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DEGENERATE TIME-DEPENDENT VARIATIONAL INEQUALITIES WITH APPLICATIONS TO TRAFFIC EQUILIBRIUM PROBLEMS

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Abstract — The aim of this paper is to study the continuity of the solutions to degenerate time-dependent variational inequalities. In order to obtain the continuity of the solution, a previous continuity result (see [1]) for strongly monotone variational inequalities and an appropriate use of the convergence set in Mosco's sense play an important role. The continuity result allows us to provide a discretization procedure for the calculation of the solution to the variational inequality which expresses the time-dependent traffic network equilibrium problem.

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1. Introduction

The theory of variational inequalities, born in Italy in the sixties, was influenced by physical problems. Actually both pioneer works of G. Fichera (see [7]) and G. Stampacchia (see [16]) were motivated by concrete problems, the first in mechanics (a problem in elasticity with a unilateral boundary condition) and the second in potential theory (in connection with capacity, a basic concept from electrostatics). The proliferous growth of the theory brought out many important contributions in pure mathematics, in fields like nonlinear partial differential equations, operator theory and calculus of variations, as well as in applied mathematics, where variational inequalities have proved to be essential in a wide range of problems in mechanics, engineering, mathematical programming, control and optimization, etc.

In the beginning of the eighties of the 20^{th} century, it was proved by M. J. Smith (see [17]) and S. Dafermos (see [4]) that the traffic network equilibrium problem can be formulated in terms of a finite-dimensional variational inequality and, hence, it is possible to study in this way, existence, uniqueness, stability of traffic equilibria and to compute the solutions.

Many other problems arising from the economic world, as the spatial price equilibrium problem, the oligopolistic market equilibrium problem, the migration problem and many others (see [15]), were subsequently formulated in terms of finite-dimensional variational inequalities and, by means of this theory, solved.

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In the end of the nineties of the 20^{th} century, the traffic network equilibrium problem with feasible path flows which have to satisfy time-dependent capacity constraints and demands has been formulated by P. Daniele, A. Maugeri, and W. Oettli (see [5,6]) and also by T. L. Friesz, D. Bernstein, T. E. Smith, R. L. Tobin, and B. W. Wie (see [8]), as an evolutionary variational inequality, for which existence theorems and computational procedures are given.

This paper aims to obtain a continuity result for the solutions to degenerate evolutionary variational inequalities associated to a degenerate operator. Hence this result generalizes the previous one obtained in [1] in the core of strongly monotone operators. Our result is related to the convex sets $\mathbf{K}(t)$, $t \in [0,T]$, which fulfil Mosco's convergence property. The set of constraints related to the time-dependent traffic equilibrium problem and many other equilibrium problems fulfils this condition, then the solution is continuous. The paper is organized as follows. In Section 2, we introduce the time-dependent variational inequality which models the time-dependent traffic equilibrium problem. In Section 3, we prove that the solution to time-dependent variational inequality associated to linear degenerate operator is a continuous mapping from the time interval $[0,T]$ to the Euclidean space $\mathbb{R}^m_+$ (see Theorem 3.2). In Section 4, we apply our result to a rather general two-dimensional traffic network. Finally, in Section 5, we apply the result shown to the traffic equilibrium problem and the associated variational inequality. In order to calculate the solution of a traffic equilibrium problem we use a discretization procedure and then we compute, by means of the extragradient method and the combined relaxation method, the solutions of the finite-dimensional variational inequalities. At last, we construct an approximate solution with linear interpolation.

2. The dynamic model

In order to introduce our problem, a traffic network is defined. We deal directly with the time-dependent case; for the static case, see [5,6].

A traffic network is represented by a graph $G = [N, L]$, where N is the set of nodes (i.e., cross-roads, airports, railway stations) and L is the set of directed links between the nodes. Let r be a path consisting of a sequence of links which connect an Origin-Destination (O/D) pair of nodes. Let m be the number of the paths in the network. Let $\mathcal{W}$ denote the set of the O/D pairs with typical O/D pair w_j, $|\mathcal{W}| = l$ and $m > l$. The set of paths connecting the O/D pair w_j is represented by $\mathcal{R}_j$ and the entire set of paths in the network by $\mathcal{R}$. The topology of the network is described by the pair-link incidence matrix $\Phi = (\varphi_{j,r})$, where $\varphi_{j,r}$ is 1 if path r connects the pair w and 0 otherwise. Since the feasible flows have to satisfy time-dependent capacity constraints and demand requirements, also the flow vector is a time-dependent flow vector $F(t) \in \mathbb{R}^m_+$, where t varies in the fixed time interval $[0,T]$, while the topology remains fixed. Each component $F_r(t)$ of $F(t)$ gives the flow trajectory $F : [0,T] \to \mathbb{R}^m_+$ which has to satisfy almost everywhere on $[0,T]$ the capacity constraints

$$\lambda(t) \leqslant F(t) \leqslant \mu(t)$$

and the traffic conservation law

$$\Phi F(t) = \rho(t),$$

where the bounds $\lambda \leqslant \mu$ and the demand $\rho = (\rho_j)_{w_j \in \mathcal{W}}$ are given. We assume that λ and μ belong to $L^2([0,T], \mathbb{R}^m_+)$ and that ρ lies in $L^2([0,T], \mathbb{R}^m_+)$. Assuming

$$\Phi\lambda(t) \leqslant \rho(t) \leqslant \Phi\mu(t) \quad \text{a.e. in } [0,T],$$

we obtain that the set of feasible flows

$$\mathbf{K} = \{F \in L^2([0,T], \mathbb{R}^m_+) : \ \lambda(t) \leqslant F(t) \leqslant \mu(t), \quad \Phi F(t) = \rho(t), \text{ a.e. in } [0,T]\}$$

is nonempty, as it is shown in [9]. We remark that this kind of feasible set includes the constraints set related to dynamic market, evolutionary financial equilibrium problems, electric power supply chain networks with known demands and human migration problems (see [3]). Hence, even if the regularity result, that we will show, is related in particular to a convex $\mathbf{K}$ as above, however, it is quite general because $\mathbf{K}$ is the constraint set of a lot of general equilibrium problems. Clearly $\mathbf{K}$ is a convex, closed, bounded subset of $L^2([0,T], \mathbb{R}^m_+)$. Furthermore, we give the cost trajectory C which becomes a function of the time $C : [0,T] \times \mathbb{R}^m_+ \to \mathbb{R}^m_+$. The equilibrium condition is given by a generalized version of Wardrop's condition (see [5,6]), namely,

Definition 2.1. A flow $H \in \mathbf{K}$ is a user traffic equilibrium flow if $\forall w_j \in \mathcal{W}$, $\forall q, s \in \mathcal{R}_j$ and a.e. in $[0,T]$ it results

$$C_q(t, H(t)) > C_s(t, H(t)) \Longrightarrow H_q(t) = \lambda_q(t) \quad \textit{or} \quad H_s(t) = \mu_s(t). \tag{2.1}$$

The overall flow pattern obtained according to condition (2.1) fits very well into the framework of the theory of variational inequalities. In fact, in [5] and [6] the following result has been proved:

Theorem 2.1. *A flow $H \in \mathbf{K}$ is an equilibrium pattern if and only if it satisfies the following evolutionary variational inequality:*

$$\int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle \, dt \geqslant 0, \quad \forall F \in \mathbf{K}. \tag{2.2}$$

In order to give some results of existence of equilibria, we shall recall some definitions.

Definition 2.2. $C : [0,T] \times \mathbf{K} \to L^2([0,T], \mathbb{R}^m)$ is said to be

strongly monotone if for all $F \in \mathbf{K}$ there exists $\nu > 0$ such that $\int_0^T \langle C(t, F(t)), F(t) \rangle \, dt \geqslant \nu \|F\|^2_{L^2([0,T],\mathbb{R}^m)}$;

strictly monotone if for all $F, H \in \mathbf{K}$, $F \neq H$, $\int_0^T \langle C(t, F(t)) - C(t, H(t)), F(t) - H(t) \rangle \, dt \geqslant 0$;

pseudomonotone if for all $F, H \in \mathbf{K}$

$$\int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle \, dt \geqslant 0 \Longrightarrow \int_0^T \langle C(t, F(t)), H(t) - F(t) \rangle \, dt \geqslant 0;$$

upper hemicontinuous if for all $F \in \mathbf{K}$ the function $H \to \int_0^T \langle C(t, H(t)), F(t) - H(t) \rangle \, dt$ is upper semicontinuous on $\mathbf{K}$;

upper hemicontinuous along line segments if for all $H, F \in \mathbf{K}$ the function $G \to \int_0^T \langle C(t, G(t)), F(t) - H(t) \rangle \, dt$ is upper semicontinuous on the line segment $[F, G]$.

The following general result holds:

Theorem 2.2. (see [6]) *Let $C : [0,T] \times \mathbf{K} \to L^2([0,T],\mathbb{R}^m)$ and $\mathbf{K} \subseteq L^2([0,T],\mathbb{R}^m)$ be a nonempty and convex set. Assume that*

(i) there exist $A \subseteq \mathbf{K}$ nonempty, compact and $B \subseteq \mathbf{K}$ compact, convex such that, for every $H \in \mathbf{K} \setminus A$, there exists $\widehat{H} \in B$ with $\int_0^T \langle C(t,H(t)), \widehat{H}(t) - H(t)\rangle \, dt < 0$;

either (ii) or (iii) below holds;

(ii) C is upper hemicontinuous;

(iii) C is pseudomonotone and upper hemicontinuous along line segments.

Then, there exists $H \in A$ such that

$$\int_0^T \langle C(t,H(t)), F(t) - H(t)\rangle \, dt \geqslant 0,$$

for all $F \in \mathbf{K}$.

It is well known that if C is in addition strictly monotone, then the solution to the evolutionary variational inequality is unique.

We observe that problem (2.2) (see [13]) is also equivalent to the following one:

Find $H \in \mathbf{K}$ such that

$$\langle C(t,H(t)), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.3}$$

where

$$\mathbf{K}(t) = \left\{ F(t) \in \mathbb{R}^m_+ : \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t),\ r = 1,2,\dots,m,\ \sum_{r=1}^m F_r(t)\varphi_{jr} = \rho_j(t),\ j = 1,2,\dots,l \right\},$$

for a.e. $t \in [0,T]$.

It is easy to see that if the path cost vector is linear with respect to the path flow vector, i.e., $C(t,H(t)) = A(t)H(t) + B(t)$, where $A = (A_{rs})_{r,s=1,2,\dots,m} : [0,T] \to \mathbb{R}^{m\times m}_+$ is a bounded nonnegative definite matrix-function, that is,

$$\exists M > 0 : \ \|A(t)\|_{m\times m} = \left(\sum_{r,s=1}^m A^2_{rs}(t) \right)^{1/2} \leqslant M, \quad \text{a.e. in } [0,T], \tag{2.4}$$

$$\langle A(t)F(t), F(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{a.e. in } [0,T], \tag{2.5}$$

and $B \in L^2([0,T],\mathbb{R}^m_+)$, then there exists some solution to the variational inequality (2.3). Moreover, we remark that the set $\mathbf{X}$ of solutions is a closed convex subset of $\mathbf{K}$, for Theorem 3.1 in [11]. Moreover, if A is a positive definite matrix-function, namely,

$$\exists \nu > 0 : \ \langle A(t)F(t), F(t)\rangle \geqslant \nu \|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.6}$$

we have the solution to the evolutionary variational inequality which is unique. The same result is obtained if A is a matrix-function satisfying (2.4) and the following condition:

$$\langle A(t)F(t), F(t)\rangle \geqslant \nu(t) \|F(t)\|^2_m, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.7}$$

where $\nu \in L^\infty([0,T],\mathbb{R}^+_0)$ is such that $\nexists I \subseteq [0,T]$, $\mu(I) > 0 : \nu(t) = 0$, $\forall t \in I$, with μ Lebesque's measure, namely, A is a degenerate operator. The existence of the solution to the following evolutionary variational inequality

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.8}$$

is obvious. Now, let us show that the assumption (2.7) guarantees the uniqueness of the solution to the variational inequality. In fact, ab absurdum, let us suppose that there exist two solutions $H_1, H_2 \in \mathbf{K}$ such that

$$\langle A(t)H_1(t) + B(t), F(t) - H_1(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{2.9}$$

$$\langle A(t)H_2(t) + B(t), F(t) - H_2(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T]. \tag{2.10}$$

From (2.9) and (2.10), we obtain

$$\langle A(t)H_1(t) + B(t), H_2(t) - H_1(t)\rangle \geqslant 0, \quad \text{a.e. in } [0,T],$$

$$\langle A(t)H_2(t) + B(t), H_1(t) - H_2(t)\rangle \geqslant 0, \quad \text{a.e. in } [0,T],$$

having chosen $F = H_2$ in (2.9) and $F = H_1$ in (2.10). Summing the last inequalities, it results

$$\langle A(t)[H_1(t) - H_2(t)], H_2(t) - H_1(t)\rangle \geqslant 0, \quad \text{a.e. in } [0,T],$$

then

$$\langle A(t)[H_1(t) - H_2(t)], H_1(t) - H_2(t)\rangle \leqslant 0, \quad \text{a.e. in } [0,T].$$

Since A satisfies condition (2.7), we get $\nu(t)\|H_1(t) - H_2(t)\|_m^2\, dt \leqslant 0$, a.e. in $[0,T]$. By virtue of the assumptions on ν, it follows $H_1(t) = H_2(t)$, a.e. in $[0,T]$.

3. Continuity of the variational solution

In this section, we will extend the theorem of continuity for solutions to strongly monotone evolutionary variational inequalities proved in [1] assuming, now, that the linear operator is degenerate, studying the continuity of solutions to the following evolutionary variational inequality.

Find $H \in \mathbf{K}$ such that

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.1}$$

under the assumptions (2.4) and (2.7) in relation to operator A and with $\mathbf{K}(t)$ satisfying the following assumption:

(M) $\mathbf{K}(t)$, $t \in [0,T]$, is a family of nonempty convex, closed, uniformly bounded sets of $\mathbb{R}^m$ such that $\mathbf{K}(t_n)$ converges to $\mathbf{K}(t)$ in Mosco's sense, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$, with $t_n \to t$, as $n \to +\infty$.

We recall the important concept of Mosco's convergence (see [14]).

Definition 3.1. Let $(V, \|\cdot\|)$ be an Hilbert space and $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$, as $n \to +\infty$, in Mosco's sense, if

$(M1)$ for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathbb{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all $n \in \mathbb{N}$,

$(M2)$ for any subsequence $\{H_{k_n}\}_{n\in\mathbb{N}}$ weakly converging to H in V, such that H_{k_n} lies in $\mathbf{K}_{k_n}$ for all $n \in \mathbb{N}$, then the weak limit H belongs to $\mathbf{K}$.

According to some assumptions, the set as in (2.3) fulfils these conditions, in particular the following lemma holds (see proof of Theorem 3.2 in [1]):

Lemma 3.1. *Let $\lambda, \mu \in C([0,T], \mathbb{R}^m_+)$, let $\rho \in C([0,T], \mathbb{R}^l_+)$ and let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence such that $t_n \to t \in [0,T]$, as $n \to +\infty$. Then, the sequence of sets*

$$\mathbf{K}(t_n) = \{F(t_n) \in \mathbb{R}^m_+ : \lambda(t_n) \leqslant F(t_n) \leqslant \mu(t_n),\ \Phi F(t_n) = \rho(t_n)\}, \quad n \in \mathbb{N},$$

converges to

$$\mathbf{K}(t) = \{F(t) \in \mathbb{R}^m_+ : \lambda(t) \leqslant F(t) \leqslant \mu(t),\ \Phi F(t) = \rho(t)\},$$

as $n \to +\infty$, in Mosco's sense.

Now, we recall the continuity result for strongly monotone variational inequalities (see Theorem 3.2 in [1]), namely, when the matrix-function A satisfies condition (2.6).

Theorem 3.1. *Let $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ be a positive definite matrix-function and let $B \in C([0,T], \mathbb{R}^m_+)$ be a vector-function. Suppose that $\lambda, \mu, \rho \in C([0,T], \mathbb{R}^m_+)$. Then, the evolutionary variational inequality* (3.1) *admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}^m_+)$. Moreover, the estimate*

$$\|H_1 - H_2\|_{C([0,T],\mathbb{R}^m_+)} \leqslant \nu^{-1} \|B_1 - B_2\|_{C([0,T],\mathbb{R}^m_+)}$$

holds, where ν is the constant of positive definition of matrix-function A.

Remark 3.1. Theorem 3.1 still holds true if $\mathbf{K}(t)$, $t \in [0,T]$, is a family of sets satisfying condition (M).

At first, we prove a preliminary lemma related to nonnegative matrix A satisfying condition (2.5). Hence, let us observe that, under this assumption, the set $\mathbf{X}$ of solutions to time-dependent variational inequality (3.1) is closed, convex, and nonempty. Let $I : L^2([0,T], \mathbb{R}^m_+) \to L^2([0,T], \mathbb{R}^m_+)$ be the identity operator and let us consider the following evolutionary variational inequality:

$$\langle I(t)H(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ a.e. in } [0,T], \tag{3.2}$$

which admits a unique solution $H(t) \in \mathbf{X}(t)$. Further, for every $\varepsilon > 0$, let us consider the following perturbed evolutionary variational inequality:

$$\langle [A(t) + \varepsilon I(t)]H(t) + B(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{3.3}$$

which also admits a unique continuous solution H_ε by virtue of Theorem 3.1. Then, we can prove the following preliminary result for nonnegative matrix-function:

Lemma 3.2. *Let $A \in C([0,T], \mathbb{R}^{m\times m}_+)$ be a matrix-function satisfying condition* (2.5) *and let $B \in C([0,T], \mathbb{R}^m_+)$ be a vector-function. Let $\mathbf{K}(t)$, $t \in [0,T]$, be a family of sets satisfying condition (M). If $H_\varepsilon(t)$, $\forall \varepsilon > 0$, is the unique solution to* (3.3)*, it results*

$$\lim_{\varepsilon \to 0} H_\varepsilon(t) = H(t), \quad \textit{in } [0,T],$$

and

$$\lim_{\varepsilon \to 0} \|H_\varepsilon(t) - H(t)\|^2_{L^2([0,T],\mathbb{R}^m_+)} = 0,$$

where H is a solution to the evolutionary variational inequality (3.1).

Proof. Let H be the unique solution to (3.2), therefore $H \in \mathbf{X}$ and

$$\langle I(t)H(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{X}(t), \text{ in } [0,T]. \tag{3.4}$$

Let H_ε be the unique solution to (3.3), namely, $H_\varepsilon \in \mathbf{K}$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T]. \tag{3.5}$$

Setting $F(t) = H_\varepsilon(t)$, for $t \in [0,T]$, in (3.1) and $F(t) = H(t)$, for $t \in [0,T]$, in (3.5) and adding we get

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle + \varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geqslant 0, \tag{3.6}$$

in $[0,T]$. By assumption (2.5), it follows

$$\langle A(t)[H(t) - H_\varepsilon(t)], H_\varepsilon(t) - H(t)\rangle \leqslant 0, \quad \text{in } [0,T],$$

then, by (3.6), we obtain

$$\varepsilon\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geqslant 0, \quad \text{in } [0,T],$$

and dividing by $\varepsilon > 0$, it results

$$\langle H_\varepsilon(t), H(t) - H_\varepsilon(t)\rangle \geqslant 0, \quad \text{in } [0,T]. \tag{3.7}$$

Taking into account (3.7), one has

$$\|H_\varepsilon(t)\|_m^2 \leqslant \langle H_\varepsilon(t), H(t)\rangle \leqslant \|H(t)\|_m \|H_\varepsilon(t)\|_m, \quad \text{in } [0,T],$$

then

$$\|H_\varepsilon(t)\|_m \leqslant \|H(t)\|_m, \quad \text{in } [0,T].$$

Since $H(t) \in \mathbf{X}(t) \subseteq \mathbf{K}(t)$, in $[0,T]$, and $\mathbf{K}(t)$, $t \in [0,T]$, is a family of uniformly bounded sets of $\mathbb{R}^m$, it results

$$\|H(t)\|_m \leqslant C, \quad \text{in } [0,T],$$

with C a constant independent on $t \in [0,T]$, then

$$\|H_\varepsilon(t)\|_m \leqslant C, \quad \forall \varepsilon > 0, \text{ in } [0,T].$$

Hence there exists a subsequence $\{H_\eta(t)\}_\eta$ converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, in [0,T], and thus

$$\lim_{\eta \to 0} H_\eta(t) = \overline{H}(t), \quad \text{in } [0,T].$$

Taking into account that $\mathbf{K}(t)$ is a closed set of $\mathbb{R}^m$ and $\{H_\eta(t)\}_\eta \subseteq \mathbf{K}(t)$, then

$$\overline{H}(t) \in \mathbf{K}(t), \text{ in } [0,T].$$

It remains to prove that

$$\overline{H}(t) = H(t), \quad \text{in } [0,T].$$

Hence, considering (3.5) with $\varepsilon = \eta$, we get

$$\langle A(t)H_\eta(t) + B(t), F(t) - H_\eta(t)\rangle + \eta\langle H_\eta(t), F(t) - H_\eta(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \tag{3.8}$$

in $[0,T]$, and taking into account that

$$\lim_{\eta\to 0}\langle H_\eta(t), H_\eta(t)\rangle = \langle \overline{H}(t), \overline{H}(t)\rangle, \quad \text{in } [0,T],$$

and that

$$\lim_{\eta\to 0}\langle A(t)H_\eta(t)+B(t), H_\eta(t)\rangle = \langle A(t)\overline{H}(t)+B(t), \overline{H}(t)\rangle, \quad \text{in } [0,T],$$

from (3.8), we obtain

$$\langle A(t)\overline{H}(t)+B(t), F(t)-\overline{H}(t)\rangle \geqslant 0, \quad \forall F(t)\in \mathbf{K}(t), \text{ in } [0,T]. \tag{3.9}$$

Then (3.9) implies that $\overline{H}$ is a solution to (3.1), in $[0,T]$, namely $\overline{H}\in \mathbf{X}$. If the solution to (3.1) is unique, then the proof is concluded. Now, we suppose that the solution to (3.1) is not unique. Setting $\varepsilon=\eta$ in (3.7), we get

$$\langle H_\eta(t), H(t)-H_\eta(t)\rangle \geqslant 0, \quad \text{in } [0,T],$$

and passing to the limit as $\eta\to 0$, we obtain

$$\langle \overline{H}(t), H(t)-\overline{H}(t)\rangle \geqslant 0, \quad \text{in } [0,T]. \tag{3.10}$$

Rewriting (3.4) with $F=\overline{H}\in \mathbf{X}$, it results

$$\langle H(t), \overline{H}(t)-H(t)\rangle \geqslant 0, \quad \text{in } [0,T], \tag{3.11}$$

and adding (3.10) and (3.11), we have

$$\langle \overline{H}(t)-H(t), H(t)-\overline{H}(t)\rangle \geqslant 0, \quad \text{in } [0,T].$$

Then

$$\langle \overline{H}(t)-H(t), H(t)-\overline{H}(t)\rangle = 0, \quad \text{in } [0,T],$$

that implies

$$\overline{H}(t) = H(t), \quad \text{in } [0,T].$$

In this way, we have shown that every subsequence converges to the same limit $H(t)$ and hence

$$\lim_{\varepsilon\to 0} H_\varepsilon(t) = H(t), \quad \text{in } [0,T],$$

from this it follows

$$\lim_{\varepsilon\to 0} |H_\varepsilon(t)-H(t)|^2 = 0, \quad \text{in } [0,T].$$

Moreover, we remark that

$$|H_\varepsilon(t)-H(t)|^2 \leqslant 2\big(|H_\varepsilon(t)|^2+|H(t)|^2\big) \leqslant 4C^2, \quad \text{in } [0,T],$$

then, by virtue of Lebesque's Theorem we have

$$\lim_{\varepsilon\to 0} \|H_\varepsilon(t)-H(t)\|^2_{L^2([0,T],\mathbb{R}^m_+)} = 0.$$

□

Now, we present the main result for degenerate variational inequalities (2.8), namely, when the matrix-function A verifies the conditions (2.4) and (2.7).

Theorem 3.2. *Let $A \in C([0,T], \mathbb{R}_+^{m\times m})$ be a matrix-function satisfying condition* (2.7) *and let $B \in C([0,T], \mathbb{R}_+^m)$ be a vector-function. Let $\mathbf{K}(t)$, $t \in [0,T]$, be a family of sets satisfying condition (M). Then, the evolutionary variational inequality*

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \quad in\ [0,T], \tag{3.12}$$

admits a unique solution $H \in \mathbf{K}$ such that $H \in C([0,T], \mathbb{R}_+^m)$.

Proof. Let $t \in [0,T]$ be fixed and let $\{t_n\}_{n\in\mathbb{N}} \subseteq [0,T]$ be a sequence, with $t_n \to t$, as $n \to +\infty$.

Let us consider the solution $H(t)$ to variational inequality (3.12) and the solution $H(t_n)$, $\forall n \in \mathbb{N}$, to the following variational inequalities:

$$\langle A(t_n)H(t_n) + B(t_n), F(t_n) - H(t_n)\rangle \geqslant 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}. \tag{3.13}$$

Let $H_\varepsilon(t)$ be the unique solution to the strongly monotone perturbed variational inequality (3.3), namely, $H_\varepsilon(t) \in \mathbf{K}(t)$ and

$$\langle [A(t) + \varepsilon I(t)]H_\varepsilon(t) + B(t), F(t) - H_\varepsilon(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t),\ \text{in } [0,T]. \tag{3.14}$$

Since $H_\varepsilon(t)$ is continuous in $[0,T]$, we have that solutions $H(t_n)$, $\forall n \in \mathbb{N}$, to the following evolutionary variational inequalities:

$$\langle [A(t_n) + \varepsilon I(t_n)]H_\varepsilon(t_n) + B(t_n), F(t_n) - H_\varepsilon(t_n)\rangle \geqslant 0, \quad \forall F(t_n) \in \mathbf{K}(t_n), \tag{3.15}$$

$\forall n \in \mathbb{N}$, converge to $H_\varepsilon(t)$, as $n \to +\infty$. Setting $F(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, in (3.15) and $F(t_n) = H_\varepsilon(t_n)$, $\forall n \in \mathbb{N}$, in (3.13) and adding we get, $\forall n \in \mathbb{N}$,

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle + \varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geqslant 0. \tag{3.16}$$

We remark that for condition (2.7) on the matrix-function A we have

$$\langle A(t_n)[H_\varepsilon(t_n) - H(t_n)], H(t_n) - H_\varepsilon(t_n)\rangle \leqslant 0, \quad \forall n \in \mathbb{N}.$$

Then, from (3.16) it follows $\varepsilon\langle H_\varepsilon(t_n), H(t_n) - H_\varepsilon(t_n)\rangle \geqslant 0$, $\forall n \in \mathbb{N}$, and processing as in the proof of Lemma 3.2, we get

$$\|H_\varepsilon(t_n)\|_m \leqslant C, \quad \forall \varepsilon > 0,\ \forall n \in \mathbb{N}, \tag{3.17}$$

where C is a constant independent on ε and on $n \in \mathbb{N}$.

For Lemma 3.2, it follows $\lim_{\varepsilon\to 0} H_\varepsilon(t_n) = \widetilde{H}(t_n)$, $\forall n \in \mathbb{N}$, with $\widetilde{H}(t_n) \in \mathbf{K}(t_n)$, $\forall n \in \mathbb{N}$, and such that

$$\langle A(t_n)\widetilde{H}(t_n) + B(t_n), F(t_n) - \widetilde{H}(t_n)\rangle \geqslant 0, \quad \forall F(t_n) \in \mathbf{K}(t_n),\ \forall n \in \mathbb{N}.$$

Since the solution to (3.13) is unique, one has $\widetilde{H}(t_n) = H(t_n)$, $\forall n \in \mathbb{N}$, and, passing to the limit as $\varepsilon \to 0$ in (3.17), it results $\|H(t_n)\|_m \leqslant C$, $\forall n \in \mathbb{N}$. Hence the sequence $\{H(t_n)\}_{n\in\mathbb{N}}$ is bounded, then there exists a subsequence $\{H(t_{k_n})\}_{n\in\mathbb{N}}$, with $H(t_{k_n}) \in \mathbf{K}(t_{k_n})$, $\forall n \in \mathbb{N}$, converging in $\mathbb{R}^m$ to an element $\overline{H}(t)$ of $\mathbb{R}^m$, namely $\lim_{n\to+\infty} H(t_{k_n}) = \overline{H}(t)$. Moreover, by (3.13) it obtains

$$\langle A(t)\overline{H}(t) + B(t), F(t) - \overline{H}(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t),$$

and, for the uniqueness of the solution to (3.12), it follows $\overline{H}(t) = H(t)$. The same result holds for each subsequence and therefore $\lim_{n\to+\infty} H(t_n) = H(t)$, namely, our assert. The proof is now complete. □

Remark 3.2. Theorem 3.2, by virtue of Lemma 3.1, holds if

$$\mathbf{K} = \{F \in L^2([0,T], \mathbb{R}^m_+) : \ \lambda(t) \leqslant F(t) \leqslant \mu(t), \quad \Phi F(t) = \rho(t), \quad \text{a.e. in } [0,T]\},$$

so the solutions to evolutionary equilibrium problems turn out to be continuous.

4. A general two-dimensional example

We consider a two-dimensional traffic equilibrium problem with linear and degenerate cost function. Hence, let us consider four continuous functions a_1, a_2, b_1, b_2 defined in $[0,T]$, such that $a_1(t) > 0$ in $[0,T]$, $a_2(t) > 0$ in $]0,T]$, $a_2(0) = 0$ and $b_1, b_2 \geqslant 0$ in $[0,T]$. Moreover, we suppose that $a_1(t) \neq a_2(t)$, $\forall t \in [0,T]$. Let us define the following cost vector-function:

$$C(t, H(t)) = A(t)H(t) + B(t) = \begin{pmatrix} a_1(t) & 2\sqrt{a_1(t)a_2(t)} \\ 0 & a_2(t) \end{pmatrix} \begin{pmatrix} H_1(t) \\ H_2(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

We remark that $A(t)$ is a degenerate matrix for $t = 0$, and, for $t > 0$ the matrix is positive definite, moreover, it satisfies the following condition:

$$\langle A(t)[F(t) - H(t)], F(t) - H(t)\rangle = \big\{\sqrt{a_1(t)}[F_1(t) - H_1(t)] +$$
$$\sqrt{a_2(t)}[F_2(t) - H_2(t)]\big\}^2 \geqslant [a_1(t) \wedge a_2(t)]\|F(t) - H(t)\|^2_m,$$

$\forall F, H \in L^2([0,T], \mathbb{R}^2)$, in $[0,T]$. Now, we study the solutions to the next evolutionary variational inequality

$$\langle A(t)H(t) + B(t), F(t) - H(t)\rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ in } [0,T], \tag{4.1}$$

where

$$\mathbf{K} = \{F \in L^2([0,T], \mathbb{R}^2) : \ F(t) \geqslant 0, \quad F_1(t) + F_2(t) = \rho(t), \text{ in } [0,T]\},$$

with $\rho \in C([0,T], \mathbb{R}_+)$. In our case, (4.1) becomes

$$[a_1(t)H_1(t) + 2\sqrt{a_1(t)a_2(t)}H_2(t) + b_1(t)][F_1(t) - H_1(t)] + [a_2(t)H_2(t) + b_2(t)]\times$$
$$[F_2(t) - H_2(t)] \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{in } [0,T]. \tag{4.2}$$

Then applying our main result, Theorem 3.2, we get that the solution to (4.2) is continuous in $[0,T]$. Now we determine the solution to (4.2) by using the direct method (see [12]). At first, we derive

$$F_2(t) = \rho(t) - F_1(t), \quad \text{in } [0,T], \tag{4.3}$$

$$\widetilde{\mathbf{K}} = \{F_1 \in L^2([0,T], \mathbb{R}) : \ F_1(t) \geqslant 0, \quad F_1(t) \leqslant \rho(t), \text{ in } [0,T]\}, \tag{4.4}$$

and we consider the following variational inequality:

$$\big\{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2 H_1(t) + [2\sqrt{a_1(t)a_2(t)} - a_2(t)]\rho(t) + b_1(t) - b_2(t)\big\}\times$$
$$[F_1(t) - H_1(t)] \geqslant 0, \quad \forall F(t) \in \widetilde{\mathbf{K}}(t), \text{ in } [0,T]. \tag{4.5}$$

In the first step, we see if there exist solutions to the system

$$\begin{cases} [\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2 H_1(t) + [2\sqrt{a_1(t)a_2(t)} - a_2(t)]\rho(t) + b_1(t) - b_2(t) = 0, \\ H_1 \in \widetilde{\mathbf{K}}. \end{cases} \tag{4.6}$$

For $t = 0$, (4.6) becomes

$$\begin{cases} a_1(0)H_1(0) + b_1(0) - b_2(0) = 0, \\ H_1(0) \in \widetilde{\mathbf{K}}(0), \end{cases} \tag{4.7}$$

then, we obtain

$$\begin{cases} H_1(0) = [b_2(0) - b_1(0)]/a_1(0), \\ H_2(0) = [b_1(0) - b_2(0) + a_1(0)\rho(0)]/a_1(0), \end{cases} \tag{4.8}$$

provided that

$$\begin{cases} b_1(0) \leqslant b_2(0), \\ \rho(0) \geqslant [b_2(0) - b_1(0)]/a_1(0). \end{cases}$$

Moreover, for $t \in]0, T]$, we get

$$\begin{cases} H_1(t) = \dfrac{b_2(t) - b_1(t) - [2\sqrt{a_1(t)a_2(t)} - a_2(t)]\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2}, & \forall t \in]0, T], \\ H_2(t) = \dfrac{b_1(t) - b_2(t) + a_1(t)\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2}, & \forall t \in]0, T], \end{cases} \tag{4.9}$$

provided that

$$\begin{cases} \sqrt{a_1(t)} > \sqrt{a_2(t)}/2, & \forall t \in]0, T], \\ b_1(t) \leqslant b_2(t), & \forall t \in]0, T], \\ \dfrac{b_2(t) - b_1(t)}{a_1(t)} \leqslant \rho(t) \leqslant \dfrac{b_2(t) - b_1(t)}{2\sqrt{a_1(t)a_2(t)} - a_2(t)}, & \forall t \in]0, T]. \end{cases}$$

Then the unique solution to (4.2) is given by (4.8) in $t = 0$ and by (4.9) in $]0, T]$, and, of course, it is continuous in $]0, T]$. We can easily verify that the solution is continuous also in $t = 0$, in fact we have

$$\lim_{t \to 0^+} H_1(t) = \lim_{t \to 0^+} \frac{b_2(t) - b_1(t) - [2\sqrt{a_1(t)a_2(t)} - a_2(t)]\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2} = H_1(0),$$

and

$$\lim_{t \to 0^+} H_2(t) = \lim_{t \to 0^+} \frac{b_1(t) - b_2(t) + a_1(t)\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2} = H_2(0).$$

Now, if $\rho(t) < [b_2(t) - b_1(t)]/a_1(t)$, $\forall t \in [0, T]$, $b_1(t) < b_2(t)$, $\forall t \in [0, T]$, and $\sqrt{a_1(t)} > \sqrt{a_2(t)}/2$ $\forall t \in [0, T]$, we have, for $t = 0$,

$$\begin{cases} H_1(0) = [b_2(0) - b_1(0)]/a_1(0), \\ H_2(0) = [b_1(0) - b_2(0) + a_1(0)\rho(0)]/a_1(0), \end{cases} \tag{4.10}$$

and for $t \in]0, T]$, we get

$$\begin{cases} H_1(t) = \dfrac{b_2(t) - b_1(t) - [2\sqrt{a_1(t)a_2(t)} - a_2(t)]\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2}, & \forall t \in]0, T], \\ H_2(t) = \dfrac{b_1(t) - b_2(t) + a_1(t)\rho(t)}{[\sqrt{a_1(t)} - \sqrt{a_2(t)}]^2}, & \forall t \in]0, T]. \end{cases} \tag{4.11}$$

The same calculations made before prove, also in this case, that the solution to the evolutionary variational inequality (4.2) is continuous in $[0,T]$.

On the contrary, if $\sqrt{a_1(t)} > \sqrt{a_2(t)}/2$, $\forall t \in [0,T]$, $b_1(t) > b_2(t)$, $\forall t \in [0,T]$, and $\forall \rho \in C([0,T],\mathbb{R}_+)$, the unique solution to (4.2) is given by

$$\begin{cases} H_1(t) = 0, & \forall t \in [0,T], \\ H_2(t) = \rho(t), & \forall t \in [0,T] \end{cases}$$

and also in this case the continuity of the solution can be verified.

In each case, we have found that the solution to the evolutionary variational inequality (4.2) is continuous in $[0,T]$.

5. Traffic network numerical example

Now, we introduce a method to solve the evolutionary variational inequalities related to a linear degenerate operator.

We consider the following evolutionary variational inequality: find $H \in \mathbf{K}$ such that

$$\langle C(H(t)), F(t) - H(t) \rangle \geqslant 0, \quad \forall F(t) \in \mathbf{K}(t), \text{ a.e. in } [0,T], \tag{5.1}$$

where $C(H(t)) = A(t)H(t) + B(t)$, a.e. in $[0,T]$, with A satisfying the following condition:

$$\langle A(t)F(t), F(t) \rangle \geqslant \nu(t)\|F(t)\|_m^2, \quad \forall F(t) \in \mathbf{K}(t), \quad \text{a.e. in } [0,T], \tag{5.2}$$

where $\nu \in L^\infty([0,T],\mathbb{R}_0^+)$ is such that $\nexists I \subseteq [0,T]$, $\mu(I) > 0$: $\nu(t) = 0$, $\forall t \in I$, and

$$\mathbf{K} = \Big\{ F \in L^2([0,T],\mathbb{R}_+^m) : \ \lambda_r(t) \leqslant F_r(t) \leqslant \mu_r(t), \quad r = 1,2,\ldots,m,$$
$$\sum_{r=1}^m F_r(t)\varphi_{jr} = \rho_j(t), \ j = 1,2,\ldots,l, \text{ a.e. in } [0,T] \Big\}.$$

We suppose that the assumptions of Theorem 3.2 are satisfied and hence the solution H belongs to $C([0,T],\mathbb{R}_+^m)$. As a consequence, (5.1) holds for each $t \in [0,T]$, namely,

$$\langle C(H(t)), F(t) - H(t) \rangle \geqslant 0, \quad \forall t \in [0,T].$$

A method to solve variational inequalities is the extragradient method, but it can be applied to evolutionary variational inequalities after a discretization procedure has been made.

In the following, applying a discretization procedure, we will use the extragradient method, and we will compute the solution of a variational inequality associated to a transportation network.

Consider now a partition of $[0,T]$, such that: $0 = t_0 < t_1 < \ldots < t_i < \ldots < t_N = T$. Then, for each value t_i, for $i = 0,1,\ldots,N$, we consider the static variational inequality

$$\langle C(H(t_i)), F(t_i) - H(t_i) \rangle \geqslant 0, \quad \forall F(t_i) \in \mathbf{K}(t_i), \tag{5.3}$$

where $C(H(t_i)) = A(t_i)H(t_i) + B(t_i)$ and

$$\mathbf{K}(t_i) = \Big\{ F(t_i) \in \mathbb{R}_+^m : \lambda_r(t_i) \leqslant F_r(t_i) \leqslant \mu_r(t_i), \ r = 1,\ldots,m, \ \sum_{r=1}^m F_r(t_i)\varphi_{jr} = \rho_j(t_i), \ j = 1,\ldots,l \Big\}.$$

We compute now the solution to the finite-dimensional variational inequality (5.3) using the extragradient method. The algorithm, as it is well known, starting from any $H^0(t_i) \in \mathbf{K}(t_i)$ fixed, generates a succession $\{H^k(t_i)\}_{n\in\mathbb{N}}$ such that

$$\overline{H}^k(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(H^k(t_i))),$$

$$H^{k+1}(t_i) = P_{\mathbf{K}(t_i)}(H^k(t_i) - \alpha C(\overline{H}^k(t_i))),$$

where $P_{\mathbf{K}(t_i)}(\cdot)$ denotes the orthogonal projection map onto $\mathbf{K}(t_i)$ and α is constant for all iterations.

In [2] and [18] the convergence of the extragradient method is proved under the following hypothesis: C is a monotone and Lipschitz continuous mapping and $\alpha \in (0, 1/L)$, where L is the Lipschitz constant. A drawback is the choice of α when L is unknown. Indeed, if α is too small, the convergence is slow; when α is too large, there might be no convergence at all. After an iterative procedure, we can construct a function by linear interpolation.

We remark that this problem can be solved by the usual descent methods, without calculation of the Lipschitz constant for the analysis of the convergence. In the following, we extend a combined relaxation method to the calculus of solution to time-dependent variational inequalities.

This method (see [10]) runs as follows. After a partition of real interval $[0, T]$, the algorithm, to solve finite-dimensional variational inequality (5.3), starts from any $H^0(t_i) \in \mathbf{K}(t_i)$ fixed, from a sequence $\{\gamma_k\}$ satisfying the following conditions:

$$\gamma_k \in [0, 2],\ k = 0, 1, \ldots; \quad \sum_{k=0}^{+\infty} \gamma_k(2 - \gamma_k) = \infty,$$

and from numbers $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\widetilde{\theta} > 0$ chosen. It finds m as the smallest number in $\mathbb{Z}_+$ such that

$$\langle C(H^k(t_i)) - C(G^{k,m}(t_i)), H^k(t_i) - G^{k,m}(t_i)\rangle \leqslant (1 - \alpha)(\widetilde{\theta}\beta^m)^{-1}\|G^{k,m}(t_i) - H^k(t_i)\|_m^2,$$

where $G^{k,m}(t_i)$ is a solution to the auxiliary problem of finding $\overline{G}(t_i) \in \mathbf{K}(t_i)$ such that

$$\langle C(H^k(t_i)) + (\widetilde{\theta}\beta^m)^{-1}(\overline{G}(t_i) - H^k(t_i)), H(t_i) - \overline{G}(t_i)\rangle \geqslant 0, \quad \forall H(t_i) \in \mathbf{K}(t_i).$$

Set $\theta_k := \beta^m\widetilde{\theta}$, $F^k(t_i) := G^{k,m}(t_i)$. If $H^k(t_i) = F^k(t_i)$ or $G(F^k(t_i)) = 0$, the algorithm stops, else, setting

$$t^k(t_i) := C(F^k(t_i)) - C(H^k(t_i)) - \theta_k^{-1}(F^k(t_i) - H^k(t_i)), \quad c^k(t_i) := C(F^k(t_i)),$$

$$\sigma^k(t_i) := \alpha\theta_k^{-1}\|F^k(t_i) - H^k(t_i)\|_m^2/\|t^k(t_i)\|_m^2, \quad H^{k+1}(t_i) := P_{\mathbf{K}(t_i)}(H^k(t_i) - \gamma_k\sigma^k(t_i)c^k(t_i)),$$

the iteration repeats itself. After a linear interpolation, the approximate equilibrium solution is constructed.

In [10], it is shown that this method is convergent to a solution of the finite-dimensional variational inequality problem under the only assumption that C is locally Lipschitz continuous and monotone.

Now, we consider a transportation network pattern for the network shown in Fig. 1. The network consists of six nodes and eight links. We assume that the O/D pairs are represented by $w_1 = (P_1, P_5)$ and $w_2 = (P_2, P_6)$, which are respectively connected by the following paths:

$$w_1 : \begin{cases} R_1 = (P_1, P_2) \cup (P_2, P_5) \\ R_2 = (P_1, P_4) \cup (P_4, P_5), \end{cases} \qquad w_2 : \begin{cases} R_3 = (P_2, P_3) \cup (P_3, P_6) \\ R_4 = (P_2, P_5) \cup (P_5, P_6) \\ R_5 = (P_2, P_5) \cup (P_5, P_3) \cup (P_3, P_6). \end{cases}$$

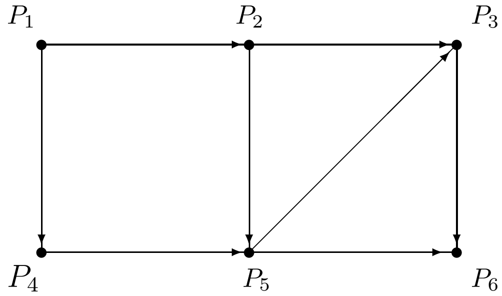

Fig. 1. A network model

We consider the cost vector-function on path C defined by

$$C : L^2([0,2], \mathbb{R}^5_+) \to L^2([0,2], \mathbb{R}^5_+),$$

$$\begin{aligned} &C_1(H(t)) = (2t+1)H_1(t) + t^2 H_4(t) + (t+3)H_5(t) + 3t + 1, \\ &C_2(H(t)) = (t+1)^2 H_2(t) + t^2 + 3, \\ &C_3(H(t)) = (2t+3)H_3(t) + (t^2+2)H_5(t) + 2t, \\ &C_4(H(t)) = t^2 H_1(t) + (t+4)H_4(t) + t^2 H_5(t), \\ &C_5(H(t)) = (t+3)H_1(t) + (t^2+2)H_3(t) + t^2 H_4(t) + (3t^2+2)H_5(t) + t + 2. \end{aligned}$$

The set of feasible flows is given by

$$\mathbf{K} = \Big\{ F \in L^2([0,2], \mathbb{R}^5_+) : \; (0,0,0,0,0) \leqslant (F_1(t), F_2(t), F_3(t), F_4(t), F_5(t)) \leqslant$$
$$(20t+15, 30t+10, 20t+15, 40t+19, 30t+21), \; F_1(t) + F_2(t) = 2t+3,$$
$$F_3(t) + F_4(t) + F_5(t) = 6t+5, \quad \text{in } [0,2] \Big\}.$$

It is easy to verify that the cost vector-function satisfies condition (5.2). Moreover, it results that

$$\|C(H(t)) - C(F(t))\|^2_m = [(2t+1)(H_1(t) - F_1(t)) + t^2(H_4(t) - F_4(t)) +$$
$$(t+3)(H_5(t) - F_5(t))]^2 + [(t+1)^2(H_2(t) - F_2(t))]^2 + [(2t+3)(H_3(t) - F_3(t)) +$$
$$(t^2+2)(H_5(t) - F_5(t)]^2 + [t^2(H_1(t) - F_1(t)) + (t+4)(H_4(t) - F_4(t)) + t^2(H_5(t) - F_5(t)]^2 +$$
$$[(t+3)(H_1(t) - F_1(t)) + (t^2+2)(H_3(t) - F_3(t)) + t^2(H_4(t) - F_4(t)) + (3t^2+2)(H_5(t) - F_5(t))]^2 \leqslant$$
$$[3(2t+1)^2 + 3t^4 + 4(t+3)^2](H_1(t) - F_1(t))^2 + (t+1)^4(H_2(t) - F_2(t))^2 +$$
$$[2(2t+3)^2 + 4(t^2+2)](H_3(t) - F_3(t))^2 + [3t^4 + 3(t+4)^2 + 4t^4](H_4(t) - F_4(t))^2 +$$
$$[3(t+3)^2 + 2(t^2+2)^2 + 3t^4 + 4(3t^2+2)^2](H_5(t) - F_5(t))^2 \leqslant 979\|H(t) - F(t)\|^2_m,$$

for any $H(t), F(t) \in \mathbf{K}(t)$ and for $t \in [0,2]$. As a consequence, the extragradient method is convergent for $\alpha \in (0, 0.031)$, for the property of C. We can compute an approximate curve of equilibria, by selecting $t_i \in \{k/15 : \; k \in \{0,1,\dots,30\}\}$. Using a simple MatLab computation and choosing the initial point $H^0(t_i) = (t_i+1, t_i+2, 2t_i+2, 2t_i+2, 2t_i+1)$ to start the iterative method, we obtain the equilibria consisting of the points, which are written in Table 1. Repeating the computational experience with the combined relaxation method,

Table 1. **Numerical results**

t_i	$H_1(t_i)$	$H_2(t_i)$	$H_3(t_i)$	$H_4(t_i)$	$H_5(t_i)$
0	2.4997175	0.5002824	2.8571429	2.1428571	0
1/15	2.3593725	0.7739608	3.0329378	2.3670622	0
2/15	2.2605846	1.0060820	3.2090342	2.5909658	0
1/5	2.1909429	1.2090571	3.3852142	2.8147858	0
4/15	2.1425800	1.3907534	3.5614116	3.0385884	0
1/3	2.1094881	1.5571786	3.7376283	3.2623717	0
2/5	2.0875250	1.7124750	3.9138980	3.4861020	0
7/15	2.0736694	1.8596639	4.0902629	3.7097371	0
8/15	2.0656801	2.0009866	4.2667629	3.9332371	0
3/5	2.0618720	2.1381280	4.4434288	4.1565712	0
2/3	2.0609642	2.2723691	4.6202803	4.3797197	0
11/15	2.0619749	2.4046918	4.7973251	4.6026749	0
4/5	2.0641447	2.5358553	4.9745592	4.8254408	0
13/15	2.0668817	2.6664516	5.1519674	5.0480326	0
14/15	2.0697226	2.7969441	5.3295248	5.2704752	0
1	2.0723025	2.9276975	5.5071972	5.4928028	0
16/15	2.0743332	3.0590001	5.6849427	5.7150573	0
17/15	2.0755860	3.1910807	5.8627124	5.9372876	0
6/5	2.0758771	3.3241229	6.0404507	6.1595493	0
19/15	2.0750606	3.4582727	6.2180975	6.3819025	0
4/3	2.0730165	3.5936502	6.3955863	6.6044137	0
7/5	2.0696484	3.7303516	6.5728470	6.8271530	0
22/15	2.0648773	3.8684560	6.7498056	7.0501944	0
23/15	2.0586367	4.0080300	6.9263838	7.2736162	0
8/5	2.0508703	4.1491297	7.1025000	7.4975000	0
5/3	2.0415295	4.2918039	7.2780691	7.7219309	0
26/15	2.0305705	4.4360962	7.4530019	7.9469981	0
9/5	2.0178891	4.5821109	7.6271046	8.1728954	0
28/15	2.0038977	4.7294357	7.8008529	8.3991471	0
29/15	1.9878496	4.8788171	7.9733178	8.6266822	0
2	1.9700377	5.0299623	8.1447570	8.8552430	0

we obtain the same results with a different speed of the convergence, as Table 2 shows. In detail, we report the number of iterations (iter), the number of function evaluations (nf), the number of projections (np), and the time of computation (ctime), expressed by seconds, for different methods. The stopping criterion is $\|r(H^k(t_i))\|_5 = \|H^k(t_i) - H^{k-1}(t_i)\|_5 \leqslant 10^{-6}$, for $i = 1, 2, \dots, 30$.

The interpolation of equilibria points yields the curves of equilibria, how we can see in Fig. 2.

Table 2. **Analysis of the convergence of the extragradient method and of the combined relaxation method**

	Extragradient Method			Combined Relaxation Method		
t_i	iter	np/nf	ctime	iter	np/nf	ctime
0	3551	7104/7105	10.9220	258	259/518	0.4690
1/15	3185	6370/6371	9.5940	207	208/416	0.3280
2/15	2913	5826/5827	8.7500	183	184/368	0.2970
1/5	5043	10087/10088	15.9060	178	179/358	0.2810
4/15	4819	9638/9639	15.1720	2195	2197/4393	3.4690
1/3	4691	9382/9383	13.7810	1751	1753/3505	2.7810
2/5	4616	9232/9233	13.5000	2021	2023/4045	3.2190
7/15	4532	9064/9065	13.2350	2685	2687/5373	4.1410
8/15	4378	8756/8757	12.6870	2563	2565/5129	3.9370
3/5	4134	8268/8269	12.2030	2409	2411/4821	3.6880
2/3	3826	7652/7653	11.2190	2224	2226/4451	3.4060
11/15	3498	6996/6997	10.3280	2005	2007/4013	3.1090
4/5	3186	6372/6373	9.2660	1800	1802/3603	2.7970
13/15	2909	5818/5819	8.4840	1635	1637/3273	2.5470
14/15	2673	5346/5347	7.8750	1473	1475/2949	2.2960
1	2475	4950/4951	7.5310	1333	1335/2669	2.0790
16/15	2309	4618/4619	6.8440	1211	1213/2425	1.9060
17/15	2169	4338/4339	6.5780	1102	1104/2207	1.7340
6/5	2048	4096/4097	6.5780	999	1001/2001	1.5630
19/15	1943	3886/3887	5.8750	917	919/1837	1.4370
4/3	1848	3696/3697	5.5160	806	808/1615	1.2810
7/5	1760	3520/3521	5.2340	647	649/1297	1.0310
22/15	1677	3354/3355	5.0000	506	508/1015	0.8130
23/15	1595	3190/3191	4.7660	507	509/1017	0.8120
8/5	1509	3018/3019	4.6250	443	445/889	0.7030
5/3	1410	2820/2821	4.3280	366	368/735	0.5790
26/15	1271	2542/2543	3.7660	267	269/537	0.4220
9/5	1540	3081/3082	4.8280	165	167/333	0.2650
28/15	2132	4264/4265	6.3750	96	98/195	0.1560
29/15	2367	4734/4735	7.0780	80	82/163	0.1410
2	2486	4972/4973	7.4380	3044	3047/6092	4.8280

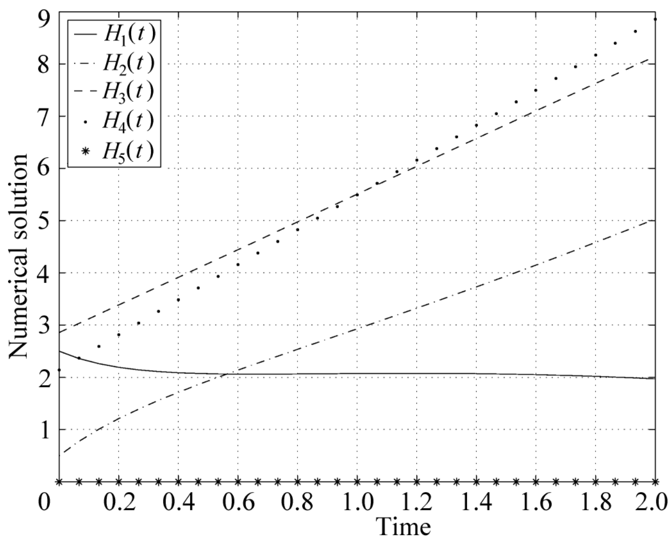

Fig. 2. Network pattern of the numerical example

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