppose that in (1) $f \in \text{Range}(A)$, $\text{Range}(A) \neq \overline{\text{Range}(A)}$ and instead of the right-hand side f we are given only an approximation $f_\delta \in X$ such that $\|f - f_\delta\| \leq \delta$, $\delta > 0$.

Our aim is to recover the normal solution $x^\dagger$ of (1) (i.e., the solution of (1) with minimal norm in X) with the best possible accuracy under the assumption

$$x^\dagger \in M_{\nu, \rho}(A) = \{x : x = |A|^\nu v, \|v\| \leq \rho\}. \quad (2)$$

Here $\rho > 0$ is the known parameter, the unknown parameter ν belongs to the interval $[1, \nu_1]$, $1 < \nu_1 < \infty$, $|A| = (A^*A)^{1/2}$ and A^* is the adjoint operator of A .

Now the class of operators A will be defined, on which we pursue the investigation. Thus, we take an orthonormal basis $E = \{e_k\}_{k=1}^\infty$ in X and define a P_m as the orthoprojector onto the linear span of elements $e_1, e_2, \dots, e_m$, i.e., $P_m f = \sum_{k=1}^m (f, e_k) e_k$. By $\mathcal{H}^r$, $r = 1, 2, \dots$, denote the class of compact linear operators A with $\|A\| \leq 1$ acting in X and satisfying the conditions

$$\|(I - P_m)A\| \leq \beta_r m^{-r}, \quad \|A(I - P_m)\| \leq \beta_r m^{-r} \quad (3)$$

at any $m = 1, 2, \dots$ and some constant $\beta_r > 0$.

An important example of (1) with $A \in \mathcal{H}^r$ is given by the first kind Fredholm integral equation

$$Ax(t) \equiv \int_0^1 k(t, \tau)x(\tau)d\tau = f(t),$$

where the integral operators A, A^* act from $X = L_2(0, 1)$ into the Sobolev space $W_2^r[0, 1]$ of r -time differentiable functions. In particular, condition (3) is satisfied by the operators with kernels $k(t, \tau)$ that have r bounded in $L_2(0, 1)$ derivatives with respect to both variables. The following function systems can be used as the basis E satisfying (3): the orthonormal system of Haars functions (at $r = 1$), the subspace of trigonometric functions (in the case of 1-periodic coefficients $k(t, \tau), f(t)$) and the orthonormal system of Legendre polynomials considered on the interval $[0, 1]$.

2. Statement of the problem

As is well known [9, p.14], the accuracy of recovery of solutions $x^\dagger$ filling the set $M_{\nu, \rho}(A)$ cannot be less than $\rho^{1/\nu+1}\delta^{\nu/\nu+1}$. The class of equations (1) with the operators $A \in \mathcal{H}^r$ and the right-hand side $f \in AM_{\nu, \rho}(A) = \{f : f = Au, \quad u \in M_{\nu, \rho}(A)\}$, where $\nu \in (0, \nu_1]$, is denoted by Ψ_{ρ, ν_1}^r . The goal of the present paper is to work out an efficient approach to the finite-dimensional solution of (1) which is economic in the sense of information expenses (see (8)) and allows to guarantee the optimal order of accuracy $O(\delta^{\nu/\nu+1})$ on the class Ψ_{ρ, ν_1}^r .

In [2], the authors discussed the same problem under the Tikhonov regularization with the source condition $f \in AM_{\nu, \rho}(A), 0 < \nu \leq 1$. In some sense, the present paper is an extension of [2] with more general result.

3. Proposed approach to solve (1)

By virtue of compactness of the operator A problem (1) is ill-posed. For this reason, constructing stable approximations of $x^\dagger$ requires the application of some special regularization methods (see [8]). Following [1], by the regularization method we shall mean an operator $R_\alpha = R_\alpha(A) : X \rightarrow X$ such that as an approximate solution of (1) one takes the element $x_\alpha = R_\alpha(A)f_\delta$, where $\alpha > 0$ is called the regularization parameter and $R_\alpha(A) = g_\alpha(A^*A)A^*$. Here $g_\alpha(\lambda)$ is the Borel measurable function on the interval $[0, 1]$ and satisfies the following conditions:

$$\sup_{0 \leq \lambda < \infty} \lambda^\nu |1 - \lambda g_\alpha(\lambda)| \leq \chi_\nu \alpha^\nu, \quad (4)$$

$$\sup_{0 \leq \lambda < \infty} \lambda^{1/2} |g_\alpha(\lambda)| \leq \chi_* \alpha^{-1/2}, \quad (5)$$

where

$$0 \leq \nu \leq \nu_*, \quad \nu_1 \leq 2\nu_* - 1, \quad (6)$$

$\chi_\nu (\chi_0 = 1)$, χ_* are some positive constants independent of α , and ν_* is referred to as the qualification of the method R_α . There are many well-known methods satisfying (4)–(6). In particular, we can mention Tikhonov's method (with $g_\alpha(\lambda) = (\alpha + \lambda)^{-1}$ at $\nu_* = 1$), Landweber's method (with $g_\alpha(\lambda) = \lambda^{-1}[1 - (1 - \mu\lambda)^{1/\alpha}]$ at $\nu_* = \infty, 0 < \mu < 2$), and Showalter's method (with $g_\alpha(\lambda) = \lambda^{-1}(1 - \exp(-\lambda/\alpha))$ at $\nu_* = \infty$).

To realize finite-dimensional approximations for equation (1), it is necessary first to discretize its coefficients. To this end, we propose a modified projection scheme of discretization, whose main point is to replace the operator A and the element f_δ by their finite-dimensional analogues

$$A_n = \sum_{k=1}^n (P_{2^k} - P_{2^{k-1}}) A P_{2^{2n-k}} + P_1 A P_{2^{2n}}, \quad P_{2^n} f_\delta = \sum_{k=1}^{2^n} (f_\delta, e_k) e_k, \quad (7)$$

where the basis $E = \{e_k\}_{k=1}^\infty$ appears in the definition of $\mathcal{H}^r$. Hereinafter by the discrete information on (1) we mean a set of values of inner products

$$(Ae_j, e_i), \quad (f_\delta, e_i) \quad (8)$$

that are needed to construct coefficients (7). To reduce the content of discrete information (8), we have to choose in the best way the discretization parameter n depending on the known values of δ, r, β_r, ρ (see (10)).

Note that the modified projection scheme (7) was used first in [4] to solve equation (1) in the case of the known parameter $\nu = 2$.

The proposed approach to the realization of finite-dimensional approximations to $x^\dagger$ consists in the following. An approximate solution $\hat{x}_\alpha$ is determined by the rule

$$\hat{x}_\alpha = R_\alpha(A_n)P_{2^n}f_\delta. \quad (9)$$

Here the regularization method R_α satisfies conditions (4)–(6), A_n has the form of (7), and as the value of the discretization parameter, one takes the least integer $n > 0$ such that

$$2^{-2rn}n \leq 2\delta/c_1, \quad (10)$$

where $c_1 = 2^{2r}(2^{2r} - 1)^{-1}\beta_r^2\rho$. As a stop rule will be used the discrepancy principle [3]. In other words the proposed computational procedure is stopped as soon as the parameter α satisfies the following condition

$$b_1\delta \leq \|P_{2^n}f_\delta - A_n\hat{x}_\alpha\| \leq b_2\delta, \quad 2 < b_1 \leq b_2. \quad (11)$$

The regularized projection method (7), (9)–(11) will be denoted as $(R_\alpha, A_n, b_1, b_2)$.

4. Main result

In what follows we shall need some properties of the approximation of the operator A_n .

Lemma 4.1. *Let $A \in \mathcal{H}^r, x^\dagger \in M_{\nu,\rho}(A), \nu \in [1, \nu_1]$. If the discretization parameter n is chosen according to (10), then*

$$\|A_n x^\dagger - P_{2^n}f_\delta\| \leq 2\delta, \quad \||A|^\nu - |A_n|^\nu\| \leq z(\nu)c_3\delta^{\nu/\nu+1},$$

where $c_3 = 2^{1-2r}(1 + 2^{r+2})(2^{2r} - 1)$ and the function $z(\nu)$ is bound in $(0, \infty]$.

Proof. Using Lemma 2 [7] and relation (10), we obtain

$$\|A_n x^\dagger - P_{2^n}f_\delta\| \leq \|A_n x^\dagger - P_{2^n}f\| + \|P_{2^n}(f - f_\delta)\| \leq c_1 2^{-2rn} + \delta \leq 2\delta.$$

With the help of Lemma 1 [7] and Lemma 4.1 [6] one can estimate the norm of the operator $|A|^\nu - |A_n|^\nu$. Namely, it holds

$$\||A|^\nu - |A_n|^\nu\| \leq z(\nu)\|A^*A - A_n^*A_n\|^{\min\{\nu/2, 1\}} \leq z(\nu)((\beta_r^2 + c_2n)2^{-2rn})^{\min\{\nu/2, 1\}}$$

with $c_2 = 2^{r+2}\beta_r^2$. Keeping in mind (10), we have

$$\||A|^\nu - |A_n|^\nu\| \leq z(\nu)\left(\frac{2(\beta_r^2 + c_2)}{c_1}\delta\right)^{\min\{\nu/2, 1\}} = \begin{cases} z(\nu)\left(\frac{2(\beta_r^2 + c_2)}{c_1}\delta\right)^{\nu/2} \leq z(\nu)c_3\delta^{\nu/\nu+1} & \text{at } \nu \in [1, 2), \\ z(\nu)\frac{2(\beta_r^2 + c_2)}{c_1}\delta \leq z(\nu)c_3\delta^{\nu/\nu+1} & \text{at } \nu \in [2, \nu_1]. \end{cases}$$

This completes the proof. $\square$

Theorem 4.1. *The optimal order of accuracy $O(\delta^{\nu/\nu+1})$ is achieved within the framework of $(R_\alpha, A_n, b_1, b_2)$ on the class Ψ_{ρ, ν_1}^r .*

Proof. To establish the present statement, we apply the technique used earlier in proving Theorem 3.3 [6] and Theorem 4.1 [5]. We can write the representation for the error of method $(R_\alpha, A_n, b_1, b_2)$

$$x^\dagger - \hat{x}_\alpha = R_{\alpha,n}(A_n x^\dagger - P_{2^n} f_\delta) + S_{\alpha,n} x^\dagger,$$

where $R_{\alpha,n} = g_\alpha(A_n^* A_n) A_n^*$, $S_{\alpha,n} = I - g_\alpha(A_n^* A_n) A_n^* A_n$. From (5) it follows

$$\|R_{\alpha,n}\| = \|g_\alpha(A_n^* A_n) A_n\| \leq \chi_* \alpha^{-1/2}.$$

Then

$$\|x^\dagger - \hat{x}_\alpha\| \leq \|S_{\alpha,n} x^\dagger\| + 2\chi_* \alpha^{-1/2} \delta. \quad (12)$$

Using (4), we find

$$\|I - A_n R_{\alpha,n}\| = \|I - g_\alpha(A_n A_n^*) A_n A_n^*\| \leq \sup_{0 \leq \lambda < \infty} |1 - \lambda g_\alpha(\lambda)| \leq 1.$$

Hence with the help of (11) and the representation

$$A_n S_{\alpha,n} x^\dagger = (A_n - A_n R_{\alpha,n} A_n) x^\dagger = (P_{2^n} f_\delta - A_n \hat{x}_\alpha) + (I - A_n R_{\alpha,n})(A_n x^\dagger - P_{2^n} f_\delta), \quad (13)$$

one obtains

$$b_1 \delta - \|A_n x^\dagger - P_{2^n} f_\delta\| \leq \|A_n S_{\alpha,n} x^\dagger\| \leq b_2 \delta + \|A_n x^\dagger - P_{2^n} f_\delta\|. \quad (14)$$

From the left-hand side of (14) and Lemma 4.1 it follows that

$$b_1 \delta - 2\delta \leq \|A_n S_{\alpha,n} x^\dagger\|, \quad \alpha^{-1/2} \delta \leq \alpha^{-1/2} (b_1 - 2)^{-1} \|A_n S_{\alpha,n} x^\dagger\|. \quad (15)$$

To estimate the norm of the operator $A_n S_{\alpha,n}$, we apply the polar decomposition $A_n = U(A_n^* A_n)^{1/2}$, $\|U\| = 1$. Then $A_n S_{\alpha,n} = U(1 - |A_n| g_\alpha(|A_n|^2) |A_n|) |A_n| = U S_{\alpha,n} |A_n|$. Next, with the help of relation (4) and Lemma 4.1 we find

$$\begin{aligned} \alpha^{-1/2} \|A_n S_{\alpha,n} x^\dagger\| &= \alpha^{-1/2} \|A_n S_{\alpha,n} |A|^\nu v\| \leq \\ &\rho \alpha^{-1/2} \|A_n S_{\alpha,n} |A_n|^\nu\| + \|A_n S_{\alpha,n} (|A|^\nu - |A_n|^\nu)\| \leq \rho \alpha^{-1/2} \|U S_{\alpha,n} |A_n|^{\nu+1}\| + \\ &\|U S_{\alpha,n} |A_n|\| \| |A|^\nu - |A_n|^\nu \| \leq \rho \chi_{(\nu+1)/2} \alpha^{\nu/2} + \rho \chi_{1/2} c_3 z(\nu) \delta^{\nu/\nu+1}. \end{aligned} \quad (16)$$

Let us estimate

$$\|S_{\alpha,n} x^\dagger\| = \|S_{\alpha,n} |A|^\nu v\| \leq \rho \|S_{\alpha,n} |A_n|^\nu\| + \rho \|S_{\alpha,n}\| \| |A|^\nu - |A_n|^\nu \| \leq \rho \chi_{\nu/2} \alpha^{\nu/2} + \rho z(\nu) c_3 \delta^{\nu/\nu+1}. \quad (17)$$

Substituting (16) into (15), we obtain

$$\alpha^{-1/2} \delta \leq (b_1 - 2)^{-1} (\rho \chi_{(\nu+1)/2} \alpha^{\nu/2} + \rho \chi_{1/2} c_3 z(\nu) \delta^{\nu/\nu+1}).$$

Using (12) and the above bounds for values $\alpha^{-1/2} \delta$, $\|S_{\alpha,n} x^\dagger\|$, we have

$$\begin{aligned} \|x^\dagger - \hat{x}_\alpha\| &\leq \rho (\chi_{\nu/2} \alpha^{\nu/2} + z(\nu) c_3 \delta^{\nu/\nu+1}) + 2(b_1 - 2)^{-1} \chi_* (\chi_{(\nu+1)/2} \alpha^{\nu/2} + \chi_{1/2} c_3 z(\nu) \delta^{\nu/\nu+1}) = \\ &c_4 \alpha^{\nu/2} + c_5 \delta^{\nu/\nu+1}, \end{aligned}$$

where $c_4 = \rho(\chi_{\nu/2} + 2\chi_*\chi_{(\nu+1)/2}(b_1 - 2)^{-1})$, $c_5 = \rho z(\nu)c_3(2\chi_*\chi_{1/2}(b_1 - 2)^{-1} + 1)$. If $\alpha \leq \delta^{2/\nu+1}$ then the following relation holds:

$$\|x^\dagger - \hat{x}_\alpha\| \leq (c_4 + c_5)\delta^{\nu/\nu+1}.$$

Thus, the statement of Theorem is proved for $\alpha \leq \delta^{2/\nu+1}$.

For arbitrary $\alpha_1 > 0$ we take any function g_α satisfying conditions (4)–(6). It is known (see Lemma 3.2 [6]) that there is a constant $c_* > 0$ such that for all $0 \leq \lambda \leq 1$ and $\alpha \geq \alpha_1$

$$(1 - \lambda g_\alpha(\lambda))^2 \leq c_*(1 - \lambda g_{\alpha_1}(\lambda))^2 + \alpha_1^{-1}(\lambda(1 - \lambda g_\alpha(\lambda)))^2.$$

Then

$$\|S_{\alpha,n}x^\dagger\|^2 \leq c_*(\|S_{\alpha_1,n}x^\dagger\|^2 + \alpha_1^{-1}\|A_n S_{\alpha,n}x^\dagger\|^2). \quad (18)$$

Set $\alpha_1 = \delta^{2/\nu+1}$. Keeping in mind Lemma 4.1 and the right-hand side of (14), we find

$$\alpha_1^{-1}\|A_n S_{\alpha,n}x^\dagger\|^2 \leq (b_2 + 2)^2 \delta^{2\nu/\nu+1}.$$

In view of (17), we obtain

$$\|S_{\alpha_1,n}x^\dagger\|^2 \leq (\rho\chi_{\nu/2}\delta^{\nu/\nu+1} + \rho z(\nu)c_3\delta^{\nu/\nu+1})^2 = \delta^{2\nu/\nu+1}(\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2.$$

Substituting the above bounds for values $\alpha_1^{-1}\|A_n S_{\alpha,n}x^\dagger\|^2$ and $\|S_{\alpha_1,n}x^\dagger\|^2$ into (18), we get

$$\begin{aligned} \|S_{\alpha,n}x^\dagger\|^2 &\leq c_*((\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2 \delta^{2\nu/\nu+1} + (b_2 + 2)^2 \delta^{2\nu/\nu+1}) = \\ &c_*((\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2 + (b_2 + 2)^2) \delta^{2\nu/\nu+1}. \end{aligned}$$

Since $\alpha > \alpha_1 = \delta^{2/\nu+1}$, it is not difficult to see that

$$\alpha^{-1/2}\delta \leq \alpha_1^{-1/2}\delta = \delta^{\nu/\nu+1}.$$

Let us substitute the bounds for values $\alpha^{-1/2}\delta$ and $\|S_{\alpha,n}x^\dagger\|$ into (12). Then

$$\|x^\dagger - \hat{x}_\alpha\| \leq c_6 \delta^{\nu/\nu+1}$$

with $c_6 = (c_*(\rho\chi_{\nu/2} + \rho z(\nu)c_3)^2 + (b_2 + 2)^2)^{1/2} + 2\chi_*$.

Thus, the relation

$$\|x^\dagger - \hat{x}_\alpha\| \leq \delta^{\nu/\nu+1} \max\{c_6, c_4 + c_5\}$$

is valid at any $\alpha > 0$. Theorem 1 is proved. $\square$

Corollary 4.1. *To ensure the optimal order of accuracy $O(\delta^{\nu/\nu+1})$ within the framework of $(R_{\alpha,n}, A_n, b_1, b_2)$ on the class Ψ_{ρ,ν_1}^r , it is required to use $O(\delta^{-1/r} \log^{1+1/r} \delta^{-1})$ values of the inner products (8).*

Proof. Introduce the quantity $\text{Card}(n)$, which denotes the volume of inner products of the form (Ae_j, e_i) involved in constructing A_n . Obviously, the order of the total content of discrete information (8) required for the discretization scheme (7) is determined by $\text{Card}(n)$. Let us calculate

$$\text{Card}(n) = 2^{2n}(1 + n/2).$$

By virtue of the choice of the parameter n (10) we have

$$n2^{-2rn} = O(\delta), \quad 2^{2n} = O(\delta^{-1/r} n^{1/r}).$$

Then

$$\text{Card}(n) = O(n n^{1/r} \delta^{-1/r}) = O(\delta^{-1/r} \log^{1+1/r} \delta^{-1}). \quad (19)$$

$\square$

5. Additional comments

The traditional Galerkin method (i.e., $A_{m,l} = P_m A P_l$) was used in [6] to solve (1) with solutions belonging to the set $M_{\nu,\rho}(A)$. From the results of [6] it follows that the best choice of parameters m and l within the framework of the standard approach to the discretization gives the following bound:

$$O(\delta^{-(3\nu_1+1)/[2(\nu_1+1)r]})$$

of minimal information expenses of (8) on the class Ψ_{ρ,ν_1}^r , $1 < \nu_1 < \infty$. A modification of the Galerkin scheme (different from (7)) was used in [5] to solve equation (1) from $\Psi_{\rho,\nu}^r$, $\nu \in [1, 2]$. By this modification one can reduce the information expenses of (8) up to

$$O(\delta^{-13/12r} \log \delta^{-1}).$$

The last result was generalized on the class Ψ_{ρ,ν_1}^r , $1 < \nu_1 < \infty$, and improved on the logarithmic factor

$$O(\delta^{-(5\nu_1+3)/[4(\nu_1+1)r]})$$

in [7]. Comparison of the bound (19) with the above results proves the efficiency of the method $(R_\alpha, A_n, b_1, b_2)$.

6. Numerical example

The efficiency of the proposed approach to the solution of equation (1) is demonstrated with the example of the Fredholm integral equation

$$\tilde{A}x(t) \equiv \int_0^1 k(t, \tau)x(\tau)d\tau = f(t), \quad (20)$$

where $x(t), f(t) \in L_2(0, 1)$ and

$$k(t, \tau) = \begin{cases} \tau^2(3t - \tau)/6, & 0 \leq \tau \leq t \leq 1, \\ t^2(3\tau - t)/6, & 0 \leq t \leq \tau \leq 1. \end{cases}$$

It is well known [10, Chapter 2, §15] that equation (20) arises in solving the following boundary value problem:

$$f^{(IV)} = 0, \quad f(0) = f'(0) = f''(1) = f'''(1) = 0,$$

and the operator $\tilde{A}$ acts from $L_2(0, 1)$ into $W_2^4[0, 1]$. As the exact solution $x^\dagger \in M_{1,1}(\tilde{A})$, one takes the function

$$x^\dagger(t) := |\tilde{A}| \cdot 1(t) = \frac{t^4}{8} + \frac{t^2(1-t^2)}{4} - \frac{t^3(1-t)}{6}.$$

To realize the projection scheme (7), we use the orthonormal system of Legendre's polynomials considered on the interval $[0, 1]$. Equation (20) is solved with the help of Landweber's method

$$x_0 = 0, \quad x_l = x_{l-1} - \mu \tilde{A}_n^*(\tilde{A}_n x_{l-1} - P_{2^n} f_\delta), \quad l = 1, 2, \dots$$

The calculations are performed at the error level $\delta = 10^{-3}$. The discretization parameter n is chosen in accordance with (10) and is equal to 3.

The results of the test computation are presented in Table for various values of b_2 and μ , where the symbol “it” is interpreted as the number of iteration steps and the symbol “error” denotes the accuracy of approximation in the $L_2(0, 1)$ metrics. The graphs of the exact solution (bold line) and of the approximate solution obtained at $b_2 = 5/2$ and $\mu = 3/2$ are plotted in Figure.

b_2	μ	error	it
5/2	1	0.002261	238
5/2	3/2	0.00222	158
5/2	2	0.00231	119
5/2	5/2	0.00231	95
5/2	3	0.00229	79
3	1	0.01471	136
3	3/2	0.01455	91
3	2	0.01461	68
3	5/2	0.01478	54
3	3	0.01473	45

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