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SUPRACONVERGENCE OF A FINITE DIFFERENCE SCHEME FOR ELLIPTIC BOUNDARY VALUE PROBLEMS OF THE THIRD KIND IN FRACTIONAL ORDER SOBOLEV SPACES

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Abstract — In this paper, we study the convergence of the finite difference discretization of a second order elliptic equation with variable coefficients subject to general boundary conditions. We prove that the scheme exhibits the phenomenon of supraconvergence on nonuniform grids, i.e., although the truncation error is in general of the first order alone, one has second order convergence. More precisely, for $s \in (1/2, 2]$ the optimal order $O(h^s)$-convergence of the finite difference solution and its gradient appears if the exact solution is in the Sobolev — Slobodetskij space $H^{1+s}(\Omega)$. All error estimates are strictly local.

Another result of the paper is a close relationship between finite difference scheme and linear finite element methods combined with a special kind of quadrature. As a consequence, the results of the paper can be viewed as the introduction of a fully discrete finite element method for which the gradient is superclose, i.e., the error of the approximate gradient with respect to the linear interpolation of the solution u is of the second order if $u \in H^3(\Omega)$. A numerical example is given.

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1. Introduction

We consider the discretization of the differential equation

$$Au := -(au_x)_x - (bu_y)_y + cu = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2 \tag{1.1}$$

subject to the boundary conditions of the third kind

$$Bu := au_x\eta_x + bu_y\eta_y + \alpha u = \psi \quad \text{on} \quad \Gamma := \partial\Omega \tag{1.2}$$

by finite differences defined on a generally nonuniform rectangular grid $\overline{\Omega}_H$ on the domain $\overline{\Omega}$, which is assumed to be a union of rectangles. Here η_x and η_y denote the components of the outer normal on Γ.

The main aim of the paper is to study the behaviour of the finite difference solution for a sequence of variable grids $\overline{\Omega}_H$, $H \in \Lambda$, with the maximal mesh size $H_{\max}$ converging to

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zero. The grids are assumed to be quasi-uniform except for the cases of $s = 1$ and $s = 2$, where no restriction is placed on the nonuniformity. Under these circumstances the scheme is, in general, only first order consistent. Our aim is to show that nevertheless the finite difference solution and its gradient are one order more accurate. This property of the FDM is usually called supraconvergence (see [24]). More precisely, we prove optimal convergence rates $O(h^s), s \in (1/2, 2]$, of the scheme for weak solutions u belonging to the fractional order Sobolev — Slobodetskij space $H^{1+s}(\Omega)$. It is shown that the gradient is also approximated with the same order. The error estimates are strictly local as is desirable when working with nonuniform grids.

Supraconvergence results for two-dimensional elliptic problems were obtained by several authors. Some basic studies can be found in [29]. In [33] the Laplacian in a square domain subject to Neumann boundary conditions and in [5] a general second order elliptic equation in a polygonal domain subject to Dirichlet boundary conditions were considered. In both papers the solution was assumed to be smooth, i.e., that u belongs to $C^4(\overline{\Omega})$.

It is known from the finite elements that the second order convergence in the L_2-norm has already been obtained for solutions $u \in H^3(\Omega)$, which is optimal with respect to the smoothness assumed. The aim of many papers was to establish also for finite difference schemes the convergence rates that are optimal with respect to the smoothness of the solution, even in the case of a less smooth solution $u \in H^t(\Omega)$ with $t < 3$. Major steps in this direction can be found in [20], [23] and [38], where finite difference schemes on uniform meshes for different types of (positive definite) elliptic equations in a rectangular domain subject to Dirichlet boundary conditions are considered. Third kind boundary conditions are analyzed in [21], where apart from the logarithmic factor the second order convergence was proved for $u \in H^3(\Omega)$. A weaker typical smoothness assumption is $u \in H^{1+s}(\Omega)$ with $s > 0$ ensuring that the pointwise restriction of u on the mesh makes sense, but even $s > -1/2$ was considered in [28]. The convergence is usually studied in discrete analogues of Sobolev spaces. Relying on another method of analysis, domains with a curved boundary are admitted in [11]. Other authors, see [16] and [22], concentrate on handling equations with nonsmooth coefficients or on obtaining convergence in discrete L_p-norms (see [38]). An excellent overview has recently been given in [16] and also in [17], where the analysis can be found in detail. In [7] the supraconvergence was analyzed based on the maximum principle.

Finite differences on nonuniform meshes for the Laplacian in a square with solutions $u \in H^{1+s}(\Omega)$ are considered in [40] for $s = 2$ and in [3] and [19] for $s \in [1, 2]$. The idea in these papers is to add a correction to the standard finite difference scheme on uniform grids that makes the scheme second order accurate also on nonuniform meshes. This disagrees with the result of the present paper that no correction is needed to prove the same convergence order as on uniform meshes, i.e., supraconvergence takes place. Our kind of analysis works fine in the case of Dirichlet boundary conditions (see the forthcoming paper [6]). We consider here the more complicated boundary conditions of the third kind, which were studied in [3] for $s = 2$ on nonuniform meshes in rectangular domains. Also mixed derivative terms could be included in the differential operator A. For ease of presentation we restrict ourselves to the present simpler case. Problems with mixed derivatives were studied with the aid of the maximum principle for smooth solutions u in [34].

A one-dimensional version of the results obtained in this paper was published in [1]. In the one-dimensional case, several authors studied the supraconvergence (see [8–10, 15, 24, 32, 37]). Also for hyperbolic and parabolic equations the supraconvergence was considered (see [2, 18, 29, 39, 42]).

In the proofs we prefer to work with the usual norms in the Sobolev — Slobodetskij spaces thus avoiding the uncomfortable discrete versions of these norms. Also, we find it helpful in the analysis to establish equivalence with the linear finite element method on the standard triangulation $\mathcal{T}_H$ associated with the rectangular grid $\overline{\Omega}_H$ combined with a special kind of quadrature. In fact, this relation opens up the possibility of expressing the discretized boundary conditions, always a problem for finite difference methods if conditions of the third kind are involved, in a reasonable form. As a consequence, the second order convergence of the gradient in the finite difference scheme is nothing but the supercloseness ([36], [41, p. 80]) of the gradient of the fully discrete FEM approximation, i.e., it is second order accurate to the linear interpolation $Q_H u$ on $\mathcal{T}_H$ of the exact solution u. Several recovery techniques for the gradient are based on the supercloseness property (see [12–14, 25, 26, 30, 43, 44] and the references in [27]). In the supercloseness results involved in these papers the meshes are either completely uniform or a smooth transformation of a uniform mesh when working on nonuniform meshes. We want to point out the significant difference in the behaviour of the scheme on uniform and nonuniforms grids, which can be well seen from the finite difference presentation: while on the former grids the truncation error is of the second order and smoothly varying from grid point to grid point, it is of the first order and strongly oscillating on the latter. In most cases the Dirichlet conditions were considered, but in [13] and [36] also boundary conditions of the third kind are admitted. The order of the supercloseness in the latter case is then reduced to $\mathrm{O}(h^{3/2})$. In [30], the finite element scheme is fully discrete. It is obtained with the aid of a second order accurate quadrature formula, while our quadrature formulas are only of the first order. Recently, the supercloseness has been studied in [31] for nonconforming finite elements.

The paper is organized as follows. In Section 2 we describe the finite difference method for problem (1.1), (1.2). In the next section an equivalent linear FEM with quadrature for which stability is easy to obtain is introduced. In Section 4 the crucial estimate for the truncation error is proved for the low regularity case $s \in (1/2, 1]$, from which, together with the stability, the first convergence result in Theorem 4.1 follows: the H^1-norm of the discretization error $Q_H(u - u_H)$ is of order $O(H^s_{\max})$ provided u is in the Sobolev — Slobodetskij space $H^{1+s}(\Omega)$. Our supraconvergence result, i.e., that the same convergence result holds also for $s \in (1, 2]$, is stated in Theorem 5.1 and proved in Section 5. In this section it is also shown that in general supraconvergence does not take place in the case of the right-hand side $f \in H^1(\Omega)$ in (1.1) if pointwise restriction on the grid $\overline{\Omega}_H$ is used in place of the integral average (2.3) below. But the pointwise restriction can be taken if $f \in H^s(\Omega), s > 1$ (see Remark 5.4). In Section 6 we give some numerical results. Some notations are given in an appendix.

2. The finite difference scheme

In this section, we set up the discretization of (1.1) and (1.2). We first introduce a generally nonequidistant rectangular grid $\overline{\Omega}_H$. Let $\mathbf{h} = \{h_j\}_{j\in\mathbb{Z}}$ and $\mathbf{k} = \{k_\ell\}_{\ell\in\mathbb{Z}}$ be two sequences of mesh sizes, i.e., of positive numbers. We define the grid

$$\mathbb{R}^x_{\mathbf{h}} = \{x_j \in \mathbb{R} : x_{j+1} = x_j + h_j,\, j \in \mathbb{Z}\}$$

with $x_0 \in \mathbb{R}$ given and the corresponding grid $\mathbb{R}^y_{\mathbf{k}}$ with the mesh size vector $\mathbf{k}$ in place of $\mathbf{h}$ and y_0 in place of x_0. Points in the middle between two adjacent grid points are denoted by $x_{j+1/2} := x_j + h_j/2$ and $x_{j-1/2} := x_j - h_{j-1}/2\,(= x_{(j-1)+1/2})$ and, respectively, in the

y-direction. Let $\mathbb{R}_H$ be the two-dimensional rectangular grid

$$\mathbb{R}_H = \mathbb{R}^x_{\mathbf{h}} \times \mathbb{R}^y_{\mathbf{k}} \subset \mathbb{R}^2$$

and define $\Omega_H := \Omega \cap \mathbb{R}_H$, $\Gamma_H := \Gamma \cap \mathbb{R}_H$, $\overline{\Omega}_H := \overline{\Omega} \cap \mathbb{R}_H \, (= \Omega_H \cup \Gamma_H)$. The grid $\overline{\Omega}_H$ is assumed to satisfy the condition that the vertices of Ω are in Γ_H.

For the formulation of the finite difference approximation we use the centered finite difference quotients

$$(\delta_x^{(1/2)} v)_{j,\ell} := \frac{v_{j+1/2,\ell} - v_{j-1/2,\ell}}{x_{j+1/2} - x_{j-1/2}} \quad \text{and} \quad (\delta_x^{(1/2)} v)_{j+1/2,\ell} := \frac{v_{j+1,\ell} - v_{j,\ell}}{x_{j+1} - x_j}.$$

Here $v_{j,\ell} := v(x_j, y_\ell)$ and $v_{j+1/2,\ell} := v(x_{j+1/2}, y_\ell)$ for the functions v defined on $\overline{\Omega}$. The operators also apply for the grid functions $v_H \in W_H$, the space of functions defined on $\overline{\Omega}_H$, if the final result makes sense. The definition of $\delta_y^{(1/2)}$ etc. is analogous. If it is convenient we also use the notation $v_P := v_H(P)$ for $P \in \overline{\Omega}_H$. Then we approximate the differential operator (1.1) by

$$A_H u_H := -\delta_x^{(1/2)}(a\delta_x^{(1/2)} u_H) - \delta_y^{(1/2)}(b\delta_y^{(1/2)} u_H) + c u_H = f_H \quad \text{in } \Omega_H. \tag{2.1}$$

We assume that the coefficients of A belong at least to $C(\overline{\Omega})$ to ensure that $A_H u_H$ is well-defined. We also assume that at least $\alpha, \psi \in C(\Gamma)$ and $f \in L_2(\Omega)$. Further assumptions will be imposed later. The right-hand side f_H in (2.1) is obtained by averaging f in the following way: for a point $P = (x_j, y_\ell) \in \overline{\Omega}_H$ let $x_P := x_j, y_P := y_\ell$ and

$$\square_P := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega, \quad \omega_P := |\square_P|, \tag{2.2}$$

where $|\square_P|$ denotes the measure of $\square_P$. Then

$$f_P := \frac{1}{\omega_P} \int\limits_{\square_P} f(x,y)\, dV. \tag{2.3}$$

In Section 5 we will also consider the possibility of taking f_H to be the pointwise restriction of f on the grid $\overline{\Omega}_H$. The pointwise restriction of the function v on the grid $\overline{\Omega}_H$ will be denoted by $R_H v$. If it is clear from the context, we often write only v in place of $R_H v$.

The right-hand side ψ of the boundary condition is simply approximated by its restriction to the points in Γ_H. In the case $s \in (1/2, 1]$, we can also take (see Remark 4.2)

$$\psi_P := \frac{1}{\sigma_P} \int\limits_{\Gamma_P} \psi(x,y)\, d\sigma, \quad P \in \Gamma_H, \tag{2.4}$$

where

$$\sigma_P := |\Gamma_P| \text{ with } \Gamma_P := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \partial\Omega \;\text{ for } P = (x_j, y_\ell) \in \Gamma_H. \tag{2.5}$$

We come to the discretization of the boundary conditions, where we distinguish three different types of boundary points: inner points on straight segments, convex and re-entrant corners of Γ_H. The following discretizations can be systematically derived from the variational formulation (3.6) in Section 3.

We start with the point $(x_j, y_\ell) \in \Gamma_H$ on the interior of the vertical segment with Ω lying locally to the right. The discretization is then

$$-(a\delta_x^{(1/2)}u_H)_{j+1/2,\ell} - \frac{h_j}{2}\Big(\delta_y^{(1/2)}(b\delta_y^{(1/2)}u_H)\Big)_{j,\ell} + \frac{h_j}{2}(cu_H)_{j,\ell} + (\alpha u_H)_{j,\ell} = \psi_{j,\ell} + \frac{h_j}{2} f_{j,\ell}. \quad (2.6)$$

Next we consider the convex corner (x_j, y_ℓ) with Ω lying locally to the right and above. The discretization is in this case given by

$$-\frac{k_\ell}{h_j + k_\ell}(a\delta_x^{(1/2)}u_H)_{j+1/2,\ell} - \frac{h_j}{h_j + k_\ell}(b\delta_y^{(1/2)}u_H)_{j,\ell+1/2} + \frac{h_j k_\ell}{2(h_j + k_\ell)}(cu_H)_{j,\ell} + (\alpha u_H)_{j,\ell} =$$

$$\psi_{j,\ell} + \frac{h_j k_\ell}{2(h_j + k_\ell)} f_{j,\ell}. \quad (2.7)$$

Finally, let (x_j, y_ℓ) be a re-entrant corner with Ω lying locally to the left and below, which leads to

$$\frac{k_\ell}{h_j + k_\ell}(a\delta_x^{(1/2)}u_H)_{j-1/2,\ell} - \frac{k_{\ell-1}(h_j + h_{j-1})}{2(h_j + k_\ell)}\Big(\delta_x^{(1/2)}(a\delta_x^{(1/2)}u_H)\Big)_{j,\ell} + \frac{h_j}{h_j + k_\ell}(b\delta_y^{(1/2)}u_H)_{j,\ell-1/2} -$$

$$\frac{h_{j-1}(k_\ell + k_{\ell-1})}{2(h_j + k_\ell)}\Big(\delta_y^{(1/2)}(b\delta_y^{(1/2)}u_H)\Big)_{j,\ell} + \frac{h_{j-1}k_{\ell-1} + h_j k_{\ell-1} + h_{j-1}k_\ell}{2(h_j + k_\ell)}(cu_H)_{j,\ell} + (\alpha u_H)_{j,\ell} =$$

$$\psi_{j,\ell} + \frac{h_{j-1}k_{\ell-1} + h_j k_{\ell-1} + h_{j-1}k_\ell}{2(h_j + k_\ell)} f_{j,\ell}. \quad (2.8)$$

The discretization in the remaining points has a corresponding form, we refrain from writing them down for all possible geometric situations. We refer to them altogether as "discrete boundary conditions". These discretizations can be rewritten in a more familiar form by introducing auxiliary gridpoints. For example, in the case of (2.6) let $u_{j-1,\ell}$ be an auxiliary variable in an auxiliary gridpoint $(x_j - h_j, y_\ell)$. If $a = 1$, then (2.6) is equivalent to the equations

$$(A_H u_H)_{j,\ell} = f_{j,\ell} \quad \text{and} \quad \frac{(u_H)_{j+1,\ell} - (u_H)_{j-1,\ell}}{2h_j} + (\alpha u_H)_{j,\ell} = \psi_{j,\ell}.$$

Here, according to the introduction of the auxiliary gridpoint, the coordinate $x_{j-1/2}$ has to be replaced by $x_j - h_j/2$ in A_H .

3. Equivalent fully discrete Galerkin method

Our analysis of the finite difference method is based on the observation that equations (2.1) and together with the discrete boundary conditions (see (2.6)–(2.8)) can be equivalently written as a linear finite element method with quadrature which is also of interest in its own.

We start with the common variational formulation of (1.1), (1.2). By $(\cdot,\cdot)_0$ and $\langle\cdot,\cdot\rangle_0$ we denote the standard inner product on $L_2(\Omega)$ and $L_2(\Gamma)$, respectively. We also use $\|\cdot\|_s$ and $|\cdot|_s$, or more explicitly $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$, for the usual norm and seminorm, respectively, in the Sobolev — Slobodetskij space $H^s(\Omega)$ for $s \geqslant 0$. Let us recall that for $\sigma \in (0,1)$

$$|v|_{\sigma,\Omega} := \left(\iint\limits_{\Omega\times\Omega} \frac{|v(x,y) - v(\xi,\eta)|^2}{|(x,y) - (\xi,\eta)|^{2+2\sigma}}\, dV\, dV\right)^{1/2} \quad (3.1)$$

and for positive $s = [s] + \sigma$ with $\sigma \in (0,1)$

$$|v|_{s,\Omega} := \Big(\sum_{|\alpha|=[s]} |D^\alpha v|^2_{\sigma,\Omega} \Big)^{1/2}, \quad \|v\|_{s,\Omega} := (\|v\|^2_{[s],\Omega} + |v|^2_{s,\Omega})^{1/2}. \tag{3.2}$$

We will need the Sobolev — Slobodetskij spaces $H^s(\Gamma)$, too. In this case, some care has to be taken if $s > 1$ since Γ is only Lipschitz. In our situation we circumvent this difficulty by defining the norm simply as the (Euclidean) sum of its well-defined H^s-norms extended over the (disjoint) straight sections of Γ. By $W^t_q(\Omega)$ with $t \in \mathbb{N}_0$ and $q \geqslant 1$ we denote the usual L_q Sobolev space with the seminorm and norm

$$|v|_{t,q} := \Big(\sum_{|\alpha|=t} \int_\Omega |D^\alpha v(x,y)|^q \, dV \Big)^{1/q}, \quad \|v\|_{t,q} := \Big(\sum_{k=0}^{t} |v|^q_{k,q} \Big)^{1/q},$$

respectively, understanding the case $q = \infty$ in the usual way.

The variational formulation of our problem is:

$$\text{find } u \in H^1(\Omega) \text{ such that } A(u,v) = (f,v)_0 + \langle \psi, v \rangle_0 \quad \text{for} \quad v \in H^1(\Omega), \tag{3.3}$$

where $A(\cdot\,,\cdot)$ is the sesquilinear form defined by

$$A(v,w) = (av_x, w_x)_0 + (bv_y, w_y)_0 + (cv, w)_0 + \langle \alpha v, w \rangle_0 \quad \text{for} \quad v, w \in H^1(\Omega). \tag{3.4}$$

We make the general assumption that the operator A in (1.1) is uniformly elliptic in Ω and, for simplicity, that $c \geqslant 0$ in Ω, $\alpha \geqslant 0$ on Γ and, additionally, that not both c and α vanish identically. Recall that the coefficients a, b, c and α are assumed to be at least continuous. Then the homogeneous problem (3.3), i.e., with $f = 0$ and $\psi = 0$, has only the solution $u = 0$.

Next we introduce discrete analogues of the inner products $(\cdot,\cdot)_0$ and $\langle \cdot,\cdot \rangle_0$ by

$$(v_H, w_H)_H := \sum_{P \in \overline{\Omega}_H} \omega_P \, (v_H)_P \, (\overline{w}_H)_P \quad \text{for} \quad v_H, w_H \in W_H \tag{3.5}$$

and

$$\langle \varphi_H, \chi_H \rangle_H := \sum_{P \in \Gamma_H} \sigma_P \, (\varphi_H)_P (\overline{\chi}_H)_P$$

for the grid functions φ_H, χ_H on Γ_H with ω_P from (2.2) and σ_P from (2.5). The fully discrete variational problem has the form

$$\text{find } u_H \in W_H \text{ such that } A_H(u_H, v_H) = (f_H, v_H)_H + \langle \psi_H, v_H \rangle_H, \quad v_H \in W_H. \tag{3.6}$$

Here $A_H(\cdot,\cdot)$ is a sesquilinear form which we are now going to define. Let $\mathcal{T}_H$ be a triangulation of Ω using the set $\overline{\Omega}_H$ as vertices. The specific choice of $\mathcal{T}_H$ does not matter for the subsequent results to hold. By $Q_H v_H$ we denote the continuous piecewise linear interpolation of v_H with respect to $\mathcal{T}_H$. Then $A_H(\cdot,\cdot)$ is given as the sum

$$A_H = a_H + b_H + c_H + \gamma_H \tag{3.7}$$

of sesquilinear forms corresponding to the different terms in the continuous variational problem (3.4). They are all constructed in a similar way on the basis of linear triangular finite elements combined with an individual quadrature for each term.

Let $\Delta \in \mathcal{T}_H$. We define $a_{\Delta,x}$ to be the value of the coefficient a at the midpoint of the edge of Δ parallel to the x-axis. Then let

$$a_H(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} a_{\Delta,x} \int\limits_{\Delta} (Q_H v_H)_x (Q_H \overline{w}_H)_x \, dV \quad \text{for} \quad v_H, w_H \in W_H. \tag{3.8}$$

Similarly, let $b_{\Delta,y}$ denote the value of the coefficient b at the midpoint of the side of Δ parallel to the y-axis and

$$b_H(v_H, w_H) := \sum_{\Delta\in\mathcal{T}_H} b_{\Delta,y} \int\limits_{\Delta} (Q_H v_H)_y (Q_H \overline{w}_H)_y \, dV \quad \text{for} \quad v_H, w_H \in W_H. \tag{3.9}$$

Finally,

$$c_H(v_H, w_H) := (c v_H, w_H)_H \quad \text{for} \quad v_H, w_H \in W_H. \tag{3.10}$$

The boundary term in (3.4) is simply discretized by

$$\gamma_H(v_H, w_H) := \langle \alpha v_H, w_H \rangle_H \quad \text{for} \quad v_H, w_H \in W_H. \tag{3.11}$$

The finite difference equations belonging to (3.6) are obtained by choosing grid functions v_H that vanish at all but one grid point in $\overline{\Omega}_H$. In this way the following proposition is seen to hold.

Proposition 3.1. *Let the sesquilinear form $A_H(\cdot\,,\cdot)$ and the operator A_H be defined by* (3.7) *and* (2.1), *respectively. Then*

$$A_H(v_H, w_H) = (A_H v_H, w_H)$$

for all v_H, $w_H \in W_H$ with $w_H = 0$ on Γ_H. Moreover, the finite difference equations (2.1) *together with the discrete boundary conditions* (*see* (2.6) – (2.8)) *are equivalent to the discrete variational problem* (3.6).

We now turn to the stability of (3.6) and consider an infinite sequence of grids $\mathbb{R}_H$ such that the maximal mesh size $H_{\max} := \max\{h_j, k_\ell : j, \ell \in \mathbb{Z}\}$ tends to zero. By Λ we denote the sequence of mesh size vectors. The sequence of grids $\overline{\Omega}_H$, $H \in \Lambda$, is called quasi-uniform if all possible quotients of mesh sizes in $\overline{\Omega}_H$ are bounded uniformly for $H \in \Lambda$.

From the ellipticity of the variational problem (3.3), also taking into account the continuity and the sign assumptions of the coefficients, the following proposition is easily seen to hold.

Proposition 3.2. *The following inequality holds for all $v_H \in W_H$ and $H_{\max}$ small enough:*

$$\|Q_H v_H\|_1 \leqslant C \sup_{0\neq w_H\in W_H} \frac{|A_H(v_H, w_H)|}{\|Q_H w_H\|_1}. \tag{3.12}$$

Here and in the following C denotes a generic constant independent of significant quantities.

4. Convergence: case $s \in (1/2, 1]$

Our error estimates are based on the inverse stability inequality (3.12) in Proposition 3.2 applied to the global discretization error $R_H u - u_H$ in place of v_H, where $R_H u \in W_H$ denotes the pointwise restriction of u to the grid $\overline{\Omega}_H$. Hence, since u_H solves (3.6), we have to estimate the truncation error

$$\tau_H := A_H(R_H u, v_H) - (f_H, v_H)_H - \langle \psi_H, v_H \rangle_H \tag{4.1}$$

in terms of $\|Q_H v_H\|_1$. This will be done in this section for the case of a solution u of low regularity.

To simplify the presentation, we introduce for $P \in \Omega^x_{1/2} := \{(x_{j+1/2}, y_\ell) \in \overline{\Omega}\}$ the coordinates $x_P := x_{j+1/2}$, $y_P := y_\ell$, the step size $h_P := h_j$, and the line segments, rectangles and differences

$$S_P := \{x_{j+1/2}\} \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega, \quad \square_P := (x_j, x_{j+1}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega,$$

$$(\Delta_x v_H)_P := v_{j+1,\ell} - v_{j,\ell}. \tag{4.2}$$

For the point set $\Omega^y_{1/2} := \{(x_j, y_{\ell+1/2}) \in \overline{\Omega}\}$ the corresponding quantities are defined. Note that the above symbols will be differently defined in the sequel depending on where the point P is situated.

Our starting point is the quantity $(f_H, v_H)_H$ in (4.1). According to the definition of f_H in (2.3), we have

$$(f_H, v_H)_H = \sum_{P \in \overline{\Omega}_H} \int_{\square_P} (Au)(x,y)\, dV\, (\overline{v}_H)_P. \tag{4.3}$$

We transform the quantities in (4.3) containing derivatives.

Lemma 4.1. *The following identity holds:*

$$\sum_{P \in \overline{\Omega}_H} \int_{\square_P} (au_x)_x\, dV\, (\overline{v}_H)_P = - \sum_{P \in \Omega^x_{1/2}} \int_{S_P} au_x\, dy (\Delta_x \overline{v}_H)_P + \sum_{P \in \Gamma_H} \int_{\Gamma_P} au_x \eta_x\, d\sigma (\overline{v}_H)_P. \tag{4.4}$$

Proof. Integrating by parts, we obtain

$$\sum_{P \in \overline{\Omega}_H} \int_{\square_P} (au_x)_x\, dV\, (\overline{v}_H)_P = \sum_{P \in \overline{\Omega}_H} \int_{\partial \square_P} au_x \eta_x\, d\sigma\, (\overline{v}_H)_P.$$

Note that η_x takes the values 1, -1, and 0 on $\partial \square_P$. Separating the integrals extended over sections of Γ from the sum and then summing by parts leads to (4.4). □

Lemma 4.2. *Let $s \in (1/2, 1)$, $u \in H^{1+s}(\Omega)$, $a \in W^1_{2/(1-s)}(\Omega)$ and assume that $\{\overline{\Omega}_H\}_{H \in \Lambda}$ is quasi-uniform. Then the following estimate holds for all $P \in \Omega^x_{1/2}$:*

$$\left| \int_{S_P} au_x\, dy - |S_P| (a \delta_x^{(1/2)} u)_P \right| \leqslant C |S_P|^s \big(\|u_x\|_{s, \square_P} + \|u_y\|_{s, \square_P} \big) \leqslant C (\operatorname{diam} \square_P)^s \|u\|_{1+s, \square_P} \tag{4.5}$$

Proof. Denoting the left-hand side of (4.5) by $|F_P|$, we obtain

$$|F_P| \leqslant \left| a_P \int_{S_P} \big((\delta_x^{(1/2)} u)_P - u_x \big)\, dy \right| + \left| \int_{S_P} (a_P u_x - au_x)\, dy \right|. \tag{4.6}$$

For the given range of s the imbeddings $H^s(\Omega) \hookrightarrow L_2(S_P)$, $H^{1+s}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W^1_{2/(1-s)}(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous (see [35, Th. 3.37 with Th. 3.30 and Th. 3.26 with Th. 3.16 and Th. A.4]). Hence

$$F_1(u) := a_P \int\limits_{S_P} \big((\delta_x^{(1/2)} u)_P - u_x \big)\, dy \tag{4.7}$$

is a bounded linear functional on $H^{1+s}(\Omega)$. It vanishes for the functions $1, x, y$. We transform $\square_P$ to the unit square, apply the generalized Bramble — Hilbert Lemma for the fractional order spaces (see [4, Th. 6.1]), and obtain after transforming back

$$|F_1(u)| \leqslant C \sup_\Omega |a(x,y)| \frac{|S_P|}{h_P} \max\{|S_P|, h_P\} \frac{1}{h_P|S_P|} \max\{|S_P|^{1+s}, h_P^{1+s}\} |u|_{1+s,\square_P}.$$

Since the grids are quasi-uniform, $F_1(u)$ can be estimated by the right-hand side of (4.5).

We will now estimate the second member of the right-hand side of (4.6). Fix $u_x \in H^s(\Omega)$ and let

$$F_2(a) := \int\limits_{S_P} (a_P u_x - a u_x)\, dy = \int\limits_{S_P} \big(a_P - a(x_P, y)\big) u_x(x_P, y)\, dy.$$

For brevity we set $r := 2/(1-s)$. The linear form $F_2(a)$ is bounded for $a \in W^1_r(\Omega)$ and vanishes for $a = 1$. With the aid of the Bramble — Hilbert Lemma we can derive in a similar way as before the bound

$$|F_2(a)| \leqslant C|S_P| \bigg((h_P^r + |S_P|^r) \frac{1}{h_P|S_P|} \bigg)^{1/r} |a|_{W^1_r(\square_P)} \times$$
$$\bigg(\frac{1}{h_P|S_P|} + \frac{1}{(h_P|S_P|)^2} \max\{h_P^{2+2s}, |S_P|^{2+2s}\} \bigg)^{1/2} \|u_x\|_{s,\square_P}.$$

Thus also F_2 can be estimated as desired and the proof is complete. □

Remark 4.1. The assertion of Lemma 4.2 holds true if the norm of u_x over the rectangle $\square_P$ is replaced by the norm over the part of $\square_P$ lying to the left or the right of the segment S_P, respectively. This is immediate from the proof. In the proof of Theorem 4.1 we will make use of this observation. A corresponding remark applies to the following Lemmas 4.3 and 4.4.

Lemma 4.3. *Let $u \in H^2(\Omega)$ and $a \in W^1_\infty(\Omega)$. Then the following estimate holds for $P \in \Omega^x_{1/2}$:*

$$\bigg| \int\limits_{S_P} a u_x\, dy - |S_P| (a \delta_x^{(1/2)} u)_P \bigg| \leqslant C \operatorname{diam} \square_P \bigg(\frac{|S_P|}{h_P} \bigg)^{1/2} \|u_x\|_{1,\square_P}. \tag{4.8}$$

Proof. Our starting point is again (4.6) and we use F_1 and F_2 as defined in the proof of Lemma 4.2. For almost all $y \in S_P, P = (x_{j+1/2}, y_\ell)$, we obtain by virtue of the Bramble — Hilbert Lemma

$$\big| a_P \big((\delta_x^{(1/2)} u)_P - u_x \big) \big| \leqslant C \sup_\Omega |a(x,y)| \int\limits_{x_j}^{x_{j+1}} |u_{xx}(x,y)|\, dx$$

and integration with respect to y together with the Schwarz inequality leads to the bound in (4.8) for $|F_1|$. Similarly, since for each fixed $u_x \in H^1(\Omega)$ the linear form F_2 is bounded for $a \in W^1_\infty(\Omega)$, the Bramble — Hilbert Lemma furnishes

$$|F_2| \leqslant C|S_P|(|S_P| + h_P) \max\{\|a_x\|_{L_\infty(\square_P)}, \|a_y\|_{L_\infty(\square_P)}\}(|S_P|h_P)^{-1/2}\|u_x\|_{1,\square_P},$$

which can be estimated as desired. □

Lemma 4.4. *Let $s \in (1/2, 1]$, $u \in H^{1+s}(\Omega)$ and $a \in W^1_{2/(1-s)}(\Omega)$. For $s \in (1/2, 1)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. Then for all $v_H \in W_H$*

$$\Big|\sum_{P\in\Omega^x_{1/2}} \int_{S_P} au_x\, dy\, (\Delta_x\overline{v}_H)_P - a_H(R_Hu, v_H)\Big| \leqslant C\Big(\sum_{P\in\Omega^x_{1/2}} (\operatorname{diam}\square_P)^{2s}\|u\|^2_{1+s,\square_P}\Big)^{1/2}\|Q_Hv_H\|_1 \leqslant C\, H^s_{\max}\|u\|_{1+s}\|Q_Hv_H\|_1. \tag{4.9}$$

Proof. A short calculation shows that the sesquilinear form a_H from (3.8) permits the representation

$$a_H(R_Hu, v_H) = \sum_{P\in\Omega^x_{1/2}} |S_P|(a\delta^{(1/2)}_x u)_P(\Delta_x\overline{v}_H)_P. \tag{4.10}$$

Since, with F_P from the proof of Lemma 4.2,

$$\Big|\sum_{P\in\Omega^x_{1/2}} F_P(\Delta_x\overline{v}_H)_P\Big|^2 \leqslant \sum_{P\in\Omega^x_{1/2}} \frac{h_P}{|S_P|}|F_P|^2 \sum_{P\in\Omega^x_{1/2}} h_P|S_P||(\delta^{(1/2)}_x v_H)_P|^2$$

the first inequality follows from Lemma 4.2 if $s \in (1/2, 1)$ and from Lemma 4.3 if $s = 1$. The second one is a consequence of $\operatorname{diam}\square_P \leqslant \sqrt{2}H_{\max}$ and $\sum_{P\in\Omega^x_{1/2}} \|u\|^2_{1+s,\square_P} \leqslant \|u\|^2_{1+s}$. □

Lemma 4.5. *Let $s \in (1/2, 1]$, $u \in H^{1+s}(\Omega)$ and $c \in W^1_{2/(1-s)}(\Omega)$. For $s \in (1/2, 1)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. Then for all $v_H \in W_H$*

$$\Big|\sum_{P\in\overline{\Omega}_H} \int_{\square_P} (cu)(x, y)\, dV(\overline{v}_H)_P - (cu, v_H)_H\Big| \leqslant C\Big(\sum_{P\in\overline{\Omega}_H} (\operatorname{diam}\square_P)^{2s}\|u\|^2_{1+s,\square_P}\Big)^{1/2}\|Q_Hv_H\|_0 \leqslant C\, H^s_{\max}\|u\|_{1+s}\|Q_Hv_H\|_0. \tag{4.11}$$

Proof. The main ingredient of the proof is the estimate

$$|F| := \Big|\int_{\square_P} (cu)(x, y)\, dV - \omega_P(cu)_P\Big| \leqslant C(\operatorname{diam}\square_P)^s\omega_P^{1/2}\|u\|_{1+s,\square_P} \tag{4.12}$$

for $P \in \overline{\Omega}_H$ from which the assertion follows using the Schwarz inequality and the relation

$$\sum_{P\in\overline{\Omega}_H} \omega_P|(v_H)_P|^2 \leqslant C\int_\Omega |(Q_Hv_H)(x, y)|^2\, dV.$$

We consider only the case $s \in (1/2, 1)$, the case $s = 1$ being similar. Let

$$F_1 := \int_{\square_P} \big(c(x, y) - c_P\big)u(x, y)\, dV, \quad F_2 := c_P\int_{\square_P} \big(u(x, y) - u_P\big)\, dV,$$

such that $|F| \leqslant |F_1|+|F_2|$. For brevity let $r := 2/(1-s)$. The imbeddings $H^{1+s}(\Omega) \hookrightarrow C(\overline{\Omega})$ and $W^1_r(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous. Fix $u \in H^{1+s}(\Omega)$. Then F_1 is a linear form that is bounded for $c \in W^1_r(\square_P)$ and vanishes for $c = 1$. Hence, the Bramble — Hilbert Lemma furnishes after a suitable scaling

$$|F_1| \leqslant C\omega_P\big(\omega_P^{-1}(\mathrm{diam}\,\square_P)^r\big)^{1/r}|c|_{W^1_r(\square_P)}\big[\omega_P^{-1}\big(1+(\mathrm{diam}\,\square_P)^2+\omega_P^{-1}(\mathrm{diam}\,\square_P)^{4+2s}\big)\big]^{1/2}\|u\|_{1+s,\square_P}$$

leading to the desired bound. Next we are going to estimate the linear functional F_2, which is for each fixed $c \in W^1_r(\square_P)$ bounded for $u \in H^{1+s}(\Omega)$ and vanishes for $u = 1$. The generalized Bramble — Hilbert Lemma shows in this situation that F_2 is already bounded with respect to the semi-norm $\big(|u|_1^2 + |u|_{1+s}^2\big)^{1/2}$. With the usual scaling procedure it follows that

$$|F_2| \leqslant C \sup_{\square_P} |c(x,y)|\omega_P\big[\omega_P^{-1}(\mathrm{diam}\,\square_P)^2|u|^2_{1,\square_P} + \omega_P^{-2}(\mathrm{diam}\,\square_P)^{4+2s}|u|^2_{1+s,\square_P}\big]^{1/2}.$$

□

With the same type of arguments one can prove the following lemma.

Lemma 4.6. *Let $s \in (1/2,1]$, $\psi \in H^s(\Gamma)$ and $\alpha \in W^1_{1/(1-s)}(\Gamma)$. Then the following estimate holds for $P \in \Gamma_H$:*

$$\Big|\int_{\Gamma_P} \alpha\psi\, d\sigma - \sigma_P(\alpha\psi)_P\Big| \leqslant C\sigma_P^{s+1/2}\,\|\psi\|_{s,\Gamma_P}. \tag{4.13}$$

Lemma 4.7. *Let $s \in (1/2,1]$ and $\psi \in H^s(\Gamma)$. Then for all $v_H \in W_H$*

$$\Big|\sum_{P\in\Gamma_H}\int_{\Gamma_P}\psi\, d\sigma\,(\overline{v}_H)_P - \langle\psi, v_H\rangle_H\Big| \leqslant C\Big(\sum_{P\in\Gamma_H}\sigma_P^{2s}\|\psi\|^2_{s,\Gamma_P}\Big)^{1/2}\|Q_Hv_H\|_1 \leqslant CH^s_{\max}\|\psi\|_{s,\Gamma}\|Q_Hv_H\|_1.$$

Proof. With the aid of Lemma 4.6, choosing $\alpha = 1$, and the Schwarz inequality we can estimate the square of the left-hand side of the asserted inequality by $C\sum_{P\in\Gamma_H}\sigma_P^{2s}\,\|\psi\|^2_{s,\Gamma_P}\times \sum_{P\in\Gamma_H}\sigma_P|(v_H)_P|^2$. Since

$$\sum_{P\in\Gamma_H}\sigma_P|(v_H)_P|^2 \leqslant C\langle Q_Hv_H, Q_Hv_H\rangle_0 \leqslant C\|Q_Hv_H\|^2_{1,\Omega}$$

the first inequality in the assertion is proved. The second one follows from the same argument as in the proof of Lemma 4.5. □

Remark 4.2. In the case that ψ is discretized by (2.4), the corresponding left-hand side of the estimate in Lemma 4.7 vanishes identically.

With the aid of Lemma 4.6 we also obtain the next estimate.

Lemma 4.8. *Let $s \in (1/2,1]$, $u \in H^{1+s}(\Omega)$ and $\alpha \in W^1_{1/(1-s)}(\Gamma)$. Then for all $v_H \in W_H$*

$$\Big|\sum_{P\in\Gamma_H}\int_{\Gamma_P}\alpha u\, d\sigma(\overline{v}_H)_P - \langle\alpha u, v_H\rangle_H\Big| \leqslant C\Big(\sum_{P\in\Gamma_H}\sigma_P^{2s}\|u\|^2_{s,\Gamma_P}\Big)^{1/2}\|Q_Hv_H\|_1 \leqslant CH^s_{\max}\|u\|_{s,\Gamma}\|Q_Hv_H\|_1.$$

We are now in the position to prove the low regularity error estimate.

Theorem 4.1. *Let $s \in (1/2, 1]$. Assume $u \in H^{1+s}(\Omega)$, $a, b, c \in W^1_{2/(1-s)}(\Omega)$, $\alpha \in W^1_{1/(1-s)}(\Gamma)$ and $\psi \in H^s(\Gamma)$. For $s \in (1/2, 1)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. Then for $H_{\max}$ small enough there exists a unique solution u_H of the finite difference equations satisfying*

$$\|Q_H(R_H u - u_H)\|_1 \leqslant C\Bigg(\sum_{P\in\overline{\Omega}_H}(\operatorname{diam}\square_P)^{2s}\|u\|^2_{1+s,\square_P} + \sum_{P\in\Gamma_H}\sigma_P^{2s}\big(\|u\|^2_{s,\Gamma_P} + |\psi|^2_{s,\Gamma_P}\big)\Bigg)^{1/2} \leqslant$$

$$C\,H^s_{\max}\big(\|u\|_{1+s} + \|\psi\|_{s,\Gamma}\big).$$

Proof. From Proposition 3.2 follows, for $H_{\max}$ small enough, the uniqueness of problem (3.6) and hence the unique existence of u_H. The asserted bound will be obtained from the same proposition by estimating τ_H from (4.1). Note that τ_H can be written in the form

$$\tau_H = a_H(u, v_H) + b_H(u, v_H) + (cu, v_H)_H + \langle \alpha u, v_H\rangle_H - (f_H, v_H)_H - \langle\psi, v_H\rangle_H.$$

Substitute $(f_H, v_H)_H$ from (4.3). Use now (4.4) and the corresponding relation for the y-derivative term $(bu_y)_y$ (let us remark that relation (4.4) can be used although u may not have second order derivatives because it is only an intermediate step in transforming the integral of f into well-defined quantities). Since (1.2) holds in $H^{s-1/2}(\Gamma)$, we have the relation

$$\int\limits_{\Gamma_P}\big(au_x\eta_x + bu_y\eta_y\big)\,d\sigma = \int\limits_{\Gamma_P}\big(\psi - \alpha u\big)\,d\sigma. \tag{4.14}$$

The asserted bound is then obtained by collecting the estimates from Lemmas 4.4, 4.5, 4.7, and 4.8. □

5. Convergence: case $s \in (1, 2]$

We are now going to prove the supraconvergence. We begin with estimating the error in replacing $(a\delta_x^{(1/2)}u)_P$ in (4.10) by $(au_x)_P$.

Lemma 5.1. *Let $s \in (1, 2]$, $u \in H^{1+s}(\Omega)$ and $a \in C(\overline{\Omega})$. For $s \in (1, 2)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. Then for all $v_H \in W_H$*

$$\sum_{P\in\Omega^x_{1/2}}|S_P|\big|\big((a\delta_x^{(1/2)}u)_P - (au_x)_P\big)(\Delta_x\overline{v}_H)_P\big| \leqslant C\Bigg(\sum_{P\in\Omega^x_{1/2}}(\operatorname{diam}\square_P)^{2s}|u_x|^2_{s,\square_P}\Bigg)^{1/2}|Q_H v_H|_1 \leqslant$$

$$CH^s_{\max}|u_x|_s|Q_H v_H|_1.$$

Proof. The proof follows similar lines as in the proofs before. We consider only the case $s \in (1, 2)$, the case $s = 2$ is similar albeit somewhat easier. Note that the imbedding $H^s(\Omega) \hookrightarrow C(\overline{\Omega})$ is continuous. Let $P = (x_{j+1/2}, y_\ell) \in \Omega^x_{1/2}$. We consider

$$(a\delta_x^{(1/2)}u)_P - (au_x)_P = a_P\Bigg(\frac{1}{h_j}\int\limits_{x_j}^{x_{j+1}}u_x(x, y_\ell)\,dx - (u_x)_P\Bigg)$$

as a linear functional in $u_x \in H^s(\Omega)$. It vanishes for polynomials of degree 1. By virtue of the generalized Bramble — Hilbert Lemma we obtain

$$\left|(a\delta_x^{(1/2)}u)_P - (au_x)_P\right| \leqslant C \sup_\Omega |a(x,y)| \frac{1}{h_P|S_P|}(h_P + |S_P|)\max\{h_P^s, |S_P|^s\}|u_x|_{s,\square_P},$$

from which the result is easily derived as before. □

Remark 5.1. The claim of Lemma 5.1 holds also true if the rectangle $\square_P$ is replaced by the upper or lower half of $\square_P$. This is immediate from the proof. We have avoided to state this fact in the wording of the lemma to keep the presentation easier. But in the proof of Theorem 5.1 we will make use of this observation. A corresponding remark applies to Lemmas 5.3 and 5.4.

The next lemma provides an essential step in obtaining supraconvergence. We need for points $P \in \Omega_{1/2}^{xy} := \{(x_{j+1/2}, y_{\ell+1/2}) \in \Omega\}$ the line segments, points and rectangles

$$S_P := \{x_{j+1/2}\}\times(y_\ell, y_{\ell+1}), \quad S_{P^-} := \{x_{j+1/2}\}\times(y_\ell, y_{\ell+1/2}), \quad S_{P^+} := \{x_{j+1/2}\}\times(y_{\ell+1/2}, y_{\ell+1}),$$

$$P^- := (x_{j+1/2}, y_\ell), \quad P^+ := (x_{j+1/2}, y_{\ell+1}), \quad \square_P := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}). \tag{5.1}$$

For points $P = (x_j, y_{\ell+1/2}) \in \Omega_{1/2}^y$ we define the following vertices and half vertical line segments of the rectangle $\square_P$ $(= (x_{j-1/2}, x_{j+1/2}) \times (y_\ell, y_{\ell+1}))$: If $P \in \Omega_{1/2}^y \backslash \Gamma$ then

$$S_P^{(1)} := \{x_{j+1/2}\} \times (y_{\ell+1/2}, y_{\ell+1}), \quad S_P^{(2)} := \{x_{j-1/2}\} \times (y_{\ell+1/2}, y_{\ell+1}),$$

$$S_P^{(3)} := \{x_{j-1/2}\} \times (y_\ell, y_{\ell+1/2}), \quad S_P^{(4)} := \{x_{j+1/2}\} \times (y_\ell, y_{\ell+1/2}),$$

$$P^{(1)} := (x_{j+1/2}, y_{\ell+1}), \quad P^{(2)} := (x_{j-1/2}, y_{\ell+1}), \quad P^{(3)} := (x_{j-1/2}, y_\ell), \quad P^{(4)} := (x_{j+1/2}, y_\ell),$$

$$h_P := x_{j+1/2} - x_{j-1/2}, \quad k_P := y_{\ell+1} - y_\ell.$$

For $P \in \Gamma_{1/2}^y := \Omega_{1/2}^y \cap \Gamma$, the set of midpoints of the vertical boundary sections, the definitions are corresponding, only with $x_{j+1/2}$ or $x_{j-1/2}$, respectively, replaced adequately by x_j. For $\Gamma_{1/2}^y$ we need also the half sections Γ_P^-, Γ_P^+ of the boundary sections below and above P and

$$\Gamma_P := \Gamma_P^- \cup \Gamma_P^+, \quad \sigma_P := y_{\ell+1} - y_\ell. \tag{5.2}$$

Fig. 1 may help to identify the sets introduced here.

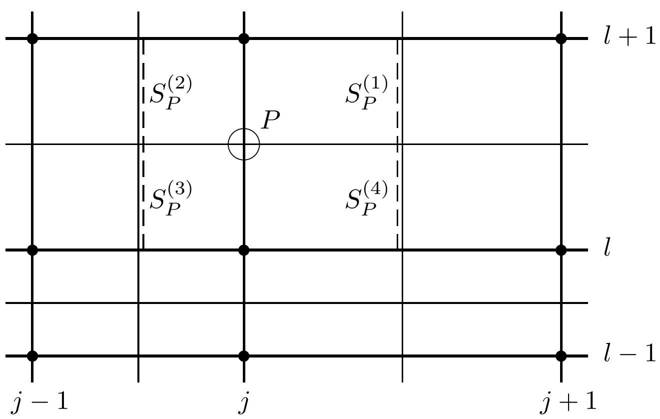

Fig. 1. Notation in the proof of Lemma 5.2 for $P \in \Omega_{1/2}^y$

Lemma 5.2. *The following identity*

$$\sum_{P\in\Omega^x_{1/2}} \Big(\int\limits_{S_P} au_x\,dy\ -|S_P|(au_x)_P\Big)(\Delta_x\overline{v}_H)_P = F_1+F_2+F_3 \tag{5.3}$$

holds, where

$$F_1=\sum_{P\in\Omega^{xy}_{1/2}}\Big(\int\limits_{S_P} au_x\,dy-\frac{k_P}{2}\big((au_x)_{P^+}+(au_x)_{P^-}\big)\Big)\frac{(\Delta_x\overline{v}_H)_{P^+}+(\Delta_x\overline{v}_H)_{P^-}}{2},$$

$$F_2=\sum_{P\in\Omega^{y}_{1/2}}\sum_{i=1}^{4}(-1)^i\Big(\int\limits_{S_P^{(i)}} au_x\,dy-\frac{k_P}{2}(au_x)_{P^{(i)}}\Big)\frac{(\Delta_y\overline{v}_H)_P}{2},$$

$$F_3=\sum_{P\in\Gamma^{y}_{1/2}}(\eta_x)_P\Big(\int\limits_{\Gamma_P^+} au_x\,d\sigma-\frac{\sigma_P}{2}(au_x)_{P^+}-\int\limits_{\Gamma_P^-} au_x\,d\sigma+\frac{\sigma_P}{2}(au_x)_{P^-}\Big)\frac{(\Delta_y\overline{v}_H)_P}{2}.$$

Proof. We divide the integral extended over S_P into two integrals over the halfsections below and above P and note that $|S_P|$ is equal to the sum of the lengths of these halfsections. A straightforward calculation yields that the left-hand side of (5.3) can be written as F_1 plus the quantity

$$\sum_{P\in\Omega^{xy}_{1/2}}\Big(\int\limits_{S_{P+}} au_xdy-\frac{k_P}{2}(au_x)_{P^+}-\int\limits_{S_{P-}} au_xdy+\frac{k_P}{2}(au_x)_{P^-}\Big)\frac{(\Delta_x\overline{v}_H)_{P^+}-(\Delta_x\overline{v}_H)_{P^-}}{2}.$$

Noting that $(\Delta_x\overline{v}_H)_{P^+}-(\Delta_x\overline{v}_H)_{P^-}=(\Delta_x\Delta_y\overline{v}_H)_P$, we perform another summation by parts, this time with respect to the x-variable, which yields the assertion. □

Lemma 5.3. *Let $s\in(1,2]$, $u\in H^{1+s}(\Omega)$ and $a\in W^2_{2/(2-s)}(\Omega)$. For $s\in(1,2)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. For all $v_H\in W_H$ the quantity F_1 in Lemma 5.2 can be estimated by*

$$|F_1|\leqslant C\Big(\sum_{P\in\Omega^{xy}_{1/2}}(\operatorname{diam}\square_P)^{2s}\|u_x\|^2_{s,\square_P}\Big)^{1/2}|Q_Hv_H|_1\leqslant CH^s_{\max}\|u_x\|_s|Q_Hv_H|_1.$$

Proof. We begin with the case $s\in(1,2)$. As a preparation, we introduce for $P\in\Omega^{xy}_{1/2}$ the quantities

$$F_{11}:=\int\limits_{S_P}\big(a-a_P-(a_y)_P(y-y_P)\big)u_x\,dy,$$

$$F_{12}:=\frac{k_P}{2}\Big[\Big(a_P-(a_y)_P\frac{k_P}{2}-a_{P^-}\Big)(u_x)_{P^-}+\Big(a_P+(a_y)_P\frac{k_P}{2}-a_{P^+}\Big)(u_x)_{P^+}\Big],$$

$$F_{13}:=a_P\Big(\int\limits_{S_P}u_x\,dy-\frac{k_P}{2}\big((u_x)_{P^+}+(u_x)_{P^-}\big)\Big),$$

$$F_{14}:=(a_y)_P\Big(\int\limits_{S_P}(y-y_P)u_x\,dy-\frac{k_P^2}{4}\big((u_x)_{P^+}-(u_x)_{P^-}\big)\Big), \tag{5.4}$$

which we are going to estimate. We use in the following that the imbeddings $W_r^2(\Omega) \hookrightarrow C^1(\overline{\Omega})$ and $H^s(\Omega) \hookrightarrow C(\overline{\Omega})$ are continuous, where $r := 2/(2-s)$.

Starting with (5.4) we note that F_{11} is a bounded bilinear form for $(a, u_x) \in W_r^2(\Omega) \times H^s(\Omega)$ that vanishes for $a = 1, x, y$. With a scaling as in the proofs before, recalling also that the grids are quasi-uniform, we derive with the aid of the generalized Bramble — Hilbert Lemma the estimate

$$|F_{11}| \leqslant C(\mathrm{diam}\,\square_P)^s\, |a|_{W_r^2(\square_P)} \|u_x\|_{s,\square_P} \leqslant C(\mathrm{diam}\,\square_P)^s \|u_x\|_{s,\square_P}.$$

A similar argument yields the same bound for F_{12}. In F_{13} we deal with the second order accurate trapezoidal rule applied to $u_x(x_P, \cdot)$ and obtain again with the aid of the generalized Bramble — Hilbert Lemma the bound

$$|F_{13}| \leqslant C \sup_{\Omega} |a(x,y)| (\mathrm{diam}\,\square_P)^s \|u_x\|_{s,\square_P} \leqslant C(\mathrm{diam}\,\square_P)^s \|u_x\|_{s,\square_P}.$$

Finally, F_{14} is a linear bounded functional with respect to $u_x \in H^s(\Omega)$ vanishing for $u_x = 1$. Reasoning as before, it is seen that the same bound as for F_{13} applies to F_{14}. Since F_1 can be written as

$$F_1 = \sum_{P \in \Omega_{1/2}^{xy}} \big(F_{11} + F_{12} + F_{13} + F_{14}\big) \frac{h_P}{2} \big[\big((Q_H \overline{v}_H)_x\big)_{P-} + \big((Q_H \overline{v}_H)_x\big)_{P+}\big]$$

the assertion is obtained after an application of the Schwarz inequality.

We now turn to the proof in the case $s = 2$ and start with the observation that we have the trapezoidal rule applied to au_x under the sum defining F_1 which is exact for the functions $1, x$ and y. The Bramble — Hilbert Lemma furnishes the bound

$$\left| \int_{S_P} a u_x \, dy - \frac{k_P}{2}\big((au_x)_{P+} + (au_x)_{P-}\big) \right| \leqslant C \left(\frac{k_P}{h_P}\right)^{1/2} (\mathrm{diam}\,\square_P)^2 |au_x|_{2,\square_P}.$$

Note that $a \in W_\infty^2(\Omega)$ and so $au_x \in H^2(\Omega)$. The proof can be completed as before. □

Lemma 5.4. *Let $s \in (1,2]$, $u \in H^{1+s}(\Omega)$ and $a \in W^2_{2/(2-s)}(\Omega)$. For $s \in (1,2)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H \in \Lambda}$ is quasi-uniform. The quantity F_2 in Lemma* 5.2 *satisfies the same estimate as F_1 in Lemma* 5.3.

Proof. We begin with the case $s \in (1,2)$. Let $P \in \Omega_{1/2}^{xy}$. We denote by $\bar{P}$ the center of the rectangle $\square_P$ and introduce the quantities

$$F_{21} := \sum_{i=1}^{4} (-1)^i \int_{S_P^{(i)}} \left(a - a_{\bar{P}} - (a_x)_{\bar{P}} \frac{h_P^{(i)}}{2} - (a_y)_{\bar{P}}(y - y_{\bar{P}})\right) u_x \, dy,$$

$$F_{22} := \sum_{i=1}^{4} (-1)^i \frac{k_P}{2} \left(a_{\bar{P}} + (a_x)_{\bar{P}} \frac{h_P^{(i)}}{2} + (a_y)_{\bar{P}} \frac{k_P^{(i)}}{2} - a_{P^{(i)}}\right) (u_x)_{P^{(i)}},$$

$$F_{23} := a_{\bar{P}} \sum_{i=1}^{4} (-1)^i \left(\int_{S_P^{(i)}} u_x \, dy - \frac{k_P}{2}(u_x)_{P^{(i)}}\right), \quad F_{24} := \sum_{i=1}^{4} (-1)^i (a_x)_{\bar{P}} \frac{h_P^{(i)}}{2} \left(\int_{S_P^{(i)}} u_x \, dy - \frac{k_P}{2}(u_x)_{P^{(i)}}\right),$$

$$F_{25} := \sum_{i=1}^{4}(-1)^i (a_y)_{\bar P}\Big(\int\limits_{S_P^{(i)}} (y - y_{\bar P}) u_x \, dy - \frac{k_P}{2}\frac{k_P^{(i)}}{2}(u_x)_{P^{(i)}} \Big),$$

where $h_P^{(i)} := h_P$ for $i = 1, 4$, $h_P^{(i)} := -h_P$ for $i = 2, 3$, and $k_P^{(i)} := k_P$ for $i = 1, 2$, $k_P^{(i)} := -k_P$ for $i = 3, 4$. The quantities F_{21} and F_{22} can be estimated in the same way as F_{11} and F_{12}, respectively, in the proof of Lemma 5.3. Note that F_{23} vanishes for $u_x = 1, x, y$, and, consequently, can be estimated as F_{13} before. Finally, F_{24} and F_{25}, considered as functionals in u_x, vanish for $u_x = 1$ and hence can be estimated as F_{13} and F_{14} before. Upon noting that $F_2 = \sum_{P\in\Omega^y_{1/2}} \sum_{j=1}^{5} F_{2j} k_P \big((Q_H v_H)_y\big)_P/2$ the proof is completed in the same way as that of Lemma 5.3.

Consider now the case $s = 2$. Let $P = (x_j, y_{\ell+1/2}) \in \Omega^y_{1/2}$. We start with the identity

$$F := \sum_{i=1}^{4}(-1)^i\Big(\int\limits_{S_P^{(i)}} a u_x \, dy - \frac{k_P}{2}(a u_x)_{P^{(i)}} \Big) =$$

$$\int\limits_{x_{j-1/2}}^{x_{j+1/2}} \Big(\int\limits_{y_\ell}^{y_{\ell+1/2}} (a u_x)_x \, dy - \int\limits_{y_{\ell+1/2}}^{y_{\ell+1}} (a u_x)_x \, dy - \frac{k_P}{2}(a u_x)_x(x, y_\ell) + \frac{k_P}{2}(a u_x)_x(x, y_{\ell+1}) \Big) \, dx.$$

The integrand of the outer integral exists for almost all $x \in (x_{j-1/2}, x_{j+1/2})$ and is the sum of errors of one-dimensional rectangle rules applied to $(a u_x)_x$ that can be estimated with the aid of the Bramble — Hilbert Lemma. We obtain

$$|F| \leqslant C \int\limits_{x_{j-1/2}}^{x_{j+1/2}} k_P^{3/2} \Big(\int\limits_{y_\ell}^{y_{\ell+1}} \big| \big(a(x,y) u_x(x,y)\big)_{xy} \big|^2 \, dy \Big)^{1/2} \leqslant C h_P^{1/2} k_P^{3/2} \|u_x\|_{2,\square_P} \leqslant C(h_P^2 + k_P^2) \|u_x\|_{2,\square_P},$$

where we made use of $a \in W^2_\infty(\Omega)$ and invoked the Schwarz and Young inequalities in the second and third step, respectively. The proof can now be completed as before. □

Remark 5.2. The proofs of Lemmas 5.3 and 5.4 simplify considerably if the coefficient a is constant.

To estimate the error related to the approximation of cu, we need two auxiliary lemmas. The proof of the first one is straightforward.

Lemma 5.5. *The following identity holds for* $e_j, w_j \in \mathbb{C}$, $j = 1, \dots, 4$*:*

$$4\sum_{i=1}^{4} e_i w_i = \sum_{i=1}^{4} e_i \sum_{i=1}^{4} w_i + (e_1 - e_2 + e_3 - e_4)(w_1 - w_2 + w_3 - w_4) +$$

$$(e_1 + e_2 - e_3 - e_4)(w_1 + w_2 - w_3 - w_4) + (e_1 - e_2 - e_3 + e_4)(w_1 - w_2 - w_3 + w_4).$$

Lemma 5.6. *Let* $s \in (1, 2]$ *and* $g \in H^s(\Omega)$*. Then for all* $v_H \in W_H$

$$\Big| \sum_{P\in\overline{\Omega}_H} \int\limits_{\square_P} g \, dV (\overline{v}_H)_P - (R_H g, v_H)_H \Big| \leqslant C \Big(\sum_{P\in\Omega^{xy}_{1/2}} (\operatorname{diam} \square_P)^{2s} \|g\|^2_{s,\square_P} \Big)^{1/2} \|Q_H v_H\|_1.$$

Proof. Let $P \in \Omega^{xy}_{1/2}$. Analogously to the case $P \in \Omega^{y}_{1/2}$ considered in Lemma 5.2 we introduce the vertices $P^{(i)}, i = 1, 2, 3, 4$, of $\square_P$ and divide $\square_P$ into four congruent subrectangles $\square_P^{(i)}$ (with vertices P and $P^{(i)}$). It is then seen that

$$\sum_{P\in\overline{\Omega}_H} \int_{\square_P} g\, dV (\overline{v}_H)_P - (R_H g, v_H)_H = \sum_{P\in\overline{\Omega}_H} \Big(\int_{\square_P} g\, dV - |\square_P| g_P \Big) (\overline{v}_H)_P =$$

$$\sum_{P\in\Omega^{xy}_{1/2}} \sum_{i=1}^{4} \Big(\int_{\square_P^{(i)}} g\, dV - |\square_P^{(i)}| g_{P^{(i)}} \Big) (\overline{v}_H)_{P^{(i)}}.$$

For $P \in \Omega^{xy}_{1/2}$ we apply Lemma 5.5 with $e_i := \int_{\square_P^{(i)}} g\, dV - |\square_P^{(i)}| g_{P^{(i)}}$, $w_i := (\overline{v}_H)_{P^{(i)}}$ and estimate the resulting quantities. Firstly, discarding a factor 4, there appears the quantity

$$\sum_{i=1}^{4} \Big(\int_{\square_P^{(i)}} g\, dV - |\square_P^{(i)}| g_{P^{(i)}} \Big) \sum_{i=1}^{4} (\overline{v}_H)_{P^{(i)}} = \Big(\int_{\square_P} g\, dV - \frac{|\square_P|}{4} \sum_{i=1}^{4} g_{P^{(i)}} \Big) \sum_{i=1}^{4} (\overline{v}_H)_{P^{(i)}}, \tag{5.5}$$

containing a two-dimensional analogue of the trapezoidal rule which can be estimated with the aid of the generalized Bramble — Hilbert Lemma by

$$C(\mathrm{diam}\square_P)^s |g|_{s,\square_P} \Big(\sum_{i=1}^{4} |\square_P| |(v_H)_{P^{(i)}}|^2 \Big)^{1/2} \leqslant C(\mathrm{diam}\square_P)^s |g|_{s,\square_P} \|Q_H v_H\|_{0,\square_P}.$$

Thus for this part the desired bound is obtained. The next quantity resulting from the application of Lemma 5.5 is

$$\sum_{i=1}^{4} (-1)^i \Big(\int_{\square_P^{(i)}} g\, dV - |\square_P^{(i)}| g_{P^{(i)}} \Big) \sum_{i=1}^{4} (-1)^i (\overline{v}_H)_{P^{(i)}}. \tag{5.6}$$

In this situation we have quadrature rules that are exact for constant functions only, but we can exploit the alternating structure in the last sum. We obtain this time the bound

$$C(\mathrm{diam}\square_P)^s \big(|g|_{s-1,\square_P} + \|g_x\|_{s-1,\square_P} + \|g_y\|_{s-1,\square_P} \big) \times$$

$$\Bigg(|\square_P| \Bigg(\Big| \frac{(v_H)_{P^{(1)}} - (v_H)_{P^{(2)}}}{h_P} \Big|^2 + \Big| \frac{(v_H)_{P^{(3)}} - (v_H)_{P^{(4)}}}{h_P} \Big|^2 \Bigg) \Bigg)^{1/2} \leqslant C(\mathrm{diam}\square_P)^s \|g\|_{s,\square_P} |Q_H v_H|_{1,\square_P}$$

furnishing the bound we need. The remaining quantities can be estimated similarly and the proof is complete. □

Lemma 5.7. *Let $s \in (1, 2]$, $u \in H^s(\Omega)$ and $c \in W^2_{2/(2-s)}(\Omega)$. Then for all $v_H \in W_H$*

$$\Big| \sum_{P\in\overline{\Omega}_H} \int_{\square_P} cu\, dV (\overline{v}_H)_P - (R_H(cu), v_H)_H \Big| \leqslant C \Big(\sum_{P\in\Omega^{xy}_{1/2}} (\mathrm{diam}\square_P)^{2s} \|u\|^2_{s,\square_P} \Big)^{1/2} \|Q_H v_H\|_1.$$

Proof. The first part of the proof is the same as that of Lemma 5.6 until formula (5.5) with $g = cu$. We continue the present proof by estimating (5.5). First the coefficient c is approximated by its first order Taylor polynomial with the midpoint $\bar{P}$ of $\square_P$ as a center. In the step corresponding to (5.6), the factor c of u is approximated by $c_{\bar{P}}$. The course of the proof is now similar to that of Lemma 5.4. We do not give the details. □

Lemma 5.8. *Let $s \in (1,2]$, $\phi \in H^s(\Gamma)$ and $\alpha \in W^2_{1/(2-s)}(\Gamma)$. Then the following estimate holds for all $v_H \in W_H$:*

$$\Big| \sum_{P\in\Gamma^y_{1/2}} \Big[\int_{\Gamma_P} \alpha\phi\, d\sigma - \frac{k_P}{2}((\alpha\phi)_{P^+} + (\alpha\phi)_{P^-})((\overline{v}_H)_{P^+} + (\overline{v}_H)_{P^-}) \Big] \Big| \leqslant$$

$$C\Big(\sum_{P\in\Gamma^y_{1/2}} k_P^{2s}\, \|\phi\|^2_{s,\Gamma_P} \Big)^{1/2} \|Q_H v_H\|_1 \leqslant C H^s_{\max}\, \|\phi\|_{s,\Gamma} \|Q_H v_H\|_1.$$

Proof. The main step in the proof is to show the estimate

$$|G| := \Big| \int_{\Gamma_P} \alpha\phi\, d\sigma - \frac{k_P}{2}\big((\alpha\phi)_{P^+} + (\alpha\phi)_{P^-}\big) \Big| \leqslant C k_P^{1/2+s} \|\phi\|_{s,\Gamma_P}. \tag{5.7}$$

To this end we introduce the quantities

$$G_1 := \int_{\Gamma_P} \big(\alpha - \alpha_P - (\alpha_\sigma)_P(\sigma - y_P)\big)\phi\, d\sigma,$$

$$G_2 := \frac{k_P}{2}\Big[\Big(\alpha_P + (\alpha_\sigma)_P\frac{k_P}{2} - \alpha_{P^+}\Big)\phi_{P^+} + \Big(\alpha_P - (\alpha_\sigma)_P\frac{k_P}{2} - \alpha_{P^-}\Big)\phi_{P^-}\Big],$$

$$G_3 := \alpha_P\Big(\int_{\Gamma_P} \phi\, d\sigma - \frac{k_P}{2}(\phi_{P^+} + \phi_{P^-})\Big), \quad G_4 := (\alpha_\sigma)_P\Big(\int_{\Gamma_P} (\sigma - y_P)\phi\, d\sigma - \frac{k_P^2}{4}(\phi_{P^+} - \phi_{P^-})\Big)$$

that sum up to G. Here, α_σ denotes the derivative of α along the boundary. With the same arguments as already used before the estimate (5.7) is obtained. □

We are now in the position to prove the supraconvergence. We denote by □ the rectangles belonging to $\Omega^{xy}_{1/2}$ (see (5.1)) and by S the sections belonging to $\Gamma^x_{1/2}$ and $\Gamma^y_{1/2}$ (see (5.2)).

Theorem 5.1. *Let $s \in (1,2], u \in H^{1+s}(\Omega), a,b,c \in W^2_{2/(2-s)}(\Omega), \psi \in H^s(\Gamma)$ and $\alpha \in W^2_{1/(2-s)}(\Gamma)$. For $s \in (1,2)$ assume additionally that the sequence of grids $\{\overline{\Omega}_H\}_{H\in\Lambda}$ is quasi-uniform. Then for $H_{\max}$ small enough there exists a unique solution u_H of the finite difference equations* (2.1) *together with the discrete boundary conditions* (*see* (2.6) – (2.8)) *satisfying*

$$\|Q_H(R_H u - u_H)\|_1 \leqslant C\Big(\sum_{\square\subset\Omega} (\operatorname{diam}\square)^{2s} \|u\|^2_{1+s,\square} + \sum_{S\subset\Gamma} |S|^{2s}\big(\|u\|^2_{s,S} + |\psi|^2_{s,S}\big)\Big)^{1/2} \leqslant$$

$$C\, H^s_{\max}\big(\|u\|_{1+s,\Omega} + |\psi|_{s,\Gamma}\big).$$

Proof. The proof follows the same lines as that of Theorem 4.1. We only have to estimate the truncation error

$$\tau_H = a_H(u, v_H) + b_H(u, v_H) + (cu, v_H)_H + \langle \alpha u, v_H\rangle_H - (f_H, v_H)_H - \langle \psi, v_H\rangle_H$$

in terms of the claimed bound. We begin with the part containing $(au_x)_x$ in $(f_H, v_H)_H$, which is transformed according to Lemma 4.1. For the moment we consider only the quantities related to subdomains of Ω and leave those related to boundary sections to the second part of the proof. So we form the difference of $a_H(u, v_H)$ with the first quantity on the right-hand side of (4.4). Recall the representation (4.10) in which we can replace $a\delta_x^{(1/2)}u$ by au_x at the expense of an error that is estimated in Lemma 5.1. We then apply Lemma 5.2 and estimate the quantities F_1 and F_2 with the aid of Lemmas 5.3 and 5.4. In these calculations only the boundary quantity F_3 in Lemma 5.2 is left over and needs further consideration (later in this proof). The error coming from the second order y-derivatives part $(bu_y)_y$ in the differential operator A is estimated similarly. The last error left over comes along with $(cu, v_H)_H$. It can be estimated with the aid of Lemma 5.7 in the form needed.

We come now to the boundary-related parts of τ_H which we collect next. In the proof given so far we left aside the second member of the right-hand side of (4.4) and F_3 from Lemma 5.2. In both of them we replace $au_x\eta_x$ with the aid of (1.2) by $\psi - \alpha u =: \phi$. Together with the corresponding boundary contributions in τ_H from vertical boundary sections we end up with

$$\sum_{P\in\Gamma_H^y}\int_{\Gamma_P}\phi d\sigma(\overline{v}_H)_P - \sum_{P\in\Gamma_{1/2}^y}\Big(\int_{\Gamma_P^+}\phi\, d\sigma - \frac{k_P}{2}\phi_{P+} - \int_{\Gamma_P^-}\phi\, d\sigma + \frac{k_P}{2}\phi_{P-}\Big)\frac{(\Delta_y\overline{v}_H)_P}{2} - \langle\phi, v_H\rangle_H^{(y)}, \quad (5.8)$$

where Γ_H^y and $\langle\cdot,\cdot\rangle_H^{(y)}$ denote the part of Γ_H and $\langle\cdot,\cdot\rangle_H$, respectively, extended over the vertical sections of Γ. The identity

$$\sum_{P\in\Gamma_H^y}\int_{\Gamma_P}\phi d\sigma(\overline{v}_H)_P - \langle\phi, v_H\rangle_H^{(y)} = \sum_{P\in\Gamma_{1/2}^y}\Big(\int_{\Gamma_P}\phi\, d\sigma - \frac{k_P}{2}\big(\phi_{P+} + \phi_{P-}\big)\Big)\frac{(\overline{v}_H)_{P+} + (\overline{v}_H)_{P-}}{2} +$$
$$\sum_{P\in\Gamma_{1/2}^y}\Big(\int_{\Gamma_P^+}\phi\, d\sigma - \frac{k_P}{2}\phi_{P+} - \int_{\Gamma_P^-}\phi\, d\sigma + \frac{k_P}{2}\phi_{P-}\Big)\frac{(\overline{v}_H)_{P+} - (\overline{v}_H)_{P-}}{2}$$

shows that only the composite trapezoidal rule is left over in (5.8) which can be estimated according to Lemma 5.8 by

$$C\Big(\sum_{P\in\Gamma_{1/2}^y}k_P^{2s}\|\phi\|_{s,\Gamma_P}^2\Big)^{1/2}\|Q_H v_H\|_1 \leqslant C\Big(\sum_{P\in\Gamma_{1/2}^y}k_P^{2s}\big(\|u\|_{s,\Gamma_P}^2 + \|\psi\|_{s,\Gamma_P}^2\big)\Big)^{1/2}\|Q_H v_H\|_1.$$

The horizontal boundary sections give rise to corresponding estimates. Altogether the proof is complete. □

By interpolating the result of Theorem 4.1 for $s = 1$ and of Theorem 5.1 for $s = 2$ we obtain the following corollary which holds without the assumption of quasi-uniformity of the grid. Note that the local error estimates in Theorem 5.1 are not obtained using interpolation. Also the nonclosed range of exponents $s \in (1/2, 2]$ is not accessible by interpolation.

Corollary 5.1. *Let $s \in [1,2]$, $u \in H^{1+s}(\Omega)$, $a, b, c \in W_\infty^2(\Omega)$, $\psi \in H^s(\Gamma)$ and $\alpha \in W_\infty^2(\Gamma)$. Then for $H_{\max}$ small enough there exists a unique solution u_H of the finite difference equations* (2.1) *with discrete boundary conditions* (2.6) – (2.8) *satisfying*

$$\|Q_H(R_H u - u_H)\|_1 \leqslant C\, H_{\max}^s\big(\|u\|_{1+s,\Omega} + |\psi|_{s,\Gamma}\big).$$

Remark 5.3. If the right-hand side f of (1.1) is in $H^s(\Omega), s \in (1,2]$, then its approximation (2.3) can be replaced by the pointwise restriction to the grid $\overline{\Omega}_H$ without changing the convergence rate. This follows from the observation that according to Lemma 5.6 the corresponding perturbation of the right-hand side of (3.6) can be estimated by

$$|(f_H - R_H f, v_H)_H| = \Big|\sum_{P\in\overline{\Omega}_H}\Big(\int_{\square_P} f(x,y)\, dV - \omega_P f(x_P, y_P)\Big)(\overline{v}_H)_P\Big| \leqslant$$

$$C\Bigg(\sum_{P\in\Omega^{xy}_{1/2}}(\operatorname{diam}\square_P)^{2s}\|f\|^2_{s,\square_P}\Bigg)^{1/2}\|Q_Hv_H\|_1.$$

Remark 5.4. If f is not smooth enough, then the use of the pointwise restriction in the approximation of f in (1.1) may spoil the supraconvergence. We give an example. Consider the Neumann boundary value problem

$$-\Delta u + u = f \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma, \tag{5.9}$$

where $\Omega := (0,1)^2$. We discretize (5.9) on a sequence of uniform grids $\overline{\Omega}_H$, $H \in \Lambda$, where Λ, indexed by m, is here the sequence of uniform mesh size vectors H with step sizes $h_m = k_m := 1/m, m \in \mathbb{N}$. We will show that the convergence cannot be quadratic for all $f \in H^1(\Omega)$. To this end a sequence $\{f^{(m)} \in H^1(\Omega)\}_{m\in\mathbb{N}}$ of the right-hand sides in (5.9) is constructed such that the corresponding exact and discrete solutions $u^{(m)}$ and $u_H^{(m)}$, respectively, satisfy

$$\lim_{m\to\infty} h_m^{-2}\|u^{(m)} - Q_H u_H^{(m)}\|_0 = \infty \quad \text{while} \quad \{\|f^{(m)}\|_1\}_{m\in\mathbb{N}} \text{ is bounded.} \tag{5.10}$$

Then, as a consequence of the uniform boundedness principle, the maps

$$H^1(\Omega) \ni f \mapsto h_m^{-2}\big(u - Q_H u_H\big) \in L_2(\Omega), \quad H \in \Lambda,$$

cannot be pointwise bounded. Since $u \in H^3(\Omega)$ then $h_m^{-2}\{Q_H(u-u_H)\}_{H\in\Lambda}$ is not pointwise bounded either. To verify (5.10), take

$$f^{(m)}(x,y) = \frac{1}{m}(1-\cos(2\pi m x)), \quad u^{(m)}(x,y) = \frac{1}{m}\bigg(1 - \frac{\cos(2\pi m x)}{1+(2\pi m)^2}\bigg).$$

From $R_H f^{(m)} = 0$ it follows that $u_H^{(m)} = 0$ and a direct calculation shows (5.10) to hold true. (Note that $\{\|f^{(m)}\|_2\}_{m\in\mathbb{N}}$ is not bounded.)

6. Numerical experiment

In the following we give the result of some numerical calculations. We consider the problem

$$Au := -u_{xx} - u_{yy} + u = f \quad \text{in} \quad (0,\pi)^2 \tag{6.1}$$

subject to Neumann boundary conditions. The right-hand side f is taken to be the Fourier series

$$f(x,y) := \sum_{j,k\in\mathbb{Z}} f_{j,k}e^{i(jx+ky)}, \quad f_{j,k} := (1+j^2+k^2)^{-s/2}\log(2+j^2+k^2)^r, \tag{6.2}$$

where $r = -0.6$ and $s \in \mathbb{R}$ will be chosen from some range of exponents. The solution of (6.1) is $u(x,y) := \sum_{j,k\in\mathbb{Z}} u_{j,k}e^{i(jx+ky)}$, $u_{j,k} := (1+j^2+k^2)^{-(2+s)/2}\log(2+j^2+k^2)^r$. We determined numerically $\|u\|_0 = 5.0$. It can be seen that for $r < -0.5$ the norm

$$\Bigg(\sum_{j,k\in\mathbb{Z}}(1+j^2+k^2)^{1+t}|u_{j,k}|^2\Bigg)^{1/2} \tag{6.3}$$

(which is equivalent to $\|u\|_{1+t}$) is finite for $t = s$, such that $u \in H^{1+s}((0,\pi)^2)$. But the series in (6.3) is divergent for $t > s$ and hence $u \notin H^{1+t}((0,\pi)^2)$ for $t > s$.

We discretize the problem with the finite difference scheme in Section 2 on an equidistant grid with mesh size h taking as the discrete right-hand side f_H both the averaged restriction (2.3) and the pointwise restriction of f. The grid function f_H can be written as a finite Fourier series $f_H(x,y) := \sum_{j,k=-N+1}^{N} F_{j,k} e^{i(jx+ky)}$ for $(x,y) \in (0,\pi)^2_H$, $N = \pi/h$, where the coefficients $F_{j,k}$ are obtained numerically from the given $f_{j,k}$ in (6.2) by summing up the aliasing terms. The finite difference solution is then

$$u_H(x,y) := \sum_{j,k=-N+1}^{N} \frac{F_{j,k}}{1+\operatorname{sinc}^2(jh/2)+\operatorname{sinc}^2(kh/2)} e^{i(jx+ky)} \quad \text{for } (x,y) \in (0,\pi)^2_H,$$

where $\operatorname{sinc}(x) := \sin(x)/x$. The error norm $|Q_H(u-u_H)|_1$ is then easily obtained with the aid of the Fourier coefficients of the finite Fourier series. Instead of $|Q_H(u-u_H)|_0$ we calculate the equivalent norm $|u-u_H|_H$ belonging to the discrete inner product (3.5), which is also easily obtained from the Fourier coefficients. In fact, if $(u-u_H)(x,y) = \sum_{j,k=-N+1}^{N} E_{j,k} e^{i(jx+ky)}$ for $(x,y) \in (0,\pi)^2_H$ then we work with the equivalent norm $\left(\sum_{j,k=0}^{N} |E_{j,k}|^2\right)^{1/2}$. As can be seen from Figs 2 and 3, in using the averaged restriction (2.3) of f the convergence order in the

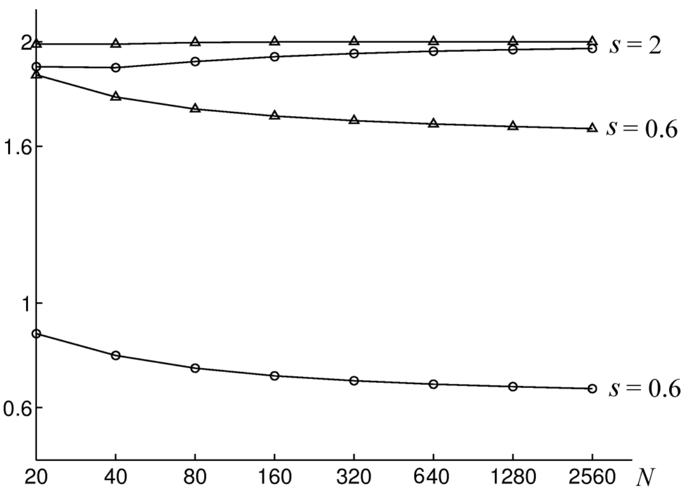

Fig. 2. Order of convergence of L_2-errors (triangles) and H^1-errors (circles) for the averaged restriction of f as a function of N

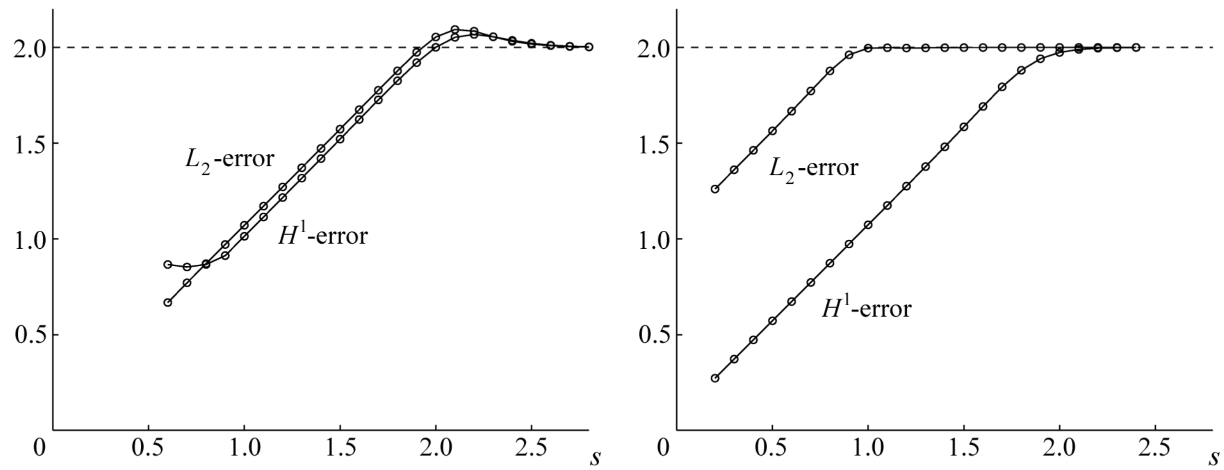

Fig. 3. Order of convergence for the pointwise (left) and averaged restrictions of f as a function of s

L_2-norm exceeds the estimated one in Theorems 4.1 and 5.1 for $s < 2$, for $s \leqslant 1$ even by one order, while the order for the pointwise restriction of f behaves in accordance with the bounds. We think that the higher convergence order observed numerically is due to the symmetries in the solution u. As can also be seen, the pointwise restriction of f shows a higher order of convergence than the worst case expectation according to Remark 5.4.

Appendix

This appendix provides a collection of some notations invoked in this paper.

- Meshes:
  $\Omega_H := \Omega \cap \mathbb{R}_H, \Gamma_H := \Gamma \cap \mathbb{R}_H, \overline{\Omega}_H := \overline{\Omega} \cap \mathbb{R}_H$.

- Basic rectangles and boundary sections in the partition:
  For $P = (x_j, y_\ell) \in \overline{\Omega}_H : \square_P := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega, \omega_P := |\square_P|$,
  $\Gamma_P := (x_{j-1/2}, x_{j+1/2}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \partial\Omega, \sigma_P := |\Gamma_P|$.

- Midpoints of horizontal gridline sections:
  For $P \in \Omega^x_{1/2} := \{(x_{j+1/2}, y_\ell) \in \overline{\Omega}\} : S_P := \{x_{j+1/2}\} \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega$,
  $\square_P := (x_j, x_{j+1}) \times (y_{\ell-1/2}, y_{\ell+1/2}) \cap \Omega, (\Delta_x v_H)_P := v_{j+1,\ell} - v_{j,\ell}$.
  Analogous definitions hold in the vertical direction.

- Centers of subdivision rectangles:
  For $P \in \Omega^{xy}_{1/2} := \{(x_{j+1/2}, y_{\ell+1/2}) \in \Omega\} : S_P := \{x_{j+1/2}\} \times (y_\ell, y_{\ell+1})$,
  $\square_P := (x_j, x_{j+1}) \times (y_\ell, y_{\ell+1}), P^- := (x_{j+1/2}, y_\ell),\ P^+ := (x_{j+1/2}, y_{\ell+1})$,
  $S_{P^-} := \{x_{j+1/2}\} \times (y_\ell, y_{\ell+1/2}),\ S_{P^+} := \{x_{j+1/2}\} \times (y_{\ell+1/2}, y_{\ell+1})$.

- For $P = (x_j, y_{\ell+1/2}) \in \Omega^y_{1/2} \backslash \Gamma$:
  $S^{(1)}_P := \{x_{j+1/2}\} \times (y_{\ell+1/2}, y_{\ell+1}), S^{(2)}_P := \{x_{j-1/2}\} \times (y_{\ell+1/2}, y_{\ell+1})$,
  $S^{(3)}_P := \{x_{j-1/2}\} \times (y_\ell, y_{\ell+1/2}), S^{(4)}_P := \{x_{j+1/2}\} \times (y_\ell, y_{\ell+1/2})$ (see Fig. 1),
  $P^{(1)} := (x_{j+1/2}, y_{\ell+1}), P^{(2)} := (x_{j-1/2}, y_{\ell+1}), P^{(3)} := (x_{j-1/2}, y_\ell), P^{(4)} := (x_{j+1/2}, y_\ell)$,
  $h_P := x_{j+1/2} - x_{j-1/2}, k_P := y_{\ell+1} - y_\ell$.

  For $P = (x_j, y_{\ell+1/2}) \in \Gamma^y_{1/2} := \Omega^y_{1/2} \cap \Gamma$ with the interior of Ω lying to the right of P: replace $x_{j-1/2}$ by x_j in the definitions before.
  $\Gamma^-_P := \{x_j\} \times (y_\ell, y_{\ell+1/2}), \Gamma^+_P := \{x_j\} \times (y_{\ell+1/2}, y_{\ell+1}), \Gamma_P := \Gamma^-_P \cup \Gamma^+_P, \sigma_P := y_{\ell+1} - y_\ell$.

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