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A FINITE VOLUME METHOD FOR THE TWO-DIMENSIONAL STATIONARY NAVIER — STOKES SYSTEM

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Abstract — In this paper we extend to the stationary incompressible Navier — Stokes system in dimension two the results obtained in [14, 26] for a cell-centered finite volume method applied to the stationary incompressible Stokes system. Here the nonlinear term is discretized as in [15]. We prove that the energy error norm is bounded by h, where h is the mesh size, under the standard assumption that the datum is small enough with respect to the viscosity parameter. Numerical tests on examples with analytic solutions and on standard benchmark problems from fluid mechanics are presented and confirm the theoretical results.

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1. Introduction

The incompressible Navier — Stokes system is one of the main equations studied in fluid mechanics. There is extensive litterature on this system, let us quote [27] for finite difference methods, [18, 29, 30] for finite element methods and [9–11, 13, 15] for finite volume methods. The main goal of all these methods is to allow an accurate discretization and a rapid implementation of the problem. The main difficulty comes from the nonlinear (convective) term. Therefore an adequate discretization should be reliable, easily implementable and especially stable-properties that are generally satisfied by finite volume methods. The second difficulty is the coupling between the velocity and the pressure, which leads to the use of discretization spaces satisfying some compatibility conditions (for instance, for finite difference or finite volume methods, by using overlapping grids [27] or for finite element methods, by checking the so-called inf-sup condition [4, 18]).

The finite volume methods appeared in the sixties and are widely used to approximate many problems of Physics or Mechanics. Convergence analysis of such schemes started in the eighties and is still in progress (see, e.g., [2, 14, 16, 20, 24, 25, 31] and the references cited there). These days this is a topic of huge extension. Among the recent topics of interest we may cite

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• Extension of standard schemes like the centered one [14] to unstructured meshes. In this respect let us quote [11, 28], where the idea is to obtain a compromise between finite volume methods and finite element methods, [3, 7], where the authors discretize the normal flux using the tangential (numerical) derivative, and finally [12], where the discretization of continuous operators is investigated.

• Treatement of more complex and realistic equations like the Stokes or Navier — Stokes systems [10, 15, 26].

Therefore the aim of this paper is to extend to the Navier — Stokes equations the method initially introduced for the Stokes system in [14] and studied in [21, 26]. Here we consider the incompressible Navier — Stokes system in two dimension and use the formulation of the problem in the primitive variables. The main ideas that we develop may be summarized as follows:

• For meshes made of triangles, we approach the velocity in the space of piecewise constant functions and the pressure in the $\mathbb{P}^1$-conforming function space. For meshes made of rectangles, we enlarge a little bit these spaces (see below for details, cf. also [26]).

• The discretization of the momentum equation of conservation is made by using the Green's formula, the nonlinear term being treated by using reconstruted values of the discrete velocity at the nodes of the mesh.

• The incompressibility constraint is considered by duality (to be more precise, by taking as test functions the functions in the $\mathbb{P}^1$-conforming space).

Our error analysis was performed as in [26] and consists in considering our scheme as a nonconforming approximation of the continuous problem. This allows us to use the nonlinear variant of the second Strang lemma [4].

Our paper is organized as follows. In the second section, we introduce the stationary incompressible Navier — Stokes system and recall some notation and definitions used throughout the paper. Section 3 is devoted to the introduction of our numerical scheme. Section 4 concerns the proof of the exitence and uniqueness of the results of the discrete problem. In Section 5 we obtain the error estimate in the energy norm. Section 6 is devoted to the introduction of the extension of our method to unstructured meshes. Finally numerical tests are presented in Section 7, which confirm our predicted order of convergence and underline the advantage of using of our scheme.

Throughout the whole paper the spaces $W^{s,p}(\Omega)$, with $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$, are the standard Sobolev spaces in Ω with norm $\|\cdot\|_{s,p,\Omega}$ and seminorm $|\cdot|_{s,p,\Omega}$ (as usual we drop the index p for $p = 2$, and $H^s(\Omega)$ corresponds to $W^{s,2}(\Omega)$). The space $H_0^1(\Omega)$ is defined, as usual, by $H_0^1(\Omega) := \{v \in H^1(\Omega)/v = 0 \text{ on } \Gamma\}$. In the sequel the symbol $|\cdot|$ will denote either the Euclidean norm in $\mathbb{R}^n$ ($n = 1$ or 2) or the length of a line segment or finally the area of a plane region. The symbol $\mathbf{v}$ means that v is a vector valued function, i.e., $\mathbf{v} = (v_1, v_2)^\top$, v_i being its i^{th} component. The notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b (and of the meshsize of the triangulation) such that $a \leqslant Cb$, while the notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously. Finally, notice that some definitions will be given for the scalar valued functions, their extension to vector valued functions being made componentwise.

2. Problem setting and notation

Let Ω be a polygonal domain of $\mathbb{R}^2$. Over the domain Ω, we consider the stationary incompressible Navier — Stokes problem with Dirichlet boundary conditions: Given the vector function $\mathbf{f} = (f_1, f_2)^\top$, find a solution $(\mathbf{u}, p)$ of

$$\begin{cases} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u} = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\nu > 0$ represents the viscosity of the fluid. The weak velocity-pressure formulation of (2.1) reads as follows: Find $\mathbf{u} \in V = (H_0^1(\Omega))^2$ and $q \in Q = L_0^2(\Omega) = \{r \in L^2(\Omega) : \int_\Omega r = 0\}$ from

$$\begin{cases} \nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v})_\Omega, & \forall \mathbf{v} \in V, \\ b(\mathbf{u}, q) = 0, & \forall q \in Q. \end{cases} \tag{2.2}$$

Here and below $(\cdot,\cdot)_\Omega$ means the inner product in $(L^2(\Omega))^2$ or in $L^2(\Omega)$ according to the context and

$$a(\mathbf{v}, \mathbf{w}) = \int_\Omega \nabla\mathbf{v}(x) : \nabla\mathbf{w}(x)\, dx, \quad \forall \mathbf{v}, \mathbf{w} \in V,$$

$$b(\mathbf{v}, q) = -\int_\Omega q \operatorname{div} \mathbf{v}\, dx, \quad \forall \mathbf{v} \in V, q \in Q, \quad c(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w}\, dx, \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V,$$

when for the vector valued function $\mathbf{v} = (v_1, v_2)^\top$, $\nabla\mathbf{v}$ means the gradient row by row, i.e., the matrix $\begin{pmatrix} \partial_1 v_1 & \partial_2 v_1 \\ \partial_1 v_2 & \partial_2 v_2 \end{pmatrix}$, $\nabla\mathbf{v} : \nabla\mathbf{w} = \sum_{i,j=1,2} \partial_i v_j\, \partial_i w_j$ and $\mathbf{u}\cdot\nabla\mathbf{v}\cdot\mathbf{w} = \sum_{i,j=1}^2 u_i\, D_i u_j\, w_j$.

Theorem 2.1. *Problem* (2.1) *admits at least one solution satisfying*

$$|\mathbf{u}|_{1,\Omega} \lesssim \nu^{-1}||\mathbf{f}||_{0,\Omega}. \tag{2.3}$$

This solution is unique provided that $\nu^{-2}||\mathbf{f}||_{0,\Omega}$ *is sufficiently small.*

Moreover if Ω *is convex, then any solution* $(\mathbf{u}, p)$ *of* (2.1) *belongs to* $(H^2(\Omega))^2 \times H^1(\Omega)$ *with the estimate*

$$\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega} \lesssim \|\mathbf{f}\|_{0,\Omega} + \frac{\|\mathbf{f}\|_{0,\Omega}^2}{\nu^2} + \frac{\|\mathbf{f}\|_{0,\Omega}^4}{\nu^6}. \tag{2.4}$$

Proof. For the proof of the existence and uniqueness of results we refer, e.g., to [18]. For the proof of estimate (2.4) see [23]. □

If the domain Ω is not convex, then the regularity of the solution $(\mathbf{u}, p)$ can be expressed in terms of weighted Sobolev spaces and an estimate similar to (2.4) is available [8, 19, 23].

Let us fix a conforming mesh $\mathcal{T}$ of Ω [5] made of triangles or of rectangles, called control volumes. For $K \in \mathcal{T}$, we set $\mathcal{E}_K$, the set of edges of K, $\mathcal{S}_K$, the set of vertices of K, g_K, the barycenter of K, and b_K the function defined on K by

$$b_K(x_1, x_2) := \begin{pmatrix} x_2 - x_{K2} \\ x_1 - x_{K1} \end{pmatrix},$$

when $x_K = \begin{pmatrix} x_{K1} \\ x_{K2} \end{pmatrix}$ is an arbitrary but fixed point of K ($= g_K$ in the case of a rectangular mesh).

We define the set of edges of the mesh $\mathcal{T}$ as $\mathcal{E} := \bigcup_{K\in\mathcal{T}} \mathcal{E}_K$ and the set of vertices as $\mathcal{S} := \bigcup_{K\in\mathcal{T}} \mathcal{S}_K$. Furthermore we introduce $\mathcal{E}_{\text{int}} := \{\sigma \in \mathcal{E} : \sigma \subset \Omega\}$ the set of interior edges and $\mathcal{E}_{\text{ext}} := \{\sigma \in \mathcal{E} : \sigma \subset \partial\Omega\}$ the set of exterior edges. For $s \in \mathcal{S}$, we introduce the patch $\omega_s := \cup_{K \in \mathcal{T}: s \in \bar{K}} K$. On the other hand, for $K \in \mathcal{T}$ and $s \in \mathcal{S}_K$ we denote by K_s the convex hole of the points s, g_K, m_1 and m_2, where m_1 and m_2 denote the midpoints of the edges of K having s as a vertex (see Fig. 1).

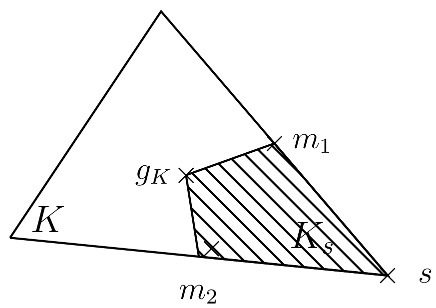


Fig. 1. Subdomain K_s, $s \in \mathcal{S}_K$, $K \in \mathcal{T}$

We further assume that $\mathcal{T}$ is a restricted admissible mesh in the sense of [14, Def. 9.4], i.e., a mesh satisfying the following assumptions (H1) and $(H2)$:

Hypothesis (*H*1): $\exists \xi > 0$ such that $h_K \leqslant \xi d(x_K, \sigma), \forall \sigma \in \mathcal{E}_K, \forall K \in \mathcal{T}$.

Hypothesis (*H*2): For all edges $\sigma \in \mathcal{E}_K$, $K \in \mathcal{T}$:
if $\sigma \in \mathcal{E}_{\text{int}}$ with $\sigma \in \bar{L}$ where $L \in \mathcal{T}$, $L \neq K$, then σ is orthogonal to $[x_K, x_L]$,
if $\sigma \in \mathcal{E}_{\text{ext}}$, then the orthogonal projection x_σ of x_K on the line containing σ belongs to σ.

Note that such a mesh is regular in Ciarlet's sense [5], i.e., it satisfies

$$\exists \xi > 0 : \max_{K \in \mathcal{T}} \{h_K / \rho_K\} \leqslant \xi, \quad \forall K \in \mathcal{T}, \tag{2.5}$$

where we recall that h_K (resp. ρ_K) is the diameter of K (resp. the maximum of the diameter of the balls included in K). For future purposes, let us set $h := \max\limits_{K \in \mathcal{T}} h_K$.

Later on, we also need the

Hypothesis (*H*3): There exists $\xi > 0$ such that $d(x_K, g_K) \leqslant \xi h_K^2$, $\forall K \in \mathcal{T}$.

The hypotheses $(H1)$ – $(H2)$ are relatively standard for the error analysis of finite volume methods (see [14]). Only $(H3)$ is specific for the Stokes system (see [21, 26]).

Note that the hypotheses $(H1)$ – $(H2)$ also imply that

$$\exists \xi \in \mathbb{N} \text{ such that Card } (\omega_s) \leqslant \xi, \quad \forall s \in \mathcal{S},$$

$$|K| \sim |L|, \quad \forall K, L \in \mathcal{T} : K \cap L \neq \varnothing, \tag{2.6}$$

where for short we write Card $(\omega_s) =$ Card $\{K \in \mathcal{T} : K \subset \omega_s\}$.

We now introduce some function spaces used throughout the paper. Let

$$V_h := \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_{h|K} \in V_K, \ \forall K \in \mathcal{T}\}, \quad Q_h := \{q_h \in Q \cap C(\bar{\Omega}) : q_{h|K} \in Q_K, \ \forall K \in \mathcal{T}\},$$

where $V_K = (\mathbb{P}_0(K))^2$, $Q_K = \mathbb{P}_1(K)$ if $\mathcal{T}$ is a triangular mesh, $V_K = (\mathbb{P}_0(K))^2 \oplus$ Span b_K, $Q_K = \mathbb{Q}_1(K)$ if $\mathcal{T}$ is a rectangular mesh.

For $v \in L^2(\Omega)$, we define the Clément interpolant $I_{Cl}(v)$ of v [6] as a unique element of $\{q_h \in C(\bar{\Omega}) : q_{h|K} \in Q_K, \forall K \in \mathcal{T}\}$ with nodal values given by $I_{Cl}(v)(s) := |\omega_s|^{-1} \int_{\omega_s} v \, dx$, $\forall s \in \mathcal{S}$.

Similarly for $\mathbf{v} \in (H^2(\Omega))^2$, we define its centered value interpolant $I_{CC}(\mathbf{v}) \in V_h$ by:

$$\begin{cases} I_{CC}(\mathbf{v})(x_K) := \mathbf{v}(x_K), & \forall K \in \mathcal{T} \text{ for a triangular mesh,} \\ I_{CC}(\mathbf{v})(x_K) := \mathbf{v}(x_K) + \alpha_K b_K, & \forall K \in \mathcal{T} \text{ for a rectangular mesh,} \end{cases}$$

where $\alpha_K := -(\Delta \mathbf{v}, b_K)_K / (\nabla b_K, \nabla b_K)_K, \forall K \in \mathcal{T}$. For $\sigma \in \mathcal{E}_K, K \in \mathcal{T}$, we recall that the approximated flux on σ (directed outside K) of $v \in V_h + (H^2(\Omega))^2$ is defined by

$$F_{K,\sigma}(\mathbf{v}) := \begin{cases} |\sigma|(\mathbf{v}(x_L) - \mathbf{v}(x_K))/d(x_K, x_L), & \text{if } \sigma = \bar{K} \cap \bar{L},\ L \in \mathcal{T}, \\ -|\sigma|\mathbf{v}(x_K)/d(x_K, \partial\Omega), & \text{if } \sigma \subset \partial\Omega. \end{cases}$$

We now introduce the bilinear form a_h and b_h of the discrete version of a and b

$$a_h : V_h + (H^2(\Omega))^2 \times V_h + (H^2(\Omega))^2 \longrightarrow \mathbb{R} \quad (\mathbf{u}, \mathbf{v}) \longmapsto \sum_{K \in \mathcal{T}} \Big\{ -\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}(\mathbf{u}) \cdot \mathbf{v}(x_K) + \int_K \nabla \mathbf{u} : \nabla \mathbf{v}\, dx \Big\},$$

$$b_h : V_h + (H^2(\Omega))^2 \times Q_h \longrightarrow \mathbb{R} \quad (\mathbf{v}, p) \longmapsto \sum_{K \in \mathcal{T}} \int_K \mathbf{v} \cdot \nabla p\, dx.$$

We then define the seminorm $|\cdot|_{1,\mathcal{T}}$ on the space V_h by taking $|\cdot|_{1,\mathcal{T}} := a_h(\cdot,\cdot)^{1/2}$. Note that by Lemma 2.2 of [26] we have the equivalence

$$|\mathbf{v}_h|^2_{1,\mathcal{T}} \sim \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = \bar{K} \cap \bar{L}}} |\mathbf{v}_h(x_K) - \mathbf{v}_h(x_L)|^2 + \sum_{\substack{\sigma \in \mathcal{E}_{\text{ext}} \\ \sigma \subset \bar{K}}} |\mathbf{v}_h(x_K)|^2 + \sum_{K \in \mathcal{T}} \int_K |\nabla \mathbf{v}_h|^2\, dx, \forall \mathbf{v}_h \in V_h. \quad (2.7)$$

3. The numerical scheme

We start with the definition of the reconstructed values at the nodes for elements from V_h.

Definition 3.1. For all $s \in \mathcal{S}$ and all $K \subset \omega_s$, we assume that a weight $A_{K,s} \in \mathbb{R}$ is given satisfying

$$\begin{cases} \exists \xi > 0 : |A_{K,s}| \leqslant \xi, \quad \forall s \in \mathcal{S}_K,\ \forall K \in \mathcal{T}, \\ \sum_{K \subset \omega_s} A_{K,s} = 1, \quad \forall s \in \mathcal{S}. \end{cases}$$

For $v_h \in V_h$, we define its reconstructed value at $s \in \mathcal{S}$ by $v_h(s) := \sum_{K \subset \omega_s} A_{K,s} v_h(x_K)$.

Examples of weights $A_{K,s}$ satisfying the above assumptions are easily built. For instance, we may take $A_{K,s} := |K| / \sum_{K' \subset \omega_s} |K'|$. A more practical choice for the weights are the ones introduced in [7] and used for unstructured meshes (used in Section 6 below)

$$A_{K,s} := (1 + \lambda_1 x_K^{(1)} + \lambda_2 x_K^{(2)}) \Big/ \Big(\operatorname{Card}(\omega_s) + \lambda_1 \sum_{K \subset \omega_s} x_K^{(1)} + \lambda_2 \sum_{K \subset \omega_s} x_K^{(2)} \Big),$$

where

$$\lambda_1 := \Big(\sum_{K \subset \omega_s} x_K^{(1)} x_K^{(2)} \sum_{K \subset \omega_s} x_K^{(2)} - \sum_{K \subset \omega_s} (x_K^{(2)})^2 \sum_{K \subset \omega_s} x_K^{(1)} \Big) \Big/ \Big(\sum_{K \subset \omega_s} (x_K^{(1)})^2 \sum_{K \subset \omega_s} (x_K^{(2)})^2 - \Big(\sum_{K \subset \omega_s} x_K^{(1)} x_K^{(2)} \Big)^2 \Big),$$

$$\lambda_2 := \Big(\sum_{K\subset\omega_s} x_K^{(1)}x_K^{(2)}\sum_{K\subset\omega_s} x_K^{(1)} - \sum_{K\subset\omega_s}(x_K^{(1)})^2\sum_{K\subset\omega_s} x_K^{(2)}\Big)\Big/\Big(\sum_{K\subset\omega_s}(x_K^{(1)})^2\sum_{K\subset\omega_s}(x_K^{(2)})^2 - \Big(\sum_{K\subset\omega_s} x_K^{(1)}x_K^{(2)}\Big)^2\Big).$$

Now we can define the discretization of the trilinear form c

$$c_h : \Big((H^2(\Omega))^2 + V_h\Big)^3 \longrightarrow \mathbb{R} \quad (\mathbf{u},\mathbf{v},\mathbf{w}) \longmapsto \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K} \mathbf{u}(x_K)\cdot\mathbf{n}_{K_s} I_{CC}(\mathbf{v})(s)\cdot\mathbf{w}(x_K)\,ds,$$

where $\mathbf{n}_{K_s}$ denotes the unit outward normal vector along the boundary of K_s.

Inspired from [15,26], the numerical scheme reads as follows: Find $(\mathbf{u}_{CC}, p_{CC}) \in V_h \times Q_h$ satisfying

$$\begin{cases} \nu a_h(\mathbf{u}_{CC},\mathbf{v}_h) + c_h(\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{v}_h) + b_h(\mathbf{v}_h,p_{CC}) = (\mathbf{f},\mathbf{v}_h)_\Omega, & \forall \mathbf{v}_h\in V_h,\\ b_h(\mathbf{u}_{CC},q_h) = 0, & \forall q_h \in Q_h. \end{cases} \tag{3.1}$$

4. Existence and uniqueness of the approximated solution

We first state or prove some technical lemmas.

Lemma 4.1 (Discrete Sobolev inequalities). *Let $\mathcal{T}$ be a mesh of Ω satisfying $(H1)$. Then we have*

$$\|\mathbf{v}_h\|_{0,q,\Omega} \lesssim q|\mathbf{v}_h|_{1,\mathcal{T}}, \quad \forall q\in[1,+\infty[, \quad \forall \mathbf{v}_h \in \mathbb{P}_0(\mathcal{T}),$$

where $\mathbb{P}_0(\mathcal{T}) = \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_{h|K}\in\mathbb{P}_0(K),\ \forall K\in\mathcal{T}\}$.

Proof. We refer to Lemma 1 of [22]. □

Lemma 4.2. *Let $\mathcal{T}$ be a mesh of Ω satisfying $(H1)$ and $(H2)$. Then for all $u \in (C^0(\bar\Omega))^2 + V_h$ we have*

$$\Big(\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}|u(s)-u(x_K)|^2\Big)^{1/2} \lesssim |u|_{1,\mathcal{T}}.$$

Proof. Let $s\in\mathcal{S}_K$ with $K\in\mathcal{T}$. Owing to the second property of the weights $A_{K,s}$ in Definition 3.1, we may write

$$u(s)-u(x_K) = \sum_{L\subset\omega_s} A_{L,s}(u(x_L)-u(x_K)) = \sum_{L\subset\omega_s} A_{L,s}\sum_{i=0}^{m_L-1}(u(x_{i+1})-u(x_i)),$$

by using temporarily an ordering of the elements of ω_s in the trigonometric sense with $x_0 = x_K$ and $x_{m_L} = x_L$. By Cauchy — Schwarz's inequality and (2.6), we have

$$|u(s)-u(x_K)|^2 \lesssim \sum_{L\subset\omega_s}|A_{L,s}|^2\sum_{i=0}^{m_L-1}|u(x_{i+1})-u(x_i)|^2.$$

We obtain the requested estimate of the Lemma by using the first property of the weights in Definition 3.1, by summing on $s\in\mathcal{S}_K$ and $K\in\mathcal{T}$ and using (2.7). □

Lemma 4.3 (Continuity of the discrete trilinear form). *Let $\mathcal{T}$ be a mesh of Ω satisfying the hypothesis $(H1)$. Then $|c_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)| \lesssim \|\mathbf{u}_h\|_{0,4,\Omega}|\mathbf{v}_h|_{1,\mathcal{T}}\|\mathbf{w}_h\|_{0,4,\Omega}$, $\forall \mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h \in V_h$ holds.*

Proof. Let $\mathbf{u}_h, \mathbf{v}_h$ and $\mathbf{w}_h \in V_h$. As we directly have $\sum_{s\in\mathcal{S}_K} \int_{\partial K_s\cap\mathring{K}} \mathbf{u}_h(x_K)\cdot\mathbf{n}_{K_s}\,ds = 0$, $\forall K \in \mathcal{T}$, we may say that

$$|c_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)| = \Big|\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\mathring{K}} \mathbf{u}_h(x_K)\cdot\mathbf{n}_{K_s}\,ds v_h^{(i)}(s) w_h^{(i)}(x_K)\Big| =$$

$$\Big|\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\mathring{K}} \mathbf{u}_h(x_K)\cdot\mathbf{n}_{K_s}\,ds\{v_h^{(i)}(s)-v_h^{(i)}(x_K)\} w_h^{(i)}(x_K)\Big| \overset{\text{Cauchy — Schwarz}}{\leqslant}$$

$$\Big(\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}|v_h^{(i)}(s)-v_h^{(i)}(x_K)|^2\Big)^{1/2}\Big(\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\Big|\int_{\partial K_s\cap\mathring{K}}\mathbf{u}_h(x_K)\cdot\mathbf{n}_{K_s}\,ds w_h^{(i)}(x_K)\Big|^2\Big)^{1/2} \overset{\text{Lemma 4.2}}{\lesssim}$$

$$|\mathbf{v}_h|_{1,\mathcal{T}}\Big(\sum_{i=1}^{2}\sum_{K\in\mathcal{T}}|K||\mathbf{u}_h(x_K)|^2|w_h^{(i)}(x_K)|^2\Big)^{1/2} \overset{\text{Cauchy — Schwarz}}{\lesssim}$$

$$|\mathbf{v}_h|_{1,\mathcal{T}}\Big(\sum_{K\in\mathcal{T}}|K||\mathbf{u}_h(x_K)|^4\Big)^{1/4}\Big(\sum_{K\in\mathcal{T}}|K||\mathbf{w}_h(x_K)|^4\Big)^{1/4}.$$

Now scaling and the finite-dimensionality arguments yield $|K||\mathbf{u}_h(x_K)|^4 \lesssim \|\mathbf{u}_h\|_{0,4,K}^4$, $\forall K \in \mathcal{T}$. This estimate in the previous one leads to the conclusion. □

The previous Lemma as well as Lemma 4.1 directly show the

Corollary 4.1. *Under the assumption of Lemma* 4.3 *we have*

$$|c_h(\mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h)| \lesssim |\mathbf{u}_h|_{1,\mathcal{T}}|\mathbf{v}_h|_{1,\mathcal{T}}|\mathbf{w}_h|_{1,\mathcal{T}}, \quad \forall \mathbf{u}_h,\mathbf{v}_h,\mathbf{w}_h \in V_h.$$

Lemma 4.4 (Inf-sup condition). *Let $\mathcal{T}$ be a mesh of Ω satisfying* (2.5). *Then*

$$\sup_{\substack{\mathbf{v}_h\in V_h\\ \mathbf{v}_h\neq 0}}\{b_h(\mathbf{v}_h,q_h)/|\mathbf{v}_h|_{1,\mathcal{T}}\} \gtrsim \|q_h\|_{0,\Omega}, \quad \forall q_h \in Q_h.$$

Proof. See Lemma 2.3 of [26] in the case of a mesh made of triangles and Lemma 3.2 of [26] in the case of rectangles. □

We are ready to show the main results of this section.

Proposition 4.1. *Let $\mathcal{T}$ be a mesh of Ω satisfying the hypotheses $(H1)$–$(H2)$. Then system* (3.1) *admits a unique solution $(\mathbf{u}_{CC}, p_{CC}) \in V_h \times Q_h$ provided that $\|\mathbf{f}\|_{0,\Omega}/\nu^2$ is small enough. This solution further satisfies the estimate*

$$|\mathbf{u}_{CC}|_{1,\mathcal{T}} \lesssim \|\mathbf{f}\|_{0,\Omega}/\nu. \tag{4.1}$$

Proof. Existence of a solution to (3.1). Let $\hat{V}_h := \{\mathbf{v}_h \in V_h : (\mathbf{v}_h, \nabla q_h)_\Omega = 0, \forall q_h \in Q_h\}$. Consider the mapping $\Lambda : \hat{V}_h \longrightarrow \hat{V}_h : \mathbf{u}_h \longmapsto \Lambda\mathbf{u}_h$ where $\Lambda\mathbf{u}_h$ is the unique element from $\hat{V}_h$ satisfying

$$\nu a_h(\Lambda\mathbf{u}_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h)_\Omega - c_h(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \hat{V}_h. \tag{4.2}$$

The existence and uniqueness of $\Lambda\mathbf{u}_h$ satisfying (4.2) is immediate since a_h is continuous and coercive on $\hat{V}_h$ (with the norm $|\cdot|_{1,\mathcal{T}}$) and the right-hand side of (4.2) defines the continuous linear form on $\hat{V}_h$ (with the same norm $|\cdot|_{1,\mathcal{T}}$). Moreover the mapping Λ is continuous owing to Corollary 4.1. Our goal is to show that there exist $r > 0$ such that

$\Lambda(B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r)) \subset B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r)$. By Brouwer fixed point Theorem (see [32, Proposition 2.6]) problem (3.1) will have at least one solution.

Taking $\mathbf{v}_h = \Lambda\mathbf{u}_h \in \hat{V}_h$ in (4.2) and using Corollary 4.1, we obtain a constant $C > 0$ such that

$$\nu|\Lambda\mathbf{u}_h|_{1,\mathcal{T}} \leqslant C(\|\mathbf{f}\|_{0,\Omega} + |\mathbf{u}_h|^2_{1,\mathcal{T}}). \tag{4.3}$$

Now we define $r := r(\|\mathbf{f}\|_{0,\Omega}, \nu, C)$ as the smallest positive root of the quadratic equation

$$X^2 - \frac{\nu}{C}X + \|\mathbf{f}\|_{0,\Omega} = 0. \tag{4.4}$$

This root r is well defined provided $\|\mathbf{f}_h\|_{0,\Omega}/\nu^2$ is small enough. A simple calculation using (4.3) and the definition of r furnish

$$\mathbf{u}_h \in B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r) \Rightarrow \Lambda\mathbf{u}_h \in B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r).$$

Then we can apply Brouwer fixed point Theorem and deduce the existence of a fixed point for Λ in $B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r)$. This fixed point $\mathbf{u}_{CC} \in \hat{V}_h$ satisfies

$$\begin{cases} \nu a_h(\mathbf{u}_{CC}, \mathbf{v}_h) + c_h(\mathbf{u}_{CC}, \mathbf{u}_{CC}, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega, \quad \forall \mathbf{v}_h \in \hat{V}_h, \\ |\mathbf{u}_{CC}|_{1,\mathcal{T}} \leqslant r(\|\mathbf{f}\|_{0,\Omega}, \nu, C) := (2C)^{-1}\left(\nu - \sqrt{\nu^2 - 4C^2\|\mathbf{f}\|_{0,\Omega}}\right), \end{cases} \tag{4.5}$$

where $C > 0$ is the constant from (4.3). Once there exists a solution $\mathbf{u}_{CC} \in \hat{V}_h$ of (4.5), the existence of $p_{CC} \in Q_h$ such that $(\mathbf{u}_{CC}, p_{CC}) \in V_h \times Q_h$ is the solution of (3.1) follows from [18, Theorem IV.1.4] and Lemma 4.4.

Proof of the uniqueness of the solution of (3.1). It suffices to show the uniqueness of the solution $\mathbf{u}_{CC} \in \hat{V}_h$ of (4.5). For that purpose we use a contradiction argument. Assume that there exist two different solutions $\mathbf{u}^1_{CC}$ and $\mathbf{u}^2_{CC}$ in $B_{\hat{V}_h,|\cdot|_{1,\mathcal{T}}}(0,r)$ of (4.5). Then we have

$$\nu|\mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}|^2_{1,\mathcal{T}} = \nu a_h(\mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}) =$$

$$|c_h(\mathbf{u}^2_{CC}, \mathbf{u}^2_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}) - c_h(\mathbf{u}^1_{CC}, \mathbf{u}^1_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC})| =$$

$$|c_h(\mathbf{u}^2_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}) + c_h(\mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}, \mathbf{u}^1_{CC}, \mathbf{u}^2_{CC} - \mathbf{u}^1_{CC})| \overset{\text{Corollary 4.1}}{\leqslant}$$

$$C(|\mathbf{u}^2_{CC}|_{1,\mathcal{T}} + |\mathbf{u}^1_{CC}|_{1,\mathcal{T}})|\mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}|^2_{1,\mathcal{T}}.$$

Since $|\mathbf{u}^2_{CC} - \mathbf{u}^1_{CC}|_{1,\mathcal{T}} > 0$, the above estimate implies that $\nu \leqslant C(|\mathbf{u}^2_{CC}|_{1,\mathcal{T}} + |\mathbf{u}^1_{CC}|_{1,\mathcal{T}})$, and therefore

$$\nu \leqslant 2Cr. \tag{4.6}$$

But r being the smallest positive root of (4.4), it satisfies $r < \nu/2C$ or equivalently $\nu > 2Cr$, which is a contradiction with (4.6).

Proof of estimate (4.1). By the definition of r (see (4.5)) we have

$$|\mathbf{u}_{CC}|_{1,\mathcal{T}} \leqslant \frac{\nu - \sqrt{\nu^2 - 4C^2\|\mathbf{f}\|_{0,\Omega}}}{2C} = \frac{4C^2\|\mathbf{f}\|_{0,\Omega}}{2C(\nu + \sqrt{\nu^2 - 4C^2\|\mathbf{f}\|_{0,\Omega}})} \leqslant \frac{4C^2\|\mathbf{f}\|_{0,\Omega}}{2C\nu} = \frac{2C\|\mathbf{f}\|_{0,\Omega}}{\nu},$$

which proves (4.1). □

5. Error estimates

Let us start with some technicalities.

Lemma 5.1. *Let* $\mathbf{v} \in H^2(\Omega)^2$ *and* $\mathcal{T}$ *be a mesh of* Ω *of size* h *and satisfying* (2.5). *Then we have*

$$\|I_{CC}(\mathbf{v})\|_{0,4,\Omega} \lesssim \|\mathbf{v}\|_{1,\Omega} + h|\mathbf{v}|_{2,\Omega}.$$

Proof. The triangular inequality yields $\|I_{CC}(\mathbf{v})\|_{0,4,\Omega} \lesssim \|\mathbf{v}\|_{0,4,\Omega}+\|\mathbf{v}-I_{CC}(\mathbf{v})\|_{0,4,\Omega}$. Using the embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we deduce that

$$\|I_{CC}(\mathbf{v})\|_{0,4,\Omega} \lesssim \|\mathbf{v}\|_{1,\Omega} + \|\mathbf{v} - I_{CC}(\mathbf{v})\|_{0,4,\Omega}. \tag{5.1}$$

To estimate the second term of this right-hand side, we start with the scaling argument ($\hat{K}$ being the reference triangle) to get $\|\mathbf{v} - \mathbf{v}(x_K)\|_{0,4,K} \lesssim |K|^{1/4}\|\hat{\mathbf{v}} - \hat{\mathbf{v}}(\hat{x}_K)\|_{0,4,\hat{K}}$. Now by the Bramble — Hilbert argument (meaningful since $W^{1,p}(\hat{K}) \hookrightarrow C^0(\bar{\hat{K}}), \forall p \in]2, +\infty[$) and hypothesis (2.5) we obtain

$$\|\mathbf{v} - \mathbf{v}(x_K)\|_{0,4,K} \lesssim |K|^{1/4}|\mathbf{v}|_{1,4,\hat{K}} \lesssim h_K|\mathbf{v}|_{1,4,K}. \tag{5.2}$$

For the triangular mesh this estimate yields

$$\|\mathbf{v} - I_{CC}(\mathbf{v})\|_{0,4,\Omega} \lesssim h|\mathbf{v}|_{1,4,\Omega} \lesssim h\|\mathbf{v}\|_{2,\Omega},$$

this last inequality following from the embedding $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. Inserting this estimate into (5.1), we arrive at the requested estimate.

For the mesh made of rectangles, we need to estimate the term $\|\alpha_K b_K\|_{0,4,K}$. But in using the definition of α_K and b_K, direct calculations yield $\|\alpha_K b_K\|_{0,4,K} \lesssim h_K^{3/2}\|\Delta \mathbf{v}\|_{0,K}$. This estimate and (5.2) lead to

$$\|\mathbf{v} - I_{CC}\mathbf{v}\|_{0,4,K} \lesssim h_K(|\mathbf{v}|_{1,4,K} + \|\mathbf{v}\|_{2,K}).$$

And we conclude as before. □

Lemma 5.2. *Let* $\mathbf{w} \in (H^2(\Omega))^2$ *and* $\mathcal{T}$ *be a mesh of* Ω *of mesh size* h *satisfying the assumptions* $(H1)$ *and* $(H2)$. *Then it holds that*

$$|-(\Delta \mathbf{w}, \mathbf{v}_h)_\Omega - a_h(I_{CC}(\mathbf{w}), \mathbf{v}_h)| \lesssim h\|\mathbf{w}\|_{2,\Omega}|\mathbf{v}_h|_{1,\mathcal{T}}, \quad \forall \mathbf{v}_h \in V_h.$$

Proof. Fix $\mathbf{v}_h \in V_h$. In the case of the mesh made of triangles, using Green's formula, we arrive at

$$-(\Delta \mathbf{w}, \mathbf{v}_h)_\Omega - a_h(I_{CC}(\mathbf{w}), \mathbf{v}_h) = \sum_{\substack{\sigma\in\mathcal{E}_{\text{int}}\\ \sigma=\bar{K}\cap\bar{L}}} \left(-\int_\sigma \mathbf{n}_{K,\sigma}\cdot\nabla \mathbf{w}\, ds + F_{K,\sigma}(I_{CC}(\mathbf{w}))\right)\cdot(\mathbf{v}_h(x_K) - \mathbf{v}_h(x_L)) -$$

$$\sum_{\substack{\sigma\in\mathcal{E}_{\text{ext}}\\ \sigma\subset\bar{K}}} \left(-\int_\sigma \mathbf{n}_{K,\sigma}\cdot\nabla \mathbf{w}\, ds + F_{K,\sigma}(I_{CC}(\mathbf{w}))\right)\cdot \mathbf{v}_h(x_K).$$

By the Cauchy — Schwarz inequality we obtain

$$|-(\Delta \mathbf{w}, \mathbf{v}_h)_\Omega - a_h(I_{CC}(\mathbf{w}), \mathbf{v}_h)| \lesssim \left(\sum_{\sigma\in\mathcal{E}} |-\int_\sigma \mathbf{n}_{K,\sigma}\cdot\nabla \mathbf{w} ds + F_{K,\sigma}(I_{CC}(\mathbf{w}))|^2\right)^{1/2} |\mathbf{v}_h|_{1,\mathcal{T}}.$$

The term $-\int_\sigma \mathbf{n}_{K,\sigma}\cdot\nabla\mathbf{w}\,ds+F_{K,\sigma}(I_{CC}(\mathbf{w}))$ is the standard consistency error encountered in centered finite volume methods. The estimate

$$\left|-\int\limits_\sigma \mathbf{n}_{K,\sigma}\cdot\nabla\mathbf{w}\,ds+F_{K,\sigma}(I_{CC}(\mathbf{w}))\right|\lesssim h\|\mathbf{w}\|_{2,K},$$

may be found in [14].

For the mesh made of rectangles, if we denote by $\mathbf{v}_{h,K}$, resp. $\mathbf{w}_{h,K}$, the component of $\mathbf{v}_h$, resp. $\mathbf{w}_h$, with respect to b_K, $K\in\mathcal{T}$, as compared to the previous case, we have the additional term

$$\sum_{K\in\mathcal{T}}\mathbf{v}_{h,K}\int\limits_K(-\Delta\mathbf{w}b_K-\mathbf{w}_{h,K}\nabla b_K\cdot\nabla b_K)\,dx.$$

But by the definition of $I_{CC}(\mathbf{w})$ we have $\int_K(-\Delta\mathbf{w}b_K-\mathbf{w}_{h,K}\nabla b_K\cdot\nabla b_K)\,dx=0$ and, consequently, this additional term is zero. □

Lemma 5.3. *Let $p\in H^1(\Omega)$ and $\mathcal{T}$ be a mesh of Ω of mesh size h satisfying (2.5). Then*

$$|(\nabla(p-I_{Cl}(p)),\mathbf{v}_h)_\Omega|\lesssim h|p|_{1,\Omega}|\mathbf{v}_h|_{1,\mathcal{T}},\quad\forall\mathbf{v}_h\in V_h.$$

Proof. Fix $\mathbf{v}_h\in V_h$. We apply the Green's formula to get

$$|(\nabla(p-I_{Cl}(p)),\mathbf{v}_h)_\Omega|=\Bigg|\sum_{\substack{\sigma\in\mathcal{E}_{\rm int}\\ \sigma=\bar K\cap\bar L}}\int\limits_\sigma(p-I_{Cl}(p))\,ds(\mathbf{v}_h(x_K)-\mathbf{v}_h(x_L))\cdot\mathbf{n}_{K,\sigma}+$$

$$\sum_{\substack{\sigma\in\mathcal{E}_{\rm ext}\\ \sigma\subset\bar K}}\int\limits_\sigma(p-I_{Cl}(p))\,ds\mathbf{v}_h(x_K)\cdot\mathbf{n}_{K,\sigma}\Bigg|\lesssim h^{1/2}\left(\sum_{\sigma\in\mathcal{E}}\|p-I_{Cl}(p)\|^2_{0,\sigma}\right)^{1/2}|\mathbf{v}_h|_{1,\mathcal{T}}\lesssim h|p|_{1,\Omega}|\mathbf{v}_h|_{1,\mathcal{T}},$$

owing to the well-known estimate $\|p-I_{Cl}p\|_{0,\sigma}\lesssim h^{1/2}|p|_{1,K}$, for all $\sigma\subset K$ (see [6, Theorem 1]). □

Theorem 5.1. *Assume that $\mathbf{f}\in(L^2(\Omega))^2$ and that Ω is convex and consider $\mathcal{T}$ as a mesh of Ω of size $h>0$ and satisfying the hypotheses $(H1)$–$(H3)$. If $\|\mathbf{f}\|_{0,\Omega}/\nu^2$ is small enough, then problems* (2.1) *and* (3.1) *have a unique solution $(\mathbf{u},p)\in(H^2(\Omega))^2\times H^1(\Omega)$ and $(\mathbf{u}_{CC},p_{CC})\in V_h\times Q_h$ respectively. Moreover, if the mesh size h is small enough, we have*

$$|\mathbf{u}-\mathbf{u}_{CC}|_{1,\mathcal{T}}+\|p-p_{CC}\|_{0,\Omega}\lesssim C(\|\mathbf{f}\|_{0,\Omega},\nu)h.$$

Proof. By setting $\mathbf{w}_h:=I_{CC}(\mathbf{u})-\mathbf{u}_{CC}\in V_h$, we may write

$$\nu|\mathbf{u}-\mathbf{u}_{CC}|^2_{1,\mathcal{T}}=\nu a_h(\mathbf{u}-\mathbf{u}_{CC},\mathbf{w}_h)=c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)-c_h(\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{w}_h)+$$

$$(\mathbf{f},\mathbf{w}_h)_\Omega-\nu a_h(\mathbf{u},\mathbf{w}_h)-c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)-b_h(\mathbf{w}_h,p)-b_h(\mathbf{w}_h,p_{CC}-p).\tag{5.3}$$

The remainder of the proof consists in estimating terms of this right-hand side.

a) Estimation of the first term of the right-hand side (5.3). Since c_h is trilinear we have

$$|c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)-c_h(\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{w}_h)|=|c_h(I_{CC}(\mathbf{u}),I_{CC}(\mathbf{u}),\mathbf{w}_h)-c_h(I_{CC}(\mathbf{u}),\mathbf{u}_{CC},\mathbf{w}_h)+$$

$$c_h(I_{CC}(\mathbf{u}),\mathbf{u}_{CC},\mathbf{w}_h)-c_h(\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{w}_h)|\leqslant|c_h(I_{CC}(\mathbf{u}),I_{CC}(\mathbf{u})-\mathbf{u}_{CC},\mathbf{w}_h)|+$$

$$|c_h(I_{CC}(\mathbf{u})-\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{w}_h)| \overset{\text{Lemma 4.3}}{\leqslant} \|I_{CC}(\mathbf{u})\|_{0,4,\Omega}|I_{CC}(\mathbf{u})-\mathbf{u}_{CC}|_{1,\mathcal{T}}\|\mathbf{w}_h\|_{0,4,\Omega}+$$

$$\|I_{CC}(\mathbf{u})-\mathbf{u}_{CC}\|_{0,4,\Omega}|\mathbf{u}_{CC}|_{1,\mathcal{T}}\|\mathbf{w}_h\|_{0,4,\Omega} \overset{\text{Lemma 4.1}}{\lesssim} |I_{CC}(\mathbf{u})-\mathbf{u}_{CC}|_{1,\mathcal{T}}(\|I_{CC}(\mathbf{u})\|_{0,4,\Omega}+$$

$$|\mathbf{u}_{CC}|_{1,\mathcal{T}})|\mathbf{w}_h|_{1,\mathcal{T}} \lesssim |I_{CC}(\mathbf{u})-\mathbf{u}_{CC}|^2_{1,\mathcal{T}}(\|\mathbf{u}\|_{1,\Omega}+h|\mathbf{u}|_{2,\Omega}+\|\mathbf{f}\|_{0,\Omega}/\nu),$$

using Lemma 5.1 and the estimate from Proposition 4.1. Using the estimates from Theorem 2.1 we obtain

$$|c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)-c_h(\mathbf{u}_{CC},\mathbf{u}_{CC},\mathbf{w}_h)| \lesssim (\|\mathbf{f}\|_{0,\Omega}/\nu + C(\|\mathbf{f}\|_{0,\Omega},\nu)h)|I_{CC}(\mathbf{u})-\mathbf{u}_{CC}|^2_{1,\mathcal{T}}. \quad (5.4)$$

b) Estimation of the second term of the right-hand side of (5.3). Using (2.1), we have

$$(\mathbf{f},\mathbf{w}_h)_\Omega - \nu a_h(\mathbf{u},\mathbf{w}_h) - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h) - b_h(\mathbf{w}_h,p) =$$

$$-\nu(\Delta\mathbf{u},\mathbf{w}_h)_\Omega - \nu a_h(\mathbf{u},\mathbf{w}_h) + (\nabla(p-I_{Cl}(p)),\mathbf{w}_h)_\Omega + (\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{w}_h)_\Omega - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h). \quad (5.5)$$

The first two terms of this right-hand side are estimated with the help of Lemmas 5.2 and 5.3. It then remains to estimate the third term. For that purpose we write

$$\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx = \sum_{K\in\mathcal{T}}\int_K(\mathbf{u}-\mathbf{u}(x_K))\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx+$$

$$\sum_{K\in\mathcal{T}}\int_K \mathbf{u}(x_K)\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h(x_K)\,dx + \sum_{K\in\mathcal{T}}\int_K \mathbf{u}(x_K)\cdot\nabla\mathbf{u}\cdot(\mathbf{w}_h-\mathbf{w}_h(x_K))\,dx.$$

In this second term, applying the Green's formula, we obtain

$$\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx = \sum_{K\in\mathcal{T}}\int_K(\mathbf{u}-\mathbf{u}(x_K))\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx+$$

$$\sum_{K\in\mathcal{T}}\int_{\partial K} \mathbf{u}(x_K)\cdot\mathbf{n}_K\mathbf{u}\cdot\mathbf{w}_h(x_K)\,ds + \sum_{K\in\mathcal{T}}\int_K \mathbf{u}(x_K)\cdot\nabla\mathbf{u}\cdot(\mathbf{w}_h-\mathbf{w}_h(x_K))\,dx.$$

Using the definition of c_h, we then arrive at

$$\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h dx - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h) = T_1+T_2+T_{\text{rect}}, \quad (5.6)$$

where for brevity we set

$$T_1 = \sum_{K\in\mathcal{T}}\int_K(\mathbf{u}-\mathbf{u}(x_K))\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx,$$

$$T_2 = \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K}\mathbf{u}(x_K)\cdot\mathbf{n}_{K_s}(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot\mathbf{w}_h(x_K)\,ds,$$

$$T_{\text{rect}} = \sum_{K\in\mathcal{T}}\int_K \mathbf{u}(x_K)\cdot\nabla\mathbf{u}\cdot(\mathbf{w}_h-\mathbf{w}_h(x_K))\,dx.$$

The first term is estimated as follows:

$$|T_1| \overset{\text{Cauchy — Schwarz}}{\leqslant} \sum_{K\in\mathcal{T}} \|\mathbf{u}-\mathbf{u}(x_K)\|_{0,K}|\mathbf{u}|_{1,4,K}\|\mathbf{w}_h\|_{0,4,K} \overset{\substack{\text{Bramble — Hilbert}\\ \text{and } W^{1,4}(\Omega)\hookrightarrow C^0(\bar\Omega)}}{\lesssim}$$

$$h^{3/2}|\mathbf{u}|^2_{1,4,\Omega}\|\mathbf{w}_h\|_{0,4,\Omega} \lesssim h\|\mathbf{u}\|^2_{2,\Omega}|\mathbf{w}_h|_{1,\mathcal{T}},$$

using the embedding $H^2(\Omega)\hookrightarrow W^{1,4}(\Omega)$. Theorem 2.1 thus furnishes

$$|T_1| \lesssim C(\|\mathbf{f}\|_{0,\Omega},\nu)h|\mathbf{w}_h|_{1,\mathcal{T}}. \tag{5.7}$$

The second term T_2 is first rewritten as follows:

$$T_2 = \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K} \mathbf{u}(x_K)\cdot\mathbf{n}_{K_s}(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot\mathbf{w}_h(x_K)\,ds =$$

$$\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K} (\mathbf{u}(x_K)-\mathbf{u})\cdot\mathbf{n}_{K_s}(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot\mathbf{w}_h(x_K)\,ds +$$

$$\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K} \mathbf{u}\cdot\mathbf{n}_{K_s}(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot\mathbf{w}_h(x_K)\,ds.$$

For this second term, as $\mathbf{u}$ is continuous across the edges, we obtain

$$T_2 = T_3+T_4,\quad T_3 = \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\int_{\partial K_s\cap\partial K} (\mathbf{u}(x_K)-\mathbf{u})\cdot\mathbf{n}_{K_s}(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot\mathbf{w}_h(x_K)\,ds,$$

$$T_4 = \sum_{K,L\in\mathcal{T}}\sum_{s\in\mathcal{S}_K\cap\mathcal{S}_L}\int_{\partial K_s\cap\partial K\cap\partial L} \mathbf{u}\cdot\mathbf{n}_K(\mathbf{u}-I_{CC}(\mathbf{u})(s))\cdot(\mathbf{w}_h(x_K)-\mathbf{w}_h(x_L))\,ds. \tag{5.8}$$

For the first term T_3, by the Cauchy — Schwarz inequality we get

$$|T_3| \leqslant \sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}\|\mathbf{u}(x_K)-\mathbf{u}\|_{0,E_s}\|\mathbf{u}-I_{CC}(\mathbf{u})(s)\|_{0,E_s}|\mathbf{w}_h(x_K)|,$$

where for brevity $E_s=\partial K_s\cap\partial K$. By scaling the arguments and the inequality $|\mathbf{w}_h(x_K)|\lesssim h_K^{-1}\|\mathbf{w}_h\|_{0,K}$, this inequality becomes $|T_3|\lesssim h\sum_{K\in\mathcal{T}}\sum_{s\in\mathcal{S}_K}|\mathbf{u}|_{1,4,K}|\mathbf{u}|_{1,4,\omega_s}\|\mathbf{w}_h\|_{0,K}$. The discrete Hölder inequality finally leads to

$$|T_3| \lesssim h\|\mathbf{u}\|^2_{1,4,\Omega}\|\mathbf{w}_h\|_{0,\Omega}. \tag{5.9}$$

For the second term T_4, we first apply the Hölder inequality to get

$$|T_4| \leqslant h^{1/2}\|\mathbf{u}\|_{0,\infty,\Omega}\sum_{K,L\in\mathcal{T}}\sum_{s\in\mathcal{S}_K\cap\mathcal{S}_L}\|\mathbf{u}-I_{CC}(\mathbf{u})(s)\|_{0,K\cap L}|\mathbf{w}_h(x_K)-\mathbf{w}_h(x_L)|.$$

By the scaling argument we obtain

$$|T_4| \lesssim h\|\mathbf{u}\|_{0,\infty,\Omega}\sum_{K,L\in\mathcal{T}}\sum_{s\in\mathcal{S}_K\cap\mathcal{S}_L}|\mathbf{u}|_{1,\omega_s}|\mathbf{w}_h(x_K)-\mathbf{w}_h(x_L)|.$$

By the discrete Cauchy — Scwarz inequality and (2.7), we obtain

$$|T_4| \lesssim h\|\mathbf{u}\|_{0,\infty,\Omega}|\mathbf{u}|_{1,\Omega}|\mathbf{w}_h|_{1,\mathfrak{T}}. \tag{5.10}$$

Estimates (5.9) and (5.10) in identity (5.8) permit writing $|T_2| \lesssim h\|\mathbf{u}\|_{2,\Omega}|\mathbf{u}|_{1,\Omega}|\mathbf{w}_h|_{1,\mathfrak{T}}$, owing to the embeddings $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ and $H^2(\Omega) \hookrightarrow W^{1,4}(\Omega)$. By Theorem 2.1, we obtain

$$|T_2| \lesssim C(\|\mathbf{f}\|_{0,\Omega}, \nu)h|\mathbf{w}_h|_{1,\mathfrak{T}}. \tag{5.11}$$

For the term T_{rect}, by the Cauchy — Schwarz inequality we may write

$$|T_{\text{rect}}| \lesssim \|\mathbf{u}\|_{0,\infty,\Omega} \sum_{K\in\mathfrak{T}} |\mathbf{u}|_{1,K}\|\mathbf{w}_h - \mathbf{w}_h(x_K)\|_{0,K}.$$

For the triangular mesh $\mathbf{w}_h - \mathbf{w}_h(x_K) = 0$ on K and then $T_{\text{rect}} = 0$. For rhe rectangular mesh $\mathbf{w}_h - \mathbf{w}_h(x_K) = \alpha_K b_K$ on K, for some $\alpha_K \in \mathbb{R}$. By direct calculations we have

$$\|\mathbf{w}_h - \mathbf{w}_h(x_K)\|_{0,K} \lesssim |\alpha_K|h_K|K|^{1/2} \lesssim |\alpha_K|h_K|b_K|_{1,K} \lesssim h_K|\mathbf{w}_h|_{1,K}.$$

These two estimates lead to $|T_{\text{rect}}| \lesssim h\|\mathbf{u}\|_{0,\infty,\Omega}|\mathbf{u}|_{1,\Omega}|\mathbf{w}_h|_{1,\mathfrak{T}}$ and by the embedding $H^2(\Omega) \hookrightarrow C(\bar{\Omega})$ and Theorem 2.1, we get

$$|T_{\text{rect}}| \lesssim C(\|\mathbf{f}\|_{0,\Omega}, \nu)h|\mathbf{w}_h|_{1,\mathfrak{T}}. \tag{5.12}$$

Inserting estimates (5.7), (5.11) and (5.12) into identity (5.6), we conclude that

$$\left| \int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{w}_h\,dx - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)\right| \lesssim C(\|\mathbf{f}\|_{0,\Omega}, \nu)h|\mathbf{w}_h|_{1,\mathfrak{T}}. \tag{5.13}$$

Going back to (5.5) and using (5.13) and Lemmas 5.2 and 5.3, we write

$$|(\mathbf{f},\mathbf{w}_h)_\Omega - \nu a_h(\mathbf{u},\mathbf{w}_h) - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h) - b_h(\mathbf{w}_h,p)| \lesssim C(\|\mathbf{f}\|_{0,\Omega}, \nu)h|\mathbf{w}_h|_{1,\mathfrak{T}}. \tag{5.14}$$

c) Estimation of the third term of the right-hand side (5.3). By the definition of b_h, we have $b_h(\mathbf{w}_h, p_{CC} - p) = (\nabla(p - p_{CC}), \mathbf{w}_h)_\Omega$. This right-hand side is estimated as follows: Using (2.1) and (3.1), we may write

$$|(\nabla(p - p_{CC}), \mathbf{w}_h)_\Omega| \leqslant \nu| - (\Delta\mathbf{u}, \mathbf{w}_h)_\Omega - a_h(\mathbf{u}_{CC}, \mathbf{w}_h)|+$$

$$|c_h(\mathbf{u}_{CC}, \mathbf{u}_{CC}, \mathbf{w}_h) - c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h)| + |c_h(\mathbf{u},\mathbf{u},\mathbf{w}_h) - (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{w}_h)_\Omega|, \quad \forall \mathbf{w}_h \in V_h.$$

The first term of this right-hand side is estimated with the help of Lemma 5.2. The second term is estimated using arguments similar to those used to obtain (5.4). The last term is estimated by (5.13). Consequently, we have

$$\sup_{\substack{\mathbf{w}_h\in V_h\\ \mathbf{w}_h\neq 0}} \frac{|(\nabla(p - p_{CC}), \mathbf{w}_h)_\Omega|}{|\mathbf{w}_h|_{1,\mathfrak{T}}} \lesssim C(\|\mathbf{f}\|_{0,\Omega}, \nu)h. \tag{5.15}$$

Estimates (5.4), (5.14) and (5.15) in (5.3) lead to

$$\left(\nu - \frac{\|\mathbf{f}\|_{0,\Omega}}{\nu} + C(\|\mathbf{f}\|_{0,\Omega}, \nu)h\right)|\mathbf{u} - \mathbf{u}_{CC}|_{1,\mathfrak{T}} \lesssim C(\mathbf{f}\|_{0,\Omega}, \nu)h,$$

which allows us to conclude that the estimate of Theorem 5.1 for $|\mathbf{u}-\mathbf{u}_{CC}|_{1,\mathcal{T}}$ provided $\|\mathbf{f}\|_{0,\Omega}/\nu$ and $C(\|\mathbf{f}\|_{0,\Omega},\nu)h$ are small enough.

We may now pass to the estimate on the pressure error norm. First Lemma 4.4 allows us to write

$$\|I_{Cl}(p)-p_{CC}\|_{0,\Omega} \lesssim \sup_{\substack{\mathbf{v}_h\in V_h\\ \mathbf{v}_h\neq 0}} \frac{|(\nabla(I_{Cl}(p)-p),\mathbf{v}_h)_\Omega|}{|\mathbf{v}_h|_{1,\mathcal{T}}} + \sup_{\substack{\mathbf{v}_h\in V_h\\ \mathbf{v}_h\neq 0}} \frac{|(\nabla(p-p_{CC}),\mathbf{v}_h)_\Omega|}{|\mathbf{v}_h|_{1,\mathcal{T}}}.$$

Using Lemma 5.3, Theorem 2.1, and estimate (5.15), we then get

$$\|I_{Cl}(p)-p_{CC}\|_{0,\Omega} \lesssim C(\|\mathbf{f}\|_{0,\Omega},\nu)h.$$

This estimate yields the requested estimate for $\|p-p_{CC}\|_{0,\Omega}$, using the well-known estimate $\|p-I_{Cl}(p)\|_{0,\Omega}\lesssim h|p|_{1,\Omega}$ [6]. The proof is then complete. □

Remark 5.1. Using the results from Theorem 5.1, we obtain the convergence error $\|\mathbf{u}-\mathbf{u}_{CC}\|_{0,q,\Omega}\lesssim h^{2/q}$, $\forall q\in[2,\infty[$. The key point is to remark that $\|\mathbf{u}-I_{CC}(\mathbf{u})\|_{0,q,\Omega}\lesssim h^{2/q}$, $\forall q\in[2,\infty[$, and then use Lemma 4.1.

6. Unstructured meshes

The main restriction of the previous scheme described in Section 3 is the lack of its flexibility. Indeed, the cohabitation of the hypotheses $(H2)$ and $(H3)$ is a strong restriction and forbids the use of unstructured meshes (essential, e.g., when local refinements of the mesh are necessary). The numerical tests from [26] have demonstrated that the hypothesis $(H3)$ is necessary. Therefore, one solution is to remove the assumption $(H2)$, but to keep the key property of the preservation of flux, we need to modify the scheme. For that purpose we use the scheme proposed in [3, 7] and then we change the definition of the bilinear form $a_h(\cdot,\cdot)$ by simply modifying the definition of the numerical flux $F_{K,\sigma}(v)$. For the sake of completeness, we recall this definition.

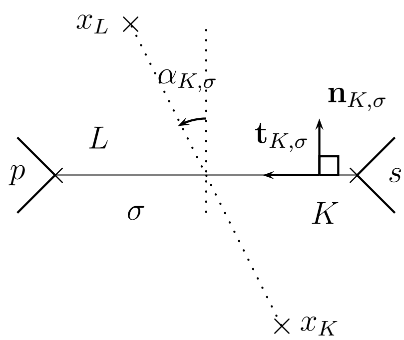


Fig. 2. Interface σ between two control volumes K and L

We consider the mesh $\mathcal{T}$ made of triangles. Let $v\in H^2(\Omega)$ and $\sigma\in\mathcal{E}_{\text{int}}$ with $\sigma=\bar K\cap\bar L$ where $K,L\in\mathcal{T}$ (see Fig. 2 for some explanations).

Note that the segment (x_K,x_L) is no longer orthogonal to σ, in other words, the condition $(H2)$ is no longer satisfied. Let $\alpha_{K,\sigma}$ be the angle between $\mathbf{n}_{K,\sigma}$ and $\overrightarrow{x_Kx_L}$. Furthermore the vector $\overrightarrow{sp}$ is oriented so that $\overrightarrow{sp}\cdot\mathbf{t}_{K,\sigma}>0$.

The key idea of the method is based on the identity

$$\nabla v\cdot\mathbf{n}_{K,\sigma} = \nabla v\cdot\frac{\overrightarrow{x_Kx_L}}{\|\overrightarrow{x_Kx_L}\|\cos(\alpha_{K,\sigma})} - \tan(\alpha_{K,\sigma})\nabla v\cdot\mathbf{t}_{K,\sigma}, \tag{6.1}$$

since $\overrightarrow{x_Kx_L}/\|\overrightarrow{x_Kx_L}\| = \cos(\alpha_{K,\sigma})\mathbf{n}_{K,\sigma}+\sin(\alpha_{K,\sigma})\mathbf{t}_{K,\sigma}$.

Using finite differences in the right-hand side of (6.1), we obtain

$$\int_\sigma \nabla v\cdot\mathbf{n}_{K,\sigma}ds \approx -|\sigma|F_{K,\sigma}(v), \tag{6.2}$$

with

$$F_{K,\sigma}(v) := \frac{v(x_L) - v(x_K)}{d(x_K, x_L)\cos(\alpha_{K,\sigma})} - \tan(\alpha_{K,\sigma})(v(p) - v(s)). \tag{6.3}$$

To give approximated values for $\{v(s)\}_{s\in\mathcal{S}}$, we use the above-mentioned weights introduced in [7]. Notice that for $s \in \mathcal{S}$, these weights are chosen in order to guarantee the exact reconstruction of the gradient of the affine function defined on the patch ω_s.

Remark 6.1. For $\sigma \subset \bar{K} \cap \partial\Omega$, we may simply take $F_{K,\sigma}(v) := -v(x_K)/d(x_K, \partial\Omega)$, as explained in [7, 14].

Remark 6.2. Note that the estimate of the consistency error similar to the results from Lemma 5.2, was proved in [7], but for functions in $W^{2,p}(\Omega)$, $\forall p > 2$. Therefore, we may expect a good convergence of the unstructured scheme if the velocity field has such a regularity.

7. Numerical tests

7.1. Analytical solution. Here we take the unit square $\Omega :=]0,1[^2$ and consider problem (2.1) with $\nu := 10^{-1}$ and the exact solution

$$\mathbf{u} := \mathbf{rot}(xy(1-x)(1-y)) \quad \text{and} \quad p := x^2 - y^2,$$

the datum $\mathbf{f}$ is fixed accordingly.

For this example we have chosen two sequences of meshes. First we take a sequence of uniform meshes made of rectangles, and implement the scheme from Section 3. Second, we take a sequence of unstructured meshes made of triangles and test the scheme evoked in Section 6. In both cases for $K \in \mathcal{T}$, the point x_K is chosen as the barycenter of K. Tables 1 and 2 furnish the obtained errors as a function of N, the number of elements of the considered mesh ($h \sim N^{-1/2}$). Figure 3 presents different orders of convergence. From these tables and Fig. 3, we see that both schemes give similar results and an order of convergence of 1 in the energy norm (in accordance with the results from Theorem 5.1) and an order of convergence of 2 for the L^2-norm of the error in the velocity, which is better than expected and is usual for such a method.

Table 1. **Errors wrt N for meshes of rectangles**

N	$\Vert I_{CC}\mathbf{u} - \mathbf{u}_{CC}\Vert_{0,\Omega}$	$\vert I_{CC}\mathbf{u} - \mathbf{u}_{CC}\vert_{1,\mathcal{T}}$	$\Vert\mathbf{u} - \mathbf{u}_{CC}\Vert_{0,\Omega}$	$\Vert p - p_{CC}\Vert_{0,\Omega}$
16	$4.70e-03$	$3.01e-02$	$4.96e-02$	$8.71e-03$
64	$1.25e-03$	$1.24e-02$	$2.51e-02$	$3.16e-03$
256	$3.31e-04$	$4.60e-03$	$1.26e-02$	$1.12e-03$
1024	$8.55e-05$	$1.64e-03$	$6.31e-03$	$3.93e-04$
4096	$2.17e-05$	$5.78e-04$	$3.15e-03$	$1.37e-04$

Table 2. **Errors wrt N for unstructured meshes of triangles**

N	$\Vert I_{CC}\mathbf{u} - \mathbf{u}_{CC}\Vert_{0,\Omega}$	$\vert I_{CC}\mathbf{u} - \mathbf{u}_{CC}\vert_{1,\mathcal{T}}$	$\Vert\mathbf{u} - \mathbf{u}_{CC}\Vert_{0,\Omega}$	$\Vert p - p_{CC}\Vert_{0,\Omega}$
44	$5.78e-03$	$6.64e-02$	$2.55e-02$	$6.68e-03$
166	$7.95e-04$	$1.73e-02$	$1.23e-02$	$1.55e-03$
612	$1.72e-04$	$6.45e-03$	$6.60e-03$	$5.21e-04$
2430	$4.03e-05$	$2.65e-03$	$3.30e-03$	$2.00e-04$
9754	$1.04e-05$	$1.36e-03$	$1.66e-03$	$8.73e-05$

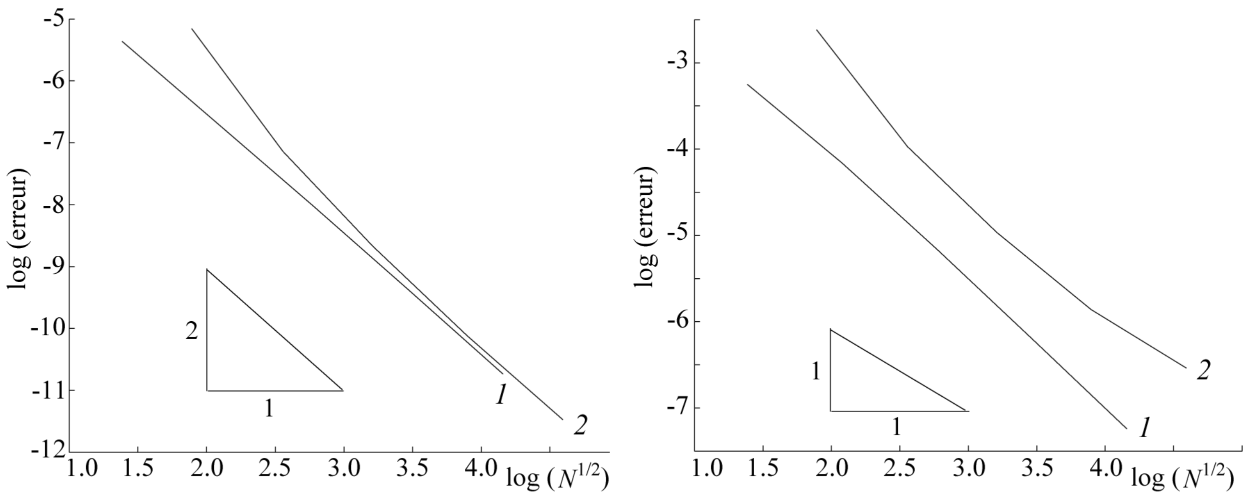


Fig. 3. Rates of convergence wrt $N^{1/2}$ for meshes of rectangles (line *1*) and triangles (line *2*) for $\|I_{CC}\mathbf{u}-\mathbf{u}_{CC}\|_{0,\Omega}$ (left) and for $|I_{CC}\mathbf{u}-\mathbf{u}_{CC}|_{1,\mathcal{T}}+\|p-p_{CC}\|_{0,\Omega}$ (right)

7.2. The lid-driven cavity. As a second test, we consider a benchmark example in fluid mechanics, namely the lid-driven cavity [13, 15, 17, 29, 30].

We again take $\Omega :=]0,1[^2$ and consider problem (2.1) with $\nu = 1/100, 1/400$ and $1/1000$, but with nonhomogeneous boundary conditions $\mathbf{u} = \mathbf{g}$, where $\mathbf{g}$ is defined by $\mathbf{g} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on $]0,1[\times\{1\}$, and $\mathbf{g} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ elsewhere.

We have implemented the scheme from Section 3 on a mesh made of 2500 rectangles refined near the top corners of Ω in order to take into account the discontinuities of the solution and to capture adequately the secondary vortices of this solution. Figure 4 illustrates the velocity field and the isolines of pressure obtained by this scheme for $\nu = 1/400$. The results appear to be approximatively identical and in accordance with the existing results from the litterature [13, 15, 17, 30]. Indeed we compare our results with those from [17]. Figure 5 represents the value of the first (left) component of the velocity field along the segment $\{0.5\}\times[0,1]$ and the second (right) component of the velocity field along the segment $[0,1]\times\{0.5\}$ for the three values of ν. Tables 3 to 5 give the obtained localization of the centers of the three vortices. From these Figures and tables, we see that our results are very close to the one from [17]. Let us further notice that an upwind scheme similar to the one from [15] has been made and furnishes quite similar results.

Table 3. **Localization of the centers of the three vortices for** $\nu = 1/100$

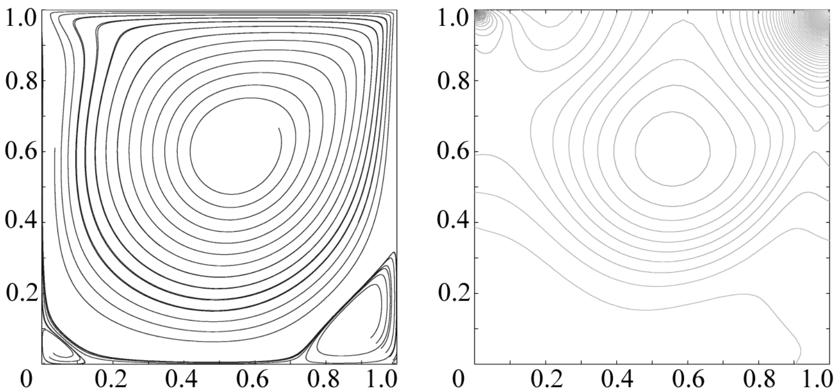
	Central vortex	Left bottom vortex	Right bottom vortex
(x,y)	$(0.62, 0.74)$	$(0.025, 0.045)$	$(0.935, 0.055)$
(x,y) [17]	$(0.6172, 0.7344)$	$(0.0313, 0.0391)$	$(0.9453, 0.0625)$

Table 4. **Localization of the centers of the three vortices for** $\nu = 1/400$

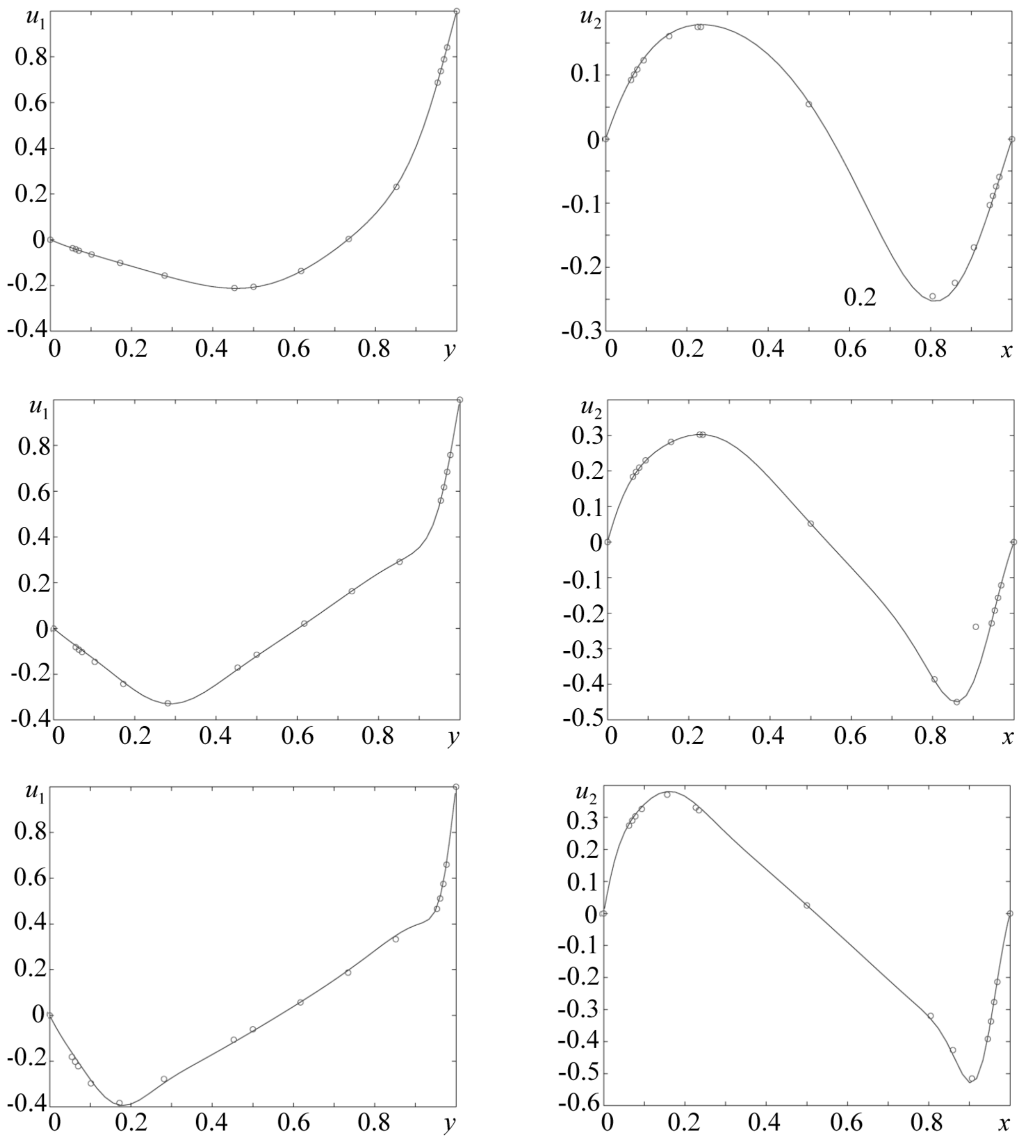
	Central vortex	Left bottom vortex	Right bottom vortex
(x,y)	$(0.56, 0.61)$	$(0.05, 0.05)$	$(0.88, 0.125)$
(x,y) [17]	$(0.5547, 0.6055)$	$(0.0508, 0.0469)$	$(0.8906, 0.1250)$

Table 5. **Localization of the centers of the three vortices for** $\nu = 1/1000$

	Central vortex	Left bottom vortex	Right bottom vortex
(x,y)	$(0.53, 0.57)$	$(0.08, 0.08)$	$(0.86, 0.12)$
(x,y) [17]	$(0.5313, 0.5625)$	$(0.0859, 0.0781)$	$(0.8594, 0.1094)$



F i g. 4. Velocity field (left) and isolines of pressure (right) for the discrete solution for $\nu = 1/400$



F i g. 5. First component of the velocity field along the segment $x = 0.5$ (left) and second component of the velocity field along the segment $y = 0.5$ (right) for our scheme (full line) and the scheme from [17] (circle) for $\nu = 1/100$ (top), $\nu = 1/400$ (middle) and $\nu = 1/1000$ (bottom)

7.3. The backward facing step. Here we consider another benchmark example in fluid mechanics: the backward facing step. This test is much harder than the previous one. We start with the description of the problem. We take the domain $\Omega :=]0,2[\times]0.5,1.5[\cup]2,28[\times]0,1.5[$ and the viscosity parameter $\nu := 1/100$. We still consider problem (2.1), but with the following boundary conditions:

$$\begin{cases} \mathbf{u} = \begin{pmatrix} -6(y-0.5)(y-1.5) \\ 0 \end{pmatrix} & \text{on }]0.5, 1.5[\times\{0\}, \\ (\nabla \mathbf{u} - pI)\cdot \mathbf{n} = \mathbf{0} & \text{on }]0, 1.5[\times\{28\}, \\ \mathbf{u} = \mathbf{0} & \text{elsewhere on } \partial\Omega, \end{cases}$$

where I denotes the identity matrix. Compared to formulation (2.1), we here consider the "Neumann" boundary condition at the exit from the domain, which physically means that the fluid is stable at the exit from the domain.

We here use a mesh made of 5800 rectangles refined near the backward facing step. Note that the nonlinear term is discretized via a compromise between the centered approach from this paper and the upwind one developed in [14]. Figures 6 (top) and 7 show, respectively, the velocity field throughout the domain Ω and near the backward facing step. The isolines of pressure are furthermore presented in Fig. 6 (bottom). The results are in agreement with the usual ones [1].

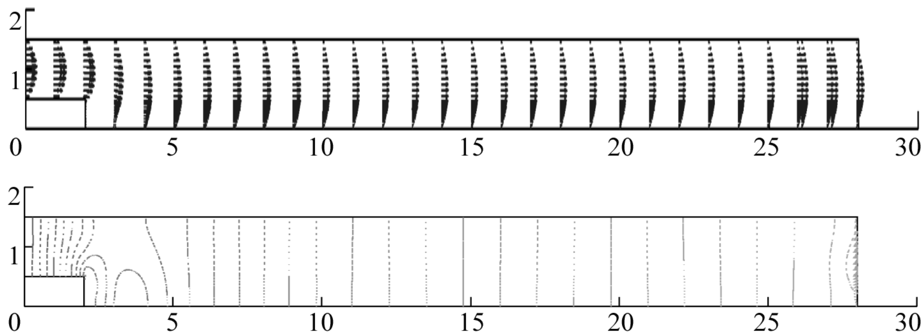


Fig. 6. Velocity field (top) and isolines of pressure (bottom) for the discrete solution

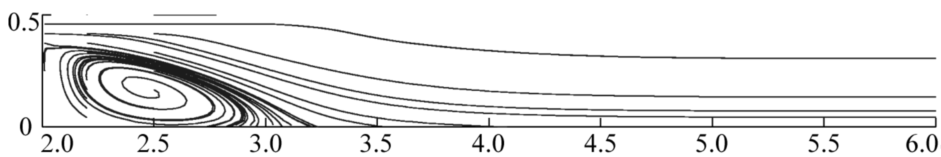


Fig. 7. Velocity field near the backward facing step

8. Conclusions

In this paper, we have introduced a new finite volume method for the approximation of the incompressible stationary Navier — Stokes system in dimension two. This method is a generalization of the methods developed in [14, 21, 26] for the Stokes system. We prove the existence and uniqueness of the results as well as the error estimate in the energy norm, under the standard assumption that the datum has to be sufficiently small compared to the viscosity parameter. A generalization of the scheme to unstructured meshes is also proposed. Some numerical tests illustrate the obtained theoretical error estimate and underline usefulness of the introduced scheme and its generalization. Nevertheless for the backward facing step problem with small viscosity parameters, the use of more adapted techniques seems to be necessary. The use of preconditioner or stabilization techniques should be probably necessary to improve the efficiency of our scheme. Such approaches as well as the extension of the scheme to three dimensional problems will be investigated in forthcoming works.

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