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INTEGRATION OF JACOBI AND WEIGHTED BERNSTEIN POLYNOMIALS USING BASES TRANSFORMATIONS

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Abstract — This paper presents methods to compute integrals of the Jacobi polynomials by the representation in terms of the Bernstein — Bézier basis. We do this because the integration of the Bernstein — Bézier form simply corresponds to applying the de Casteljau algorithm in an easy way. Formulas for the definite integral of the weighted Bernstein polynomials are also presented. Bases transformations are used. In this paper, the methods of integration enable us to gain from the properties of the Jacobi and Bernstein bases.

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1. Introduction

The Bernstein polynomials are symmetric, and the Bernstein basis form is known to be optimally stable. These properties and others make the Bernstein polynomials important for the development of Bézier curves and surfaces in Computer-Aided Geometric Design. The Bernstein — Bézier curves and surfaces have become the standard in the Computer-Aided Geometric Design context. They enjoy elegant geometric properties. For more, (see [3,5]).

The Jacobi polynomials present excellent properties in the theory of approximation of functions. Thus they are usually used in several fields of mathematics, applied science, and engineering. And, consequently, formulas for their integrals are needed. Thus we need to get the integrals of the Jacobi polynomials in terms of the Bernstein — Bézier form and vice versa. Moreover, recently the Jacobi polynomials have become very important in CAGD, for example, in the field of degree elevation and reduction, (see [8,9] and the references therein). It is thus necessary to have final results in terms of the Bernstein basis.

Usually, different CAD systems use different base. We need to integrate the Jacobi polynomials in Bernstein oriented system and vice versa.

In this paper, we discuss the issue of finding the integral of a polynomial written in the form of the Jacobi or weighted Bernstein basis. Simple and efficient methods for evaluating these integrals are presented.

In the forthcoming we use the following symbols and notations. The factorial and gamma functions of n are $n! = \Gamma(n+1)$. The combinatorial function is defined by

$$\binom{n}{\nu} = \frac{n!}{\nu!(n-\nu)!}, \quad \nu = 0, 1, \ldots, n.$$

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2. Preliminaries

In this section, we introduce the de Casteljau algorithm, the Bernstein polynomials, the Jacobi polynomials, and the bases transformations between them.

The de Casteljau algorithm is a simple and powerful tool in the field of CAGD, (see [2]). This algorithm is used to generate a polynomial $b^n(u)$ of degree n. In the following steps, we present the de Casteljau algorithm:

Given $n+1$ points $b_0, b_1, \ldots, b_n$ of the control polygon and a parameter value u^*, we set

$$b_\mu^0(u^*) = b_\mu, \quad \mu = 0, 1, \ldots, n,$$

and for $\mu = 0, 1, \ldots, n-\nu, \quad \nu = 1, \ldots, n$ we set

$$b_\mu^\nu(u^*) = (1-u^*)b_\mu^{\nu-1}(u^*) + u^* b_{\mu+1}^{\nu-1}(u^*).$$

Then $b_0^n(u^*)$ is a point on the polynomial curve $b^n(u)$ associated with the parameter u^*.

The intermediate points $b_\mu^\nu(u^*)$ are usually written in a triangular array, called the de Casteljau scheme. The de Casteljau algorithm produces a curve in the form

$$b^n(u) = \sum_{\mu=0}^{n} b_\mu B_\mu^n(u), \quad u \in [0,1], \quad b_\mu \in \mathbb{R}^d,$$

where

$$B_\mu^n(u) = \binom{n}{\mu}(1-u)^{n-\mu}u^\mu, \quad \mu = 0, 1, \ldots, n, \tag{2.1}$$

are the Bernstein polynomials of degree n on $[0,1]$. These curves are called the Bernstein-Bézier curves.

The Jacobi polynomials $P_\nu^{(\alpha,\beta)}(x)$, $\alpha, \beta > -1$, are orthogonal on $[-1,1]$ with respect to the weight function (see [10, 11])

$$w(x) = (1-x)^\alpha(1+x)^\beta, \quad \alpha, \beta > -1.$$

For the sake of symmetry behavior, they are traditionally defined on $[-1,1]$. However for our purposes in this paper, it is appropriate to shift them into $[0,1]$ to be orthogonal with respect to the weight function

$$w(u) = (1-u)^\alpha u^\beta, \quad \alpha, \beta > -1. \tag{2.2}$$

Now, we give the bases transformations between the Bernstein and the Jacobi bases. Given a polynomial $Q_n(u)$ of degree n on $[0,1]$ in terms of the Jacobi polynomials and the Bernstein polynomials by

$$Q_n(u) = \sum_{\nu=0}^{n} c_\nu P_\nu^{(\alpha,\beta)}(u) = \sum_{\mu=0}^{n} b_\mu B_\mu^n(u). \tag{2.3}$$

Let us introduce the vectors:

$$\mathbf{c}_n^t = [c_0, c_1, \ldots, c_n], \qquad \mathbf{J}_n^t = [P_0^{(\alpha,\beta)}(u), P_1^{(\alpha,\beta)}(u), \ldots, P_n^{(\alpha,\beta)}(u)],$$

$$\mathbf{b}_n^t = [b_0, b_1, \ldots, b_n], \qquad \mathbf{B}_n^t = [B_0^n(u), B_1^n(u), \ldots, B_n^n(u)].$$

Then $Q_n(u)$ can be written in the following vector-form:

$$Q_n(u) = \mathbf{c}_n^t\,\mathbf{J}_n = \mathbf{b}_n^t\,\mathbf{B}_n.$$

There exists an invertible transformation matrix M_n of dimension $n+1$ which satisfies

$$\mathbf{b}_n = M_n\,\mathbf{c}_n. \tag{2.4}$$

It is shown in [4] and [6] that the Legendre — Bernstein and Chebyshev — Bernstein bases transformations are well conditioned, respectively.

3. Integration of Jacobi polynomials

In this section, we consider a polynomial $Q_n(u)$ given in terms of the Jacobi polynomials

$$Q_n(u) = \sum_{\nu=0}^{n} c_\nu P_\nu^{(\alpha,\beta)}(u) = \mathbf{c}_n^t\,\mathbf{J}_n. \tag{3.1}$$

Our first goal is to find a representation for the indefinite integral

$$\int_0^t Q_n(u)du, \quad 0 \leqslant t \leqslant 1,$$

of $Q_n(u)$ by applying the de Casteljau algorithm.

Let us seek the expression for $Q_n(u)$ in terms of the Bernstein basis as follows:

$$Q_n(u) = \sum_{\mu=0}^{n} b_\mu B_\mu^n(u) = \mathbf{b}_n^t\,\mathbf{B}_n.$$

The entries of the matrix of transformation M_n are given (see [7]) for $\mu, \nu = 0, 1, \dots, n$,

$$M_{\mu\nu} = \frac{1}{\binom{n}{\mu}} \sum_{i=\max(0,\mu+\nu-n)}^{\min(\mu,\nu)} (-1)^{\nu-i} \binom{n-\nu}{\mu-i}\binom{\nu+\alpha}{i}\binom{\nu+\beta}{\nu-i}. \tag{3.2}$$

The integral of a Bézier curve $b^n(u) = \sum_{\mu=0}^{n} b_\mu B_\mu^n(u)$ is given by (see chapter 5 in [3])

$$\int_0^t b^n(u)du = \frac{t}{n+1}\sum_{\mu=0}^{n} b_0^\mu(t), \quad 0 \leqslant t \leqslant 1.$$

The values of $b_0^\mu(t)$ are generated by the de Casteljau algorithm. This discussion leads to the formula of the integral in the following theorem.

Theorem 3.1. *The indefinite integral of the Jacobi polynomial in* (3.1) *is given in terms of the de Casteljau algorithm by the formula*

$$\int_0^t Q_n(u)du = \frac{t}{n+1}\sum_{\mu=0}^{n} b_0^\mu(t), \quad 0 \leqslant t \leqslant 1, \tag{3.3}$$

where $\mathbf{b}_n$ *is given by* (2.4) *and* $b_0^\mu(t)$, $\mu = 0, 1, \dots, n$ *are given by the de Casteljau algorithm.*

Since the integrals of the Bernstein polynomials are given by

$$\int_0^1 B_\mu^n(u)du = \frac{1}{n+1}, \quad 0 \leqslant \mu \leqslant n.$$

Thus the integral of $Q_n(u)$ in (3.1) over $[0,1]$ is given by the following formula:

$$\int_0^1 Q_n(u)du = \frac{1}{n+1}\sum_{\mu=0}^{n} b_\mu. \tag{3.4}$$

Since the Legendre polynomials, $P_\nu(u)$, i. e., for $\alpha = \beta = 0$, satisfy

$$\int_0^1 P_\nu(u)du = 0, \quad \nu \neq 0.$$

Thus we get in particular for the integral of the Legendre polynomials,

$$c_0 = \int_0^1 Q_n(u)du = \frac{1}{n+1}\sum_{\mu=0}^{n} b_\mu. \tag{3.5}$$

This means that the coefficient c_0 of the Legendre polynomial is the average of the coefficients of the corresponding polynomial in the Bernstein form.

We consider the polynomial $Q_n(u)$ in the Jacobi form. Our second goal is to find a representation for the indefinite integral of $Q_n(u)$ in terms of the Bernstein polynomials. This can be calculated as follows:

$$\int Q_n(u)du = \int \left(\sum_{\nu=0}^{n} c_\nu P_\nu^{(\alpha,\beta)}(u)\right)du = \int (\sum_{\mu=0}^{n} b_\mu B_\mu^n(u))\,du = \sum_{\mu=0}^{n} b_\mu \int B_\mu^n(u)\,du. \tag{3.6}$$

The indefinite integral of the Bernstein polynomials, up to constant addition, are given by (see chapter 2 in [1])

$$\int B_\mu^n(u)du = \frac{-1}{n+1}\sum_{\nu=0}^{\mu} B_\nu^{n+1}(u), \quad \mu = 0, 1, \ldots, n. \tag{3.7}$$

From this it follows that

$$\int Q_n(u)du = \sum_{\mu=0}^{n} b_\mu^* B_\mu^{n+1}(u),$$

where

$$b_\mu^* = \frac{-1}{n+1}\sum_{\nu=\mu}^{n} b_\nu, \quad \mu = 0, 1, \ldots, n.$$

This is summarized in the following theorem.

Theorem 3.2. *The indefinite integral of the Jacobi polynomial* (3.1) *is given in terms of the Bernstein polynomials by the formula*

$$\int Q_n(u)du = \sum_{\mu=0}^{n} b^*_\mu B^{n+1}_\mu(u), \tag{3.8}$$

where

$$b^*_\mu = \frac{-1}{n+1}\sum_{\nu=\mu}^{n} b_\nu, \quad \mu = 0,1,\dots,n,$$

and $\mathbf{b}_n$ *is given by* (2.4).

4. Integration of weighted Bernstein polynomials

In this section, we consider the function $Q_{n,w}(u)$ which is the product of the weight function $w(u) = (1-u)^\alpha u^\beta$ by the Bernstein polynomials in the form

$$Q_{n,w}(u) = \sum_{\mu=0}^{n} b_\mu (1-u)^\alpha u^\beta B^n_\mu(u) = w(u)\, \mathbf{b}^t_n\, \mathbf{B}_n. \tag{4.1}$$

We want to find the definite integral of $Q_{n,w}(u) : \int_0^1 Q_{n,w}(u)du$. We seek the expression for $Q_{n,w}(u)$ in terms of the Jacobi basis

$$Q_{n,w}(u) = \sum_{\nu=0}^{n} c_\nu (1-u)^\alpha u^\beta P^{(\alpha,\beta)}_\nu(u) = w(u)\, \mathbf{c}^t_n\, \mathbf{J}_n. \tag{4.2}$$

The entries of the transformation matrix M^{-1}_n are given for $\mu,\nu = 0,1,\dots,n$ (see [7]) by

$$M^{-1}_{\mu\nu} = \left(\frac{\delta_{\mu 0}}{h^{(\alpha,\beta)}_0} + \frac{1-\delta_{\mu 0}}{h^{(\alpha,\beta)}_\mu}\right)\binom{n}{\nu}\sum_{i=0}^{\mu}(-1)^{\mu-i}\binom{\mu+\alpha}{i}\binom{\mu+\beta}{\mu-i}\times$$
$$B(\beta+\nu+i+1, n+\mu+\alpha-\nu-i+1), \tag{4.3}$$

where

$$\delta_{\mu 0} = \begin{cases} 1, & \text{if } \mu = 0, \\ 0, & \text{if } \mu \neq 0, \end{cases}$$

and $B(i,j)$ is the beta function given by

$$B(i,j) = \int_0^1 (1-u)^{i-1} u^{j-1} du.$$

Thus,

$$\int_0^1 Q_{n,w}(u)du = \int_0^1 \left(\sum_{\nu=0}^{n} c_\nu (1-u)^\alpha u^\beta P^{(\alpha,\beta)}_\nu(u)\right) du = \sum_{\nu=0}^{n} c_\nu \int_0^1 (1-u)^\alpha u^\beta P^{(\alpha,\beta)}_\nu(u)du.$$

In virtue of the orthogonality of the Jacobi polynomials, we get

$$\int_0^1 Q_{n,w}(u)du = c_0 \int_0^1 (1-u)^\alpha u^\beta P_0^{(\alpha,\beta)}(u)du = B(\alpha+1,\beta+1)\,c_0.$$

Thus, we have the following theorem.

Theorem 4.1. *The value of the definite integral of the weighted Bernstein polynomials in* (4.1) *is given by the formula*

$$\int_0^1 Q_{n,w}(u)du = \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,M_{n,1}^{-1}\,\mathbf{b}_n, \tag{4.4}$$

where $M_{n,1}^{-1}$ *is the first row of the matrix* M_n^{-1} *given in* (4.3).

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