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ERROR BOUNDS OF A FULLY DISCRETE PROJECTION METHOD FOR SYMM'S INTEGRAL EQUATION

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Abstract — The approximation properties of a fully discrete projection method for Symm's integral equation with a infinite smooth boundary have been investigated. For the method, error bounds have been found in the metric of Sobolev's spaces. The method turns out to be more accurate compared to the fully discrete collocation method known before.

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1. Introduction

Pseudodifferential equations arise in solving a wide range of problems containing, in particular, integral equations of classical mathematical physics (for example, Helmholtz and Laplace equations with Dirichlet conditions). At present the elliptic pseudodifferential equations appear to be the most studied. As is known, because these problems are ill-posed in space $L_2(0,1)$, their solution calls for special techniques. One way to regularize such problems consists in searching for a pair of appropriate spaces on which the equation under consideration is stable. For example, in [7] a class of problems containing elliptic pseudodifferential equations was investigated and it was proposed to consider the problem on the scale of Sobolev's spaces. In this approach, a stable solution was obtained by a method composed of a fully discrete trigonometric Galerkin scheme with a two-grid iteration method. In [4], the above approach was simplified by using a conjugate gradient-type method allowing simple iteration schemes. Later the investigation above was continued in [5]. Here the generalized minimal residual method (GMRES) was considered as applied to the same class of pseudodifferential equations. The investigation of a wide class of pseudodifferential equations, as well as the problem of recovery of solutions for particular cases of such equations remain important. So, a straightforward method for solving the first kind equations with the kernel having logarithmic singularity was proposed in work [11]. Here the discretization was realized with the help of the interpolation and collocation method on a quasi-uniform grid. In [1], [3], the following algorithms for the Symm's integral equation arising in solving the Dirichlet problem for the Laplace equation were proposed. Namely, in [1] an approximate equation was constructed with the help of the fully discrete trigonometric collocation method, but in

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[3] the fully discrete projection method was investigated. The present paper continues the investigation started in [3], and allows us to estimate the accuracy of the method of [3] in the metric of Sobolev spaces taking into account errors in both parts of the equation under consideration.

2. Description of the problem

Let us consider the Symm integral equation

$$\int_{\Gamma} \log|x-y|v(y)ds_y = g(x), \quad x \in \Gamma, \tag{2.1}$$

for a closed C^∞-smooth boundary Γ of a simple connected domain $\Omega \subset \mathbb{R}^2$. Suppose that $\gamma(t) : [0,1] \to \Gamma$ is C^∞-smooth 1-periodic parametrization of the boundary Γ such that $|\gamma'(t) \neq 0|$ for any $t \in [0,1]$. Without loss of generality it can be assumed that the logarithmic capacity of Γ is distinct from 1. It is known (see [8]) that in this case equation (2.1) is solvable and has an unique solution. As usual, we rewrite (2.1) as

$$Au := A_0 u + Bu = f, \tag{2.2}$$

where

$$(A_0 u)(t) = \int_0^1 \log|\sin\pi(t-s)|u(s)ds, \tag{2.3}$$

$$(Bu)(t) = \int_0^1 b(t,s)u(s)ds, \quad b(t,s) = \begin{cases} \log\dfrac{|\gamma(t)-\gamma(s)|}{|\sin\pi(t-s)|}, & t \neq s, \\ \log(|\gamma'(t)/\pi|), & t = s. \end{cases} \tag{2.4}$$

The eigenfunctions of the operator A_0 are trigonometric functions such that

$$A_0 e^{2\pi ikt} = \begin{cases} -(2|k|)^{-1} e^{2\pi ikt}, & k = \pm1, \pm2, \ldots, \\ -\log 2, & k = 0, \end{cases} \tag{2.5}$$

and the kernel $b(t,s)$ of the operator B represents a C^∞-smooth 1-biperiodic function. To describe the smoothness properties of $b(t,s)$, we make use of the Gevrey classes of infinitely differentiable 1-periodic functions [2, p.112]. Denote by G_β a Gevrey class of order $\beta(\beta \geqslant 1)$ of the Roumieu type in both variables. The class G_β includes the functions $b(t,s)$ that satisfy the condition

$$\|b\|^2_{\beta,\mu} := \sum_{k,l=-\infty}^{\infty} |\hat{b}(k,l)|^2 e^{2\mu(|k|^{1/\beta}+|l|^{1/\beta})} < \infty, \tag{2.6}$$

where

$$\hat{b}(k,l) = \int_0^1\int_0^1 e^{-2\pi i(kt+ls)} b(t,s)dtds$$

are the Fourier coefficients of $b(t,s)$. Note that by (2.6) for $\beta = 1$ it follows that the function $b(t,s)$ has analytic continuations in both variables into the strip $\{z : z = t + is, |s| < \mu/(2\pi)\}$ of the complex plane.

Let H^λ, $-\infty < \lambda < \infty$ denote Sobolev spaces of the 1-periodic functions by the norm

$$\|u\|_\lambda := \Big(|\hat{u}(0)|^2 + \sum_{n\neq 0, n\in\mathbb{Z}} |n|^{2\lambda}|\hat{u}(n)|^2\Big)^{1/2},$$

where $\hat{u}(n) = \int_0^1 e^{-2\pi i n t}u(t)dt$ are the Fourier coefficients of the function $u(t)$, with $H^0 = L_2(0,1)$.

Let the right-hand side f of (2.1) belong to $H^{\nu+1}$ for $\nu > 0$. Moreover, assume that instead of the exact values of the functions $f(t)$ and $\gamma(t)$ we have at our disposal some of their perturbations in the knots of the uniform grid, for which the following estimates hold true:

$$\Big(n^{-1}\sum_{j=1}^{n} |f_\delta(jn^{-1}) - f(jn^{-1})|^2\Big)^{1/2} \leqslant \delta\|f\|_{\nu+1}, \tag{2.7}$$

$$|\gamma_\varepsilon(im^{-1}) - \gamma(im^{-1})| \leqslant \varepsilon, \quad |\gamma'_\varepsilon(im^{-1}) - \gamma'(im^{-1})| \leqslant m\varepsilon, \quad i = 1,2,\ldots,m, \tag{2.8}$$

and the kernel $b_\varepsilon(t,s)$ of the perturbed operator B_ε has the form

$$b_\varepsilon(t,s) = \begin{cases} \log\dfrac{|\gamma_\varepsilon(t) - \gamma_\varepsilon(s)|}{|\sin\pi(t-s)|}, & t \neq s, \\ \log(|\gamma'_\varepsilon(t)/\pi|), & t = s. \end{cases}$$

Then according to [1], the following bound holds:

$$|b_\varepsilon(km^{-1}, lm^{-1}) - b(km^{-1}, lm^{-1})| \leqslant \begin{cases} \dfrac{c\varepsilon}{|\sin[\pi(k-l)/m]|}, & 1 \leqslant k,\quad l \leqslant m, \quad k \neq l, \\ cm\varepsilon, & k = l, \quad 1 \leqslant l \leqslant m. \end{cases}$$

3. Problem statement and method description

In the general case, to guarantee a stable solution of ill-posed problems it is required to use special regularization methods, the general principles of whose construction were stated in the theory of A. N. Tikhonov [9]. Besides, in the finite-dimensional solution of pseudodifferential equations there is one more way to achieve stability approximation, which consists in the appropriate choice of the discretization parameters. The regularization of the problem by its discretization leads to the notion of self-regularization. For example, such an approach was investigated in [10] within the framework of projection methods for which the conditions for self-regularization were described. In [12], the application of the interpolation and collocation method for solving (2.1) was justified without additional regularization, and in [3] the property of self-regularization for the fully discrete projection method was established.

As is known, Symm's integral equation (2.1) in space $H^0 = L_2(0,1)$ is unstable with respect to errors of coefficients A and f. Owing to this, small perturbations of the input data can substantially influence the solution. The typical feature of Eq. (2.1) is the property of the operator A to create isomorphism on the pair of spaces H^λ and $H^{\lambda+1}$ for all λ. This fact follows from the isomorphism of the operator $A_0 : H^\lambda \to H^{\lambda+1}$ for all $\lambda \in (-\infty,\infty)$ and the compactness of the operator $B : H^\lambda \to H^{\lambda+1}$.

The majority of methods for an approximate solution of (2.1) are constructed in view of this fact. Below a description of two such methods is given.

Let us first introduce an n-dimensional space of the trigonometrical polynomials

$$\mathcal{T}_n = \left\{ u_n : u_n = \sum_{k \in Z_n} c_k e^{2\pi i k t} \right\}, \quad Z_n = \left\{ k : -\frac{n}{2} < k \leqslant \frac{n}{2}, \quad k = 0, \pm 1, \pm 2, \dots \right\}.$$

We denote by P_n the orthogonal projector in the form

$$P_n u = \sum_{k \in Z_n} \hat{u}(k) e^{2\pi i k t} \in \mathcal{T}_n,$$

and by Q_n the interpolation projector such that $Q_n u \in \mathcal{T}_n$ and on a uniform grid the following relation takes place:

$$(Q_n u)(jn^{-1}) = u(jn^{-1}), \quad j = 1, 2, ..., n.$$

Now consider the fully discrete collocation method [1] which consists in approximating (2.1) with a C^∞-smooth 1-periodic kernel $b(t, s)$ by the equation

$$A'_{n,\varepsilon} u_{n,\varepsilon,\delta} := A_0 u_n + Q_n B'_{n,\varepsilon} u_{n,\varepsilon,\delta} = Q_n f_\delta, \quad u_n \in \mathcal{T}_n, \tag{3.1}$$

where

$$(B'_{n,\varepsilon} u)(t) = n^{-1} \sum_{j=1}^{n} b_\varepsilon(t, jn^{-1}) u(jn^{-1}).$$

For method (3.1) in [1] with $n = O((\varepsilon + \delta)^{-1/(\nu+1)})$ the error bound

$$\|u - u_{n,\varepsilon,\delta}\|_\lambda = O\left(\delta^{(\nu-\lambda)/(\nu+1)} + \varepsilon^{(\nu-\lambda)/(\nu+1)} \log \frac{1}{\delta + \varepsilon}\right) \tag{3.2}$$

was found on the scale of spaces H^λ, $-1 \leqslant \lambda \leqslant \nu$. Obviously, for $\lambda = 0$ bound (3.2) characterizes the accuracy of method (3.1) in the metric of $L_2(0, 1)$.

Further we describe the fully discrete projection method from [3] and state the corresponding theorem about the error bound for this method. According to [3], the approximate solution $u_{n,m,\varepsilon,\delta}$ of (2.1) is found from the equation

$$A_{m,\varepsilon} u_{n,m,\varepsilon,\delta} := A_0 u_{n,m,\varepsilon,\delta} + B_{m,\varepsilon} u_{n,m,\varepsilon,\delta} = Q_n f_\delta, \quad n > m, \tag{3.3}$$

where

$$(B_{m,\varepsilon} u)(t) = \int_0^1 b_{m,\varepsilon}(t, s) u(s) ds, \quad b_{m,\varepsilon}(t, s) = (Q_{m,t} \otimes Q_{m,s} b_\varepsilon)(t, s) = \sum_{k,l \in Z_m} \hat{b}_{m,\varepsilon}(k, l) e^{2\pi i (kt + ls)},$$

$$\hat{b}_{m,\varepsilon}(k, l) = m^{-2} \sum_{p,q=1}^{m} e^{-2\pi i (kp + lq)/m} b_\varepsilon(pm^{-1}, qm^{-1}).$$

Theorem 3.1. **[3]** *Assume that* $f \in H^{\nu+1}$ *and* $b(t, s)$ *satisfies* (2.6). *Besides, let estimates* (2.7), (2.8) *be true. Then for*

$$m = O\left(\frac{2}{\mu^\beta} \log^\beta \frac{1}{\varepsilon}\right) = O\left(\log^\beta \frac{1}{\varepsilon}\right), \quad n = O(\delta^{-1/(\nu+1)})$$

equation (3.3) *is uniquely solvable and*

$$\|u - u_{n,m,\varepsilon,\delta}\|_0 \leqslant c \|u\|_\nu \left(\delta^{\nu/(\nu+1)} + \varepsilon \log^{(3/2)\beta} \frac{1}{\varepsilon}\right). \tag{3.4}$$

For convenience, hereinafter by c, c_i we denote different positive constants that do not depend on m, n, δ, ε.

Note that in [1] the C^∞-smooth kernels $b(t,s)$ are considered without any additional restriction. In contrast to this, in the paper [3] the condition (2.6) on $b(t,s)$ is applied. Such a restriction allows us to introduce two discretization parameters n and m which take into account the error levels δ and ε, respectively. This way results in an improvement of the error bound (3.4), as compared to (3.2) in the case of $\lambda = 0$.

The aim of the present paper is to generalize the results from [3] by establishing error bounds for method (3.3) on the scale of spaces H^λ for all $0 \leqslant \lambda < \nu$. Besides, the corresponding rules for the choice of m, n to minimize these bounds will be obtained.

It is wellknown (see, for example, [6]) that for any n and $v_n \in \mathcal{T}_n$ the operator $A : H^\lambda \to H^{\lambda+1}$ satisfies the stability inequality

$$\|v_n\|_\lambda \leqslant d_\lambda \|Av_n\|_{\lambda+1}. \tag{3.5}$$

Note that for trigonometrical polynomials the inverse Bernstein inequality

$$\|v_n\|_\nu \leqslant \xi_{\nu-\lambda} n^{\nu-\lambda} \|v_n\|_\lambda, \quad \lambda \leqslant \nu, \quad v_n \in \mathcal{T}_n, \tag{3.6}$$

where $\xi_\lambda = 2^{-\lambda}$ is true.

In [3], it was shown that

$$A_0 u_{n,m,\delta,\varepsilon} = Q_n f_\delta - B_{m,\varepsilon} u_{n,m,\delta,\varepsilon} \in \mathcal{T}_n$$

and $u_{n,m,\delta,\varepsilon} \in \mathcal{T}_n$ is such that

$$u_{n,m,\delta,\varepsilon}(t) = \sum_{k \in Z_n} \hat{u}_{n,m,\delta,\varepsilon}(k) e^{2\pi i k t}.$$

Here the unknown coefficients $\hat{u}_{n,m,\delta,\varepsilon}(k)$ are determined from the following system of linear algebraic equations:

$$\lambda_k \hat{u}_{n,m,\delta,\varepsilon}(k) + \sum_{l \in Z_m} \hat{b}_m(k,-l) \hat{u}_{n,m,\delta,\varepsilon}(l) = \hat{f}_{\delta,n}(k), \quad k \in Z_m,$$

$$\lambda_k \hat{u}_{n,m}(k) = \hat{f}_{\delta,n}(k), \quad k \in Z_n / Z_m, \tag{3.7}$$

where $\lambda_0 = -\log 2$, $\lambda_k = -(2|k|)^{-1}$,

$$\hat{f}_{\delta,n}(k) = n^{-1} \sum_{p=1}^{n} e^{-2\pi i k p/n} f_\delta(pn^{-1}).$$

From (3.7) it follows that to determine the element $u_{n,m,\delta,\varepsilon} \in \mathcal{T}_n$, it is enough to solve the system of $m < n$ linear algebraic equations.

4. Auxiliary statements

Lemma 4.1 [3]. *Let $b(t,s)$ satisfy condition* (2.6). *Then for all $m \geqslant M_0$, where M_0 is the least integer satisfying the inequality $m > 2(\beta\nu/\mu)\beta$, the following holds true:*

$$\|B - B_m\|_{H^0 \to H^\nu} \leqslant c_1 m^\nu e^{-\chi m^{1/\beta}} \|b\|_{\beta,\mu} \tag{4.1}$$

with

$$\chi = \chi(\beta, \mu) = \mu/2^{1/\beta}, \quad (B_m u)(t) = \int_0^1 b_m(t, s)u(s)ds, \quad b_m(t, s) = (Q_{m,t} \otimes Q_{m,s} b)(t, s).$$

Lemma 4.2. *Let $b(t, s)$ satisfy condition* (2.6) *and* (2.8) *be true, then for* $0 \leqslant \lambda < \nu$

$$\|B_m - B_{m,\varepsilon}\|_{H^\lambda \to H^{\lambda+1}} \leqslant c_2 m^{3/2+\lambda}\varepsilon. \tag{4.2}$$

Proof. To prove this statement, the following estimate [3] is required:

$$\|B_m - B_{m,\varepsilon}\|_{H^0 \to H^1} \leqslant c m^{3/2}\varepsilon.$$

Using the last inequality and (3.6), we obtain

$$\|B_m - B_{m,\varepsilon}\|_{H^\lambda \to H^{\lambda+1}} \leqslant \xi_\lambda m^\lambda \|B_m - B_{m,\varepsilon}\|_{H^\lambda \to H^1} \leqslant$$

$$\xi_\lambda m^\lambda \|B_m - B_{m,\varepsilon}\|_{H^0 \to H^1} \leqslant c_2 m^{\lambda+3/2}\varepsilon.$$

which proves the lemma.

Lemma 4.3. *Let the assumptions of Theorem* 3.1 *be true. Then for large enough m the stability inequality is fulfilled for the operator $A_{m,\varepsilon} = A_0 + B_{m,\varepsilon}$ and arbitrary $v \in \mathcal{T}_n$, namely*

$$\|v\|_\lambda \leqslant d'_\lambda \|A_{m,\varepsilon} v\|_{\lambda+1}, \quad n \leqslant m, \quad 0 \leqslant \lambda < \nu.$$

Proof. First, let us prove the stability inequality for the operator A_m. Using (3.5) and (4.1), for $m \geqslant M_0$ we obtain

$$\|v\|_\lambda \leqslant d_\lambda \|Av\|_{\lambda+1} \leqslant d_\lambda \|A_m v\|_{\lambda+1} + d_\lambda \|(A - A_m)v\|_{\lambda+1} =$$

$$d_\lambda \|A_m v\|_{\lambda+1} + d_\lambda \|(B - B_m)v\|_{\lambda+1} \leqslant d_\lambda \|A_m v\|_{\lambda+1} + c_1 d_\lambda m^{\lambda+1} e^{-\chi m^{1/\beta}} \|b\|_{\beta,\mu} \|v\|_\lambda.$$

Let M_1 be the least natural number satisfying the inequality $m^{\lambda+1} e^{-\chi m^{1/\beta}} < q/(c_1 d_\lambda \|b\|_{\beta,\mu})$, where $q \in (0, 1)$. Then for all $m > m_1 := \max\{M_0, M_1\}$ the following holds:

$$\|v\|_\lambda \leqslant \tilde{d}_\lambda \|A_m v\|_{\lambda+1},$$

where $\tilde{d}_\lambda = d_\lambda/(1 - c_1 d_\lambda m_1^{\lambda+1} e^{-\chi m_1^{1/\beta}} \|b\|_{\beta,\mu})$.

Further, in view of the last formula and (4.2), we get

$$\|v\|_\lambda \leqslant \tilde{d}_\lambda \|A_m v\|_{\lambda+1} \leqslant \tilde{d}_\lambda \|A_{m,\varepsilon} v\|_{\lambda+1} + \tilde{d}_\lambda \|(A_m - A_{m,\varepsilon})v\|_{\lambda+1} =$$

$$= \tilde{d}_\lambda \|A_{m,\varepsilon} v\|_{\lambda+1} + \tilde{d}_\lambda \|(B_m - B_{m,\varepsilon})v\|_{\lambda+1} \leqslant \tilde{d}_\lambda \|A_{m,\varepsilon} v\|_{\lambda+1} + c_2 \tilde{d}_\lambda m^{\lambda+3/2}\varepsilon \|v\|_\lambda.$$

Let M_2 be the least natural number for which the inequality $m^{3/2+\lambda}\varepsilon < q/(c_2 \tilde{d}_\lambda)$ is true. Then for all $m = m(\varepsilon) > m_2 := \max\{m_1, M_2\}$ the following takes place:

$$\|v\|_\lambda \leqslant d'_\lambda \|A_{m,\varepsilon} v\|_{\lambda+1},$$

where $d'_\lambda = \tilde{d}_\lambda/(1 - c\tilde{d}_\lambda m_2^{\lambda+3/2}\varepsilon)$. This completes the proof.

Lemma 4.4. *Let the conditions of Theorem* 3.1 *and* (2.7) *be satisfied, then for* $0 \leqslant \lambda < \nu$

$$\|u_{n,m} - u_{n,m,\delta}\|_\lambda \leqslant c_3 n^{\lambda+1}\delta\|u\|_\nu, \quad c_3 = \tilde{d}_\lambda \xi_{\lambda+1}\|A\|_{H_\nu \to H_{\nu+1}}, \tag{4.3}$$

where the element $u_{n,m}$ is the solution of the equation $A_m u_{n,m} = Q_n f$, and the element $u_{n,m,\delta}$ is the solution of the equation $A_m u_{n,m,\delta} = Q_n f_\delta$.

Proof. Since $u_{n,m} - u_{n,m,\delta} \in \mathcal{T}_n$, then from (3.6) and Lemma 3 it follows that

$$\|u_{n,m} - u_{n,m,\delta}\|_\lambda \leqslant \tilde{d}_\lambda \|A_m(u_{n,m} - u_{n,m,\delta})\|_{\lambda+1} = \tilde{d}_\lambda \|A_m u_{n,m} - A_m u_{n,m,\delta}\|_{\lambda+1}$$
$$= \tilde{d}_\lambda \|Q_n f - Q_n f_\delta\|_{\lambda+1} \leqslant \xi_{\lambda+1} n^{\lambda+1}\tilde{d}_\lambda \|Q_n f - Q_n f_\delta\|_0.$$

Further we need an additional relation. According to Lemma 2.1 [1], for all $f \in H^{\nu+1}$ it is true that

$$\|Q_n(f - f_\delta)\|_0 \leqslant \delta\|f\|_{\nu+1}.$$

In view of this inequality we finally obtain

$$\|u_{n,m} - u_{n,m,\delta}\|_\lambda \leqslant \tilde{d}_\lambda \xi_{\lambda+1} n^{\lambda+1}\delta\|f\|_{\nu+1} = \tilde{d}_\lambda \xi_{\lambda+1} n^{\lambda+1}\delta\|Au\|_{\nu+1} \leqslant c_3 n^{\lambda+1}\delta\|u\|_\nu.$$

5. Main results

Below the error bound for the fully discrete projection method (3.3) will be found in the metric of spaces $H^\lambda, 0 \leqslant \lambda < \nu$.

Theorem 5.1. *Let the conditions of Theorem* 3.1, (2.7), (2.8) *be satisfied. Then for* $0 \leqslant \lambda < \nu$

$$\|u - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant c_4\|u\|_\nu (n^{-\nu+\lambda} + n^{\lambda+1}\delta + m^{\lambda+1}e^{-\chi m^{1/\beta}} + m^{3/2+\lambda}\varepsilon). \tag{5.1}$$

Proof. We will prove Theorem 2, based on the scheme used earlier when proving [3, Theorem 2.1].

Obviously, the following inequality is true:

$$\|u - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant \|u - u_{n,m}\|_\lambda + \|u_{n,m} - u_{n,m,\delta}\|_\lambda + \|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_\lambda \tag{5.2}$$

which is an error of method (3.3). The bound of the first term is established in the corollary from [3, Theorem 2.1]:

$$\|u - u_{n,m}\|_\lambda \leqslant c(n^{-\nu+\lambda} + m^{\lambda+1}e^{-\chi m^{1/\beta}})\|b\|_{\beta,\mu}\|u\|_\nu, \quad 0 \leqslant \lambda < \nu, \tag{5.3}$$

and represents an error bound of method (3.3) in the case of exact input data for Eq. (2.1). The bound of the second term is given by Lemma 4.4. Using Lemmas 4.2 and 4.3, we find the bound for the norm of the element $u_{n,m,\delta} - u_{n,m,\varepsilon,\delta} \in \mathcal{T}_n$:

$$\|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant d'_\lambda \|A_{m,\varepsilon}(u_{n,m,\delta} - u_{n,m,\varepsilon,\delta})\|_{\lambda+1} = d'_\lambda \|A_{m,\varepsilon}u_{n,m,\delta} - Q_n f_\delta\|_{\lambda+1} =$$
$$d'_\lambda \|A_{m,\varepsilon}u_{n,m,\delta} - A_m u_{n,m,\delta}\|_{\lambda+1} \leqslant d'_\lambda \|A_{m,\varepsilon} - A_m\|_{H^\lambda \to H^{\lambda+1}}\|u_{n,m,\delta}\|_\lambda =$$
$$d'_\lambda \|B_{m,\varepsilon} - B_m\|_{H^\lambda \to H^{\lambda+1}}\|u_{n,m,\delta}\|_\lambda \leqslant d'_\lambda c_2 \varepsilon m^{3/2+\lambda}\|u_{n,m,\delta}\|_\lambda. \tag{5.4}$$

It remains to estimate the norm of the element $u_{n,m,\delta}$. By means of Lemma 4 and (5.3) we get

$$\|u_{n,m,\delta}\|_\lambda \leqslant \|u_{n,m}\|_\lambda + \|u_{n,m,\delta} - u_{n,m}\|_\lambda \leqslant \|u\|_\lambda + \|u - u_{n,m}\|_\lambda + c_3 n^{\lambda+1}\delta\|u\|_\nu \leqslant$$

$$\|u\|_\lambda + c(n^{-\nu+\lambda} + m^{\lambda+1}e^{-\chi m^{1/\beta}})\|u\|_\nu + c_3 n^{\lambda+1}\delta\|u\|_\nu \leqslant$$
$$\|u\|_\nu(1 + c(n^{-\nu+\lambda} + m^{\lambda+1}e^{-\chi m^{1/\beta}}) + c_3 n^{\lambda+1}\delta) \leqslant c\|u\|_\nu.$$

Substituting the estimate obtained above into (5.4), we have

$$\|u_{n,m,\delta} - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant c\varepsilon m^{3/2+\lambda}\|u\|_\nu.$$

As a result, from (5.2) it follows that

$$\|u - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant c(n^{-\nu+\lambda} + m^{\lambda+1}e^{-\chi m^{1/\beta}})\|u\|_\nu + c_3 n^{\lambda+1}\delta\|u\|_\nu + c\varepsilon m^{3/2+\lambda}\|u\|_\nu \leqslant$$
$$\|u\|_\nu c(n^{-\nu+\lambda} + m^{\lambda+1}e^{-\chi m^{1/\beta}} + n^{\lambda+1}\delta + \varepsilon m^{3/2+\lambda}).$$

Thus, the proof of Theorem 2 is completed.

In the following statement, the rule for the choice of the discretization parameters m and n will be formulated. This rule in method (3.3) allows us to decrease the error bound (5.1) (by an order of magnitude) depending on the values of δ and ε.

Theorem 5.2. *Assume that* $f \in H^{\nu+1}$, $b(t,s)$ *satisfies* (2.6) *and estimates* (2.7), (2.8) *are true. If the discretization parameters* m *and* n *fulfil the relations*

$$m = O\left(\chi^{-\beta}\log^\beta \frac{1}{\varepsilon}\right), \tag{5.5}$$

$$n = O(\delta^{-1/(\nu+1)}), \tag{5.6}$$

then for $0 \leqslant \lambda < \nu$ *the error of method* (3.3) *has the following bound:*

$$\|u - u_{n,m,\varepsilon,\delta}\|_\lambda \leqslant c\|u\|_\nu\left(\delta^{(\nu-\lambda)/(\nu+1)} + \varepsilon\log^{\beta(3/2+\lambda)}\frac{1}{\varepsilon}\right). \tag{5.7}$$

Proof. Obviously, the sum of the first two terms in the right-hand side of (5.1) achieves the minimum (as to their orders) when their orders coincide, i.e.,

$$n^{-\nu+\lambda} \sim n^{\lambda+1}\delta.$$

From this rule (5.6) for the choice of the parameter n follows. Similar reasoning permits obtaining rule (5.5) for the choice of the discretization parameter m.

Remark 5.1. Compare bounds (3.2) and (5.7) describing the accuracy of methods (3.1) and (3.3) respectively, on the scale of spaces H^λ. For δ, both methods provide the same order of error. On the other hand, the quantity of the component depending on ε is much less (by an order of magnitude) in (5.7). In the case of $\lambda = 0$, bound (5.7) for method (3.3) was established earlier in [3] (see Theorem 3.1).

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