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ERROR ESTIMATES FOR THE LAX — FRIEDRICHS SCHEME FOR BALANCE LAWS

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Abstract — In this paper we extend the result from [9] (V. Jovanović, C. Rohde, *Error estimates for finite volume approximations of classical solutions for nonlinear systems of balance laws*, SIAM J. Numer. Anal., **43** (2006)), where, among other things, an $h^{1/2}$ — error estimate in the L^2 — norm for the elastodynamics system has been established. We first derive the general error estimate from [9, Theorem 4.4] in a setting, which is better suited for one – dimensional balance laws and afterwards we apply it to the elastodynamics system with source and to the isentropic Euler system with damping.

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1. Introduction

Let us consider the one-dimensional Cauchy problem for the balance laws

$$\partial_t u + \partial_x G(u) = B(u) \quad \text{in} \quad \mathbb{R} \times (0, T), \tag{1.1}$$

$$u(., 0) = u_0 \quad \text{in} \quad \mathbb{R}, \tag{1.2}$$

where $\mathcal{U} \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) is an open set, $G \in [C^2(\mathcal{U})]^m$ is a nonlinear flux and $B \in [C^1(\mathcal{U})]^m$ is a source term. We will assume that (1.1) is furnished with a strictly convex entropy/entropy flux pair $(\eta, q) \in C^3(\mathcal{U}) \times C^2(\mathcal{U})$. Consequently,

$$\nabla \eta^\tau DG = \nabla q^\tau, \tag{1.3}$$

(see [2]). Here the symbols ∇, D and τ denote gradient, Jacobian and transpose operators, respectively. Suppose further that there exists $\bar{u} \in \mathcal{U}$, such that $B(\bar{u}) = 0$ and $u_0 - \bar{u} \in [L^\infty(\mathbb{R})]^m \cap [H^1(\mathbb{R})]^m \cap [C^1(\mathbb{R})]^m$, where H^1 is the notation for the standard Sobolev space. The paper concerns global smooth solutions $u : \mathbb{R} \times [0, T] \to \mathbb{R}^m$ of (1.1), (1.2), with the properties stated in

Assumption 1.1. *The function u is a classical solution of the Cauchy problem* (1.1), (1.2). *In addition, we have $\int_0^T \int_{\mathbb{R}} |\partial_x u|^2 \, dx \, dt < \infty$, $\sup_{\mathbb{R} \times [0,T]} |\partial_x u| < \infty$ and there exists a convex compact set $S \subset\subset \mathcal{U}$, such that*

$$u(x, t), \quad \bar{u} \in S \quad \textit{for all} \quad (x, t) \in \mathbb{R} \times [0, T]. \tag{1.4}$$

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The numerical scheme which will be applied to the underlined system is given by

$$u_i^{n+1} = u_i^n - \lambda_i\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big] + \Delta t\, B(u_i^n), \tag{1.5}$$

$$u_i^0 = \frac{1}{\Delta x} \int\limits_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)\, dx, \tag{1.6}$$

where the numerical flux g is supposed to be consistent with G and locally Lipschitz-continuous. The mesh in the time direction is uniform: $t^n = n\Delta t$ $(n \in \mathcal{N} := \{0, 1, \ldots, N - 1\})$, for some $\Delta t > 0$ and $N \in \mathbb{N}$ with $N\Delta t = T$. The nonuniform mesh in the x — direction is characterized by $h := \sup_{i\in\mathbb{Z}} \Delta x_i < \infty$, where $\Delta x_i = x_{i+1/2} - x_{i-1/2} > 0$, for all $i \in \mathbb{Z}$. Define finally $\lambda_i := \Delta t/\Delta x_i$.

The numerical scheme (1.5), (1.6) generates a piecewise constant function $u_h : \mathbb{R} \times [0, T] \to \mathbb{R}^m$ given by

$$u_h(x, t) = u_i^n, \quad x \in [x_{i-1/2}, x_{i+1/2}), \quad t \in [t^n, t^{n+1}).$$

We suppose that u_h satisfies the following conditions:

Assumption 1.2.

$$(A1) \qquad u_i^n \in S,$$

$$\eta(u_i^{n+1}) - \eta(u_i^n) + \lambda_i[q_h(u_i^n, u_{i+1}^n) - q_h(u_{i-1}^n, u_i^n)] + 2(L\lambda_i)^2 \times$$

$$(A2) \qquad \big(|u_{i+1}^n - u_i^n|^2 + |u_i^n - u_{i-1}^n|^2\big) \leqslant \Delta t \nabla\eta(u_i^n) \cdot B(u_i^n) + C(\Delta t)^2 |u_i^n - \bar{u}|^2,$$

for all $i \in \mathbb{Z}$, $n \in \mathcal{N}$. Thereby, S is the set from (1.4), *q_h is a numerical entropy flux (a locally Lipschitz-continuous function consistent with q) and L is the Lipschitz constant of g on S.*

Notation. C denotes a constant which doesn't depend on the mesh.

We have established in [9] the error estimate

$$\|u - u_h\|_{L^2(\mathbb{R}\times[0,T])} \leqslant C h^{1/2},$$

under Assumption 1.1 and a slightly modified Assumption 1.2 (Theorem 4.4).

In addition, the conservation property

$$g(-G;\, a, b) = -g(G;\, b, a) \quad (a, b \in \mathcal{U}), \tag{1.7}$$

has been assumed. However, we will show in Section 2 that in the one-dimensional case the same error estimate can be obtained without condition (1.7) (Theorem 2.1).

In the rest of Section 2 we confine ourselves to the Lax — Friedrichs scheme and prove, under an appropriate CFL-condition, that $(A2)$ in Assumption 1.2 holds, provided $(A1)$ is satisfied (Proposition (2.1)). Earlier (see [9]), the condition $(A2)$ was derived from $(A1)$ for the elastodynamics system without a source.

Section 3 is devoted to the elastodynamics system with a source (see (3.1)). Since the only missing ingredient for the application of Theorem 2.1 is the condition $(A1)$ from Assumption 1.2, we concentrate on the problem of establishing invariant regions for the corresponding

Lax — Friedrichs scheme. The procedure relies on [7]. In the first step we suppose that $\beta = 0$ and prove that certain convex compact sets are invariant for a simplified Lax — Friedrichs scheme (see Proposition 3.1 and (3.7)). In contrast to [9], we provide here details of this proof, which is very long and tedious, because it reduces to numerous integral inequalities, some of them being nontrivial; see (3.12) and [10]. In the second step we show that these sets are also invariant for the full Lax — Friedrichs scheme with a source, under a slightly stronger CFL-condition (see the proof of Theorem 3.1).

In Section 4 we proceed with another application of Theorem 2.1. This time we turn our attention to the isentropic Euler system with damping (see (4.1)) and obtain the same error estimate as in the previous section (Theorem 4.1). All necessary assumptions on u_h can be proven in this case as well, except strict positivity of the density, which will be simply assumed.

The origins of the technique presented here go back to a stability result due to Dafermos [3] and DiPerna [4], where the L^2 – distance between the classical and the weak solution is estimated by the L^2 – distance of their initial functions. This theorem together with Vila [11] prompted the authors of [8], [9] to derive error estimates for multidimensional finite volume schemes. An earlier application of Dafermos' and DiPerna's theory to numerical schemes can be found in [1].

In the end let us note that the theory developed in [13] implies the existence of global smooth solutions of the elastodynamics system (3.1) with the properties stated in Assumption 1.1, when u_0 is "close" to $\bar{u}$. Similarly, the results from [12] can apply to the three-dimensional version of (4.1). Concerning invariant regions for (3.1), their existence is pointed out in [5]. However, establishing invariant regions for (4.1), which stay uniformly away from vacuum, is a difficult problem.

2. Error estimate

One can assume without loss of generality that

$$\eta(\bar{u}) = 0, \quad \nabla\eta(\bar{u}) = 0, \quad q(\bar{u}) = 0, \quad \nabla q(\bar{u}) = 0.$$

In addition, let us suppose that there exists a constant $c > 0$, independent of the mesh, with the property

$$c \leqslant \lambda_i, \quad \text{for all } i \in \mathbb{Z}. \tag{2.1}$$

The proof of the next theorem relies heavily on the proof of Lemmas 4.2 and 4.3 from [9].

Theorem 2.1. *Suppose that Assumptions* 1.1, 1.2 *and* (2.1) *hold for the solutions* u *and* u_h *given by* (1.1), (1.2) *and* (1.5), (1.6), *respectively. Then we have the error estimate*

$$a\int_0^T\int_{\mathbb{R}} e^{-\alpha t}|u - u_h|^2\,dx\,dt + Q_h \leqslant C(\Delta t + h), \tag{2.2}$$

where $\alpha > 0$ *is a constant depending on* S *and* $\sup\limits_{\mathbb{R}\times[0,T]} |\partial_x u|$ (*see Theorem* 2.2 *in* [9]), $a = \min\limits_{v\in S} \|\nabla^2\eta(v)\| > 0$ *and*

$$Q_h = 2\sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \theta^n (L\lambda_i)^2 \Delta x_i \big(|u_{i+1}^n - u_i^n|^2 + |u_i^n - u_{i-1}^n|^2\big).$$

Here, $\theta(t) = (T-t)e^{-\alpha t}$, $\theta^n = (1/\Delta t)\int_{t^n}^{t^{n+1}} \theta(t)\,dt$.

Proof. For the sake of simplicity we suppose that $\bar{u} = 0,\ G(\bar{u}) = 0$. Further, we define $\psi := \theta \nabla \eta(u)$. Then

$$R := \int_0^T \int_{\mathbb{R}} G(u_h)\partial_x \psi \,dx\,dt = \sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} G(u_i^n)\partial_x\psi\,dxdt =$$

$$\sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} G(u_i^n)\int_{t^n}^{t^{n+1}} \big[\psi(x_{i+1/2},t) - \psi(x_{i-1/2},t)\big]\,dt = \sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \big[G(u_i^n) - G(u_{i+1}^n)\big]\int_{t^n}^{t^{n+1}} \psi(x_{i+1/2},t)\,dt.$$

According to Proposition 4.1 and Lemma 4.2 from [9], we have

$$\int_0^T\int_{\mathbb{R}} e^{-\alpha t}|u-u_h|^2\,dxdt \leqslant L + R + \int_0^T\int_{\mathbb{R}} \theta(t)B(u_h)\cdot[\nabla\eta(u) - \nabla\eta(u_h)]\,dxdt + Ch^2, \quad (2.3)$$

where

$$L = \sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{x_{i-1/2}}^{x_{i+1/2}} \theta(t^{n+1})\big[\eta(u_i^{n+1}) - \eta(u_i^n) - \nabla\eta(u(x,t^{n+1}))\cdot(u_i^{n+1} - u_i^n)\big]\,dx.$$

Let $L = (I_1 + I_2 + I_3) + (E_1 + E_2 + E_3)$ be the splitting from the proof of [9, Lemma 4.3]. After estimating the terms I_1, I_2, I_3, E_1 in the same way as in Lemma 4.3 and E_2 by $E_2 \leqslant (Q_h/6) + C\Delta t$, it only remains to estimate

$$E_3 = -\frac{1}{\Delta t}\sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \theta\nabla\eta(u)\cdot(u_i^{n+1} - u_i^n)\,dx\,dt.$$

From (1.5) it follows that

$$E_3 = \bar{E}_3 + P - \int_0^T\int_{\mathbb{R}} \theta\nabla\eta(u)\cdot B(u_h)\,dx\,dt, \quad (2.4)$$

where

$$\bar{E}_3 = \frac{1}{\Delta x_i}\sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} \theta(t)\big[\nabla\eta(u(x,t)) - \nabla\eta(u(x_{i+1/2},t))\big]\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i)\big]dx\,dt,$$

$$P = \sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{t^n}^{t^{n+1}} \theta(t)\nabla\eta(u(x_{i+1/2},t))\,dt\cdot\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i)\big] =$$

$$\sum_{n\in\mathcal{N}}\sum_{i\in\mathbb{Z}} \int_{t^n}^{t^{n+1}} \psi(x_{i+1/2},t)\,dt\cdot\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i)\big].$$

Due to

$$|\nabla\eta(u(x,t)) - \nabla\eta(u(x_{i+1/2},t))| = \left| \int\limits_{x_{i+1/2}}^{x} \nabla^2\eta(u(s,t))\partial_x u(s,t)\, ds \right| \leqslant C \int\limits_{x_{i-1/2}}^{x_{i+1/2}} |\partial_x u(s,t)|\, ds,$$

for all $x \in [x_{i-1/2}, x_{i+1/2}]$, we obtain, using (2.1), that

$$\bar{E}_3 \leqslant C \sum_{n\in\mathcal{N}} \sum_{i\in\mathbb{Z}} \int\limits_{t^n}^{t^{n+1}} \int\limits_{x_{i-1/2}}^{x_{i+1/2}} \theta(t)|\partial_x u|\, ds\, dt\, L\big(|u_{i+1}^n - u_i^n| + |u_i^n - u_{i-1}^n|\big) \leqslant$$

$$C \sum_{n\in\mathcal{N}} \sum_{i\in\mathbb{Z}} \Bigg(\int\limits_{t^n}^{t^{n+1}} \int\limits_{x_{i-1/2}}^{x_{i+1/2}} |\partial_x u(s,t)|^2 ds\, dt \Bigg)^{1/2} \Big(2\Delta t \theta^n \Delta x_i L^2 \big(|u_{i+1}^n - u_i^n|^2 + |u_i^n - u_{i-1}^n|^2 \big) \Big)^{1/2} \leqslant$$

$$Ch^{1/2} Q_h^{1/2} \leqslant \frac{Q_h}{6} + Ch.$$

Since

$$G(u_i^n) - G(u_{i+1}^n) + g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i) =$$

$$g(u_i^n, u_i^n) - g(u_{i-1}^n, u_i) - \big(g(u_{i+1}^n, u_{i+1}^n) - g(u_i^n, u_{i+1}^n) \big),$$

then proceeding similarly as in the estimation of $\bar{E}_3$, we have

$$R + P = \sum_{n\in\mathcal{N}} \sum_{i\in\mathbb{Z}} \int\limits_{t^n}^{t^{n+1}} \big[\psi(x_{i+1/2},t) - \psi(x_{i-1/2},t) \big]\, dt \big[g(u_i^n, u_i^n) - g(u_{i-1}^n, u_i) \big] =$$

$$\sum_{n\in\mathcal{N}} \sum_{i\in\mathbb{Z}} \int\limits_{t^n}^{t^{n+1}} \int\limits_{x_{i-1/2}}^{x_{i+1/2}} \theta \nabla^2\eta(u)\partial_x u\, dxdt \cdot \big[g(u_i^n, u_i^n) - g(u_{i-1}^n, u_i) \big] \leqslant ChQ_h^{1/2} \leqslant \frac{Q_h}{6} + Ch^2.$$

Finally, making use of (2.3), (2.4), the estimates for $I_1, I_2, I_3, E_1, E_2, \bar{E}_3$ and $R+P$, we arrive at (2.2). □

We focus now on the Lax — Friedrichs scheme with the numerical flux

$$g(u_1, u_2) = \frac{1}{2}[G(u_1) + G(u_2)] + \frac{1}{2\mu}(u_1 - u_2), \tag{2.5}$$

where μ is a positive parameter. The corresponding numerical entropy flux is given by

$$q_h(u_1, u_2) := \frac{1}{2}[q(u_1) + q(u_2)] + \frac{1}{2\mu}[\eta(u_1) - \eta(u_2)]. \tag{2.6}$$

The next proposition is devoted to the condition $(A2)$ from Assumption 1.2. It is proven here in a more general setting than in [9], since it pertains to general systems with source term.

Proposition 2.1. *Suppose that (A1) from Assumption 1.2 holds for the scheme (1.5), (1.6), (2.5). Then there exist constants $\mu_0, r > 0$ depending only on S, with the following property: if $0 < \mu \leqslant \mu_0$ and $r\lambda_i/\mu \leqslant 1$ ($i \in \mathbb{Z}$), then*

$$\eta(u_i^{n+1}) - \eta(u_i^n) + \lambda_i[q_h(u_i^n, u_{i+1}^n) - q_h(u_{i-1}^n, u_i^n)]+$$

$$2\left(\tilde{L}\frac{\lambda_i}{\mu}\right)^2\left(|u_{i+1}^n - u_i^n|^2 + |u_i^n - u_{i-1}^n|^2\right) \leqslant \Delta t \nabla\eta(u_i^n)\cdot B(u_i^n) + C(\Delta t)^2|u_i^n - \bar{u}|^2,$$

for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thereby, $\tilde{L}/\mu$ is the Lipschitz constant of (2.5) on S and $\tilde{L}$ depends only on S.

Proof. By Taylor expansion, we obtain from (1.5) that

$$E_{in} := \eta(u_i^{n+1}) - \eta(u_i^n) + \lambda_i\big[q_h(u_i^n, u_{i+1}^n) - q_h(u_{i-1}^n, u_i^n)\big] =$$

$$\nabla\eta(u_i^n)\cdot\big[-\lambda_i\big(g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big)\big] + \lambda_i\big[q_h(u_i^n, u_{i+1}^n) - q_h(u_{i-1}^n, u_i^n)\big]+$$

$$\frac{1}{2}\lambda_i^2\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big]^\tau\nabla^2\eta(\xi)\,\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big] + \Delta t\nabla\eta(u_i^n)\cdot B(u_i^n)+$$

$$\frac{1}{2}(\Delta t)^2 B(u_i^n)^\tau\nabla^2\eta(\xi)B(u_i^n) - \Delta t\lambda_i B(u_i^n)^\tau\nabla^2\eta(\xi)\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big] =$$

$$E_{in}^{(1)} + E_{in}^{(2)} + \frac{1}{2}\lambda_i^2\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big]^\tau\nabla^2\eta(\xi)\,\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big]\big]+$$

$$\Delta t\nabla\eta(u_i^n)\cdot B(u_i^n) + \frac{1}{2}(\Delta t)^2 B(u_i^n)^\tau\nabla^2\eta(\xi)B(u_i^n) - \Delta t\lambda_i B(u_i^n)^\tau\nabla^2\eta(\xi)\big[g(u_i^n, u_{i+1}^n) - g(u_{i-1}^n, u_i^n)\big],$$

where

$$E_{in}^{(1)} = \lambda_i\nabla\eta(u_i^n)\big[g(u_i^n, u_i^n) - g(u_i^n, u_{i+1}^n)\big] + \lambda_i\big[q_h(u_i^n, u_{i+1}^n) - q_h(u_i^n, u_i^n)\big],$$

$$E_{in}^{(2)} = \lambda_i\nabla\eta(u_i^n)\big[g(u_{i-1}^n, u_i^n) - g(u_i^n, u_i^n)\big] + \lambda_i\big[q_h(u_i^n, u_i^n) - q_h(u_{i-1}^n, u_i^n)\big].$$

Define $H(a,b) := \lambda_i\nabla\eta(a)[g(a,a) - g(a,b)] + \lambda_i[q_h(a,b) - q_h(a,a)]$. If we plug (2.5) and (2.6) in H, we obtain

$$H(a,b) = \frac{1}{2}\nabla\eta(a)[G(a) - G(b)] - \frac{1}{2}\frac{\lambda_i}{\mu}\nabla\eta(a)(a-b) + \frac{\lambda_i}{2}[q(b) - q(a)] + \frac{1}{2}\frac{\lambda_i}{\mu}[\eta(a) - \eta(b)],$$

$$\nabla_b H(a,b) = -\frac{\lambda_i}{2}DG(b)^\tau\nabla\eta(a) + \frac{1}{2}\frac{\lambda_i}{\mu}[\nabla\eta(a) - \nabla\eta(b)] + \frac{\lambda_i}{2}\nabla q(b),$$

$$\nabla_b^2 H(a,b) = -\frac{1}{2}\frac{\lambda_i}{\mu}\left\{\nabla^2\eta(b) - \mu\frac{\partial}{\partial b}\big[DG(b)^\tau(\nabla\eta(b) - \nabla\eta(a))\big]\right\}.$$

According to (1.3), $H(a,a) = 0$ and $\nabla_b H(a,a) = 0$. Since η is strictly convex on the compact set S, we conclude that there exists $\mu_0 > 0$, such that $H(a,b) \leqslant -(\lambda_i/\mu)l|a-b|^2$ for all $a, b \in S$ and $0 < \mu \leqslant \mu_0$, where $l > 0$ depends on S. Consequently, from $E_{in}^{(1)} = H(u_i^n, u_{i+1}^n)$ and (A1) it follows that $E_{in}^{(1)} \leqslant -(\lambda/\mu)l|u_{i+1}^n - u_i^n|^2$.

Considering the function $\bar{H}(a,b) := \lambda_i\nabla\eta(a)\cdot[g(b,a) - g(a,a)] + \lambda[q_h(a,a) - q_h(b,a)]$ in the same manner, we arrive at $E_{in}^{(2)} \leqslant -(\lambda_i/\mu)l|u_i^n - u_{i-1}^n|^2$.

In view of

$$\left|\Delta t\lambda_i B(u_i^n)^T\nabla^2\eta(\xi)\left[g(u_i^n,u_{i+1}^n)-g(u_{i-1}^n,u_i^n)\right]\right| \leqslant \Delta t|B(u_i^n)|\gamma\tilde{L}\frac{\lambda_i}{\mu}(|u_{i+1}^n-u_i^n|+|u_i^n-u_{i-1}^n|)\leqslant$$

$$\frac{1}{2}\gamma\tilde{L}(\Delta t)^2|B(u_i^n)|^2+\frac{1}{2}\gamma\tilde{L}\Big(\frac{\lambda_i}{\mu}\Big)^2(|u_{i+1}^n-u_i^n|+|u_i^n-u_{i-1}^n|)^2,$$

where $\gamma=\sup_S\|\nabla^2\eta\|$, we have

$$E_{in}\leqslant\bigg(-l\frac{\lambda_i}{\mu}+2\gamma(\tilde{L})^2\Big(\frac{\lambda_i}{\mu}\Big)^2\bigg)(|u_{i+1}^n-u_i^n|^2+|u_i^n-u_{i-1}^n|^2)+\Delta t\nabla\eta(u_i^n)\cdot B(u_i^n)+C(\Delta t)^2|u_i^n-\bar{u}|^2\leqslant$$

$$-2\bigg(\tilde{L}\frac{\lambda_i}{\mu}\bigg)^2(|u_{i+1}^n-u_i^n|^2+|u_i^n-u_{i-1}^n|^2)+\Delta t\nabla\eta(u_i^n)\cdot B(u_i^n)+C(\Delta t)^2|u_i^n-\bar{u}|^2,$$

for $\lambda_i/\mu\leqslant l/[4(\tilde{L})^2\gamma]$. □

3. Elastodynamics system

As the first application of the results in the previous section we investigate the Cauchy problem for the elastodynamics system with a source:

$$\partial_t w-\partial_x v=0,\quad \partial_t v-\partial_x\sigma(w)=-\beta v, \tag{3.1}$$

$$w(x,0)=w_0(x),\quad v(x,0)=v_0(x)\quad (x\in\mathbb{R}). \tag{3.2}$$

Here $w:\mathbb{R}\times[0,T]\to[-1,\infty)$ stands for the stress and $v:\mathbb{R}\times[0,T]\to\mathbb{R}$ is the velocity. The nonlinear term $\sigma\in C^2(\mathbb{R})$ is supposed to satisfy

$$\sigma'(w)>0\ \ (w\in\mathbb{R}),\quad w\sigma''(w)>0\ \ (w\in\mathbb{R}\backslash\{0\}), \tag{3.3}$$

and $\beta>0$ is a constant. This system corresponds to the quasilinear wave equation with friction (see [5]). It is easy to verify that $\eta(w,v)=(1/2)v^2+\int^w\sigma(s)\,ds$ and $q(w,v)=-v\sigma(w)$ constitute a strictly convex entropy pair. For given $K>0$ let us introduce the set

$$S_K=\{(w,v)\subset[-1,\infty)\times\mathbb{R}\ :$$

$$|y(w,v)|\leqslant K,\ |z(w,v)|\leqslant K\}, \tag{3.4}$$

where $y(w,v)=-\int_0^w\sqrt{\sigma'(s)}\,ds+v$, $z(w,v)=-\int_0^w\sqrt{\sigma'(s)}\,ds-v$ are the the so-called Riemann invariants. From (3.3) it follows that S_K is a convex compact set (see Figure 3.1).

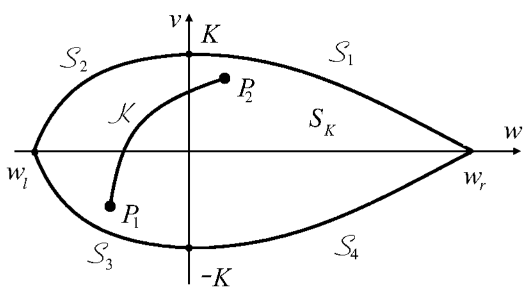

Fig. 3.1.

Define further $w_l<0$ and $w_r>0$ with

$$K=\int_{w_l}^{0}\sqrt{\sigma'}\,ds=\int_0^{w_r}\sqrt{\sigma'}\,ds. \tag{3.5}$$

The boundary of S_K may be split as $\partial S_K = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, where

$$\mathcal{S}_1 = \left\{(w,v) \,:\, w \in [0,w_r],\quad v = \varphi_1(w) := \int\limits_{w}^{w_r} \sqrt{\sigma'}\,ds\right\},$$

$$\mathcal{S}_2 = \left\{(w,v) \,:\, w \in [w_l,0],\quad v = \varphi_2(w) := \int\limits_{w_l}^{w} \sqrt{\sigma'}\,ds\right\},$$

$$\mathcal{S}_3 = \left\{(w,v) \,:\, w \in [w_l,0],\quad v = \varphi_3(w) := \int\limits_{w}^{w_l} \sqrt{\sigma'}\,ds\right\},$$

$$\mathcal{S}_4 = \left\{(w,v) \,:\, w \in [0,w_r],\quad v = \varphi_4(w) := \int\limits_{w_r}^{w} \sqrt{\sigma'}\,ds\right\}.$$

We introduce further

$$F(u_1,u) := \frac{1}{2}\big(u_1 + u\big) + \frac{\lambda}{2}\big[G(u_1) - G(u)\big], \tag{3.6}$$

where $u = (w,v)$, $u_1 = (w_1,v_1)$ and $G(u) = (-v, -\sigma(w))$. The Lax — Friedrichs scheme (1.5), (2.5) applied to (3.1) in the case where $\beta = 0$ and $\mu = 1$ has the form,

$$u_i^{n+1} = u_i^n - \frac{\lambda_i}{2}\big[G(u_{i+1}^n) - G(u_{i-1}^n)\big] + \frac{1}{2}\big(u_{i-1}^n - 2u_i^n + u_{i+1}^n\big), \tag{3.7}$$

and can be written as $u_i^{n+1} = F(u_{i-1}^n, u_{i+1}^n)$, with $\lambda = \lambda_i$. It is useful to express the mapping F by components,

$$F^1(u_1,u) = \frac{1}{2}(w_1 + w) + \frac{\lambda}{2}(v - v_1), \quad F^2(u_1,u) = \frac{1}{2}(v_1 + v) + \frac{\lambda}{2}[\sigma(w) - \sigma(w_1)]. \tag{3.8}$$

The next proposition asserts that the set S_K is invariant for the mapping F, provided

$$\lambda \sup_{u\in S_K} \sqrt{\sigma'(w)} \leqslant 1. \tag{3.9}$$

Proposition 3.1. *Suppose that* (3.9) *is satisfied. Then* $F(u_1,u) \in S_K$, *for all* $u_1, u \in S_K$.

The technique we employ here comes from [7], where a similar assertion was proven for the p-system. As the first step, we need

Lemma 3.1. *If* (3.9) *holds, then* $F(u_1,u) \in S_K$, *for all* $u_1, u \in \partial S_K$.

Proof. The proof includes the following 16 cases: $u_1 \in \mathcal{S}_i$, $u \in \mathcal{S}_j$ $(i,j \in \{1,2,3,4\})$. Due to their complexity, we shall consider here only the case where $u_1 \in \mathcal{S}_1$ and $u \in \mathcal{S}_2$. The proofs of the other cases are based upon similar ideas.

For temporarily fixed $u_1 \in \mathcal{S}_1$, $u_1 = (w_1, \varphi_1(w_1))$, $w_1 \in [0,w_r]$, let us define the functions

$$\overline{w}(w) = \frac{1}{2}(w_1 + w) + \frac{\lambda}{2}\bigg(\int\limits_{w_l}^{w} \sqrt{\sigma'}\,ds - \int\limits_{w_1}^{w_r} \sqrt{\sigma'}\,ds\bigg),$$

$$\overline{v}(w) = \frac{1}{2}\Big(\int\limits_{w_1}^{w_r} \sqrt{\sigma'}\,ds + \int\limits_{w_l}^{w} \sqrt{\sigma'}\,ds\Big) + \frac{\lambda}{2}\big(\sigma(w) - \sigma(w_1)\big), \tag{3.10}$$

for $w \in [w_l, 0]$ (compare (3.8)). Since

$$\overline{w}'(w) = \frac{1}{2} + \frac{\lambda}{2}\sqrt{\sigma'(w)} > 0,$$

then (3.10) defines a smooth curve $\mathcal{K}$ with the left end $P_1(\overline{w}(w_l), \overline{v}(w_l))$, the right end $P_2(\overline{w}(0), \overline{v}(0))$ and the slope

$$\frac{\mathrm{d}\overline{w}}{\mathrm{d}\overline{v}} = \sqrt{\sigma'(w)} > 0,$$

(see Fig. 3.1). In view of

$$\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}\overline{v}^2} = \frac{\overline{v}''\overline{w}' - \overline{v}'\overline{w}''}{\overline{w}'^3} = \frac{\sigma''}{8\overline{w}'^3}\Big(1 + \lambda\sqrt{\sigma'}\Big)\Big(\frac{1}{\sqrt{\sigma'}} + \lambda\Big) < 0,$$

for all $w \in [w_l, 0]$, we conclude that $\mathcal{K}$ is concave. Note also that due to (3.5), we have

$$\overline{w}(0) = \frac{1}{2}w_1 + \frac{\lambda}{2}\Big(\int\limits_{w_l}^{0} \sqrt{\sigma'}\,ds - \int\limits_{w_1}^{w_r} \sqrt{\sigma'}\,ds\Big) = \frac{1}{2}w_1 + \frac{\lambda}{2}\int\limits_{0}^{w_1} \sqrt{\sigma'}\,ds \geqslant 0.$$

Therefore, P_2 is located in the (w, v) — plane to the right of the v-axis. Proceeding in several steps, we will prove that $\mathcal{K} \subset S_K$, for all $w_1 \in [0, w_r]$.

1° P_2 lies below $\mathcal{S}_1$, i. e. $\overline{v}(0) \leqslant \varphi_1(\overline{w}(0))$, for all $w_1 \in [0, w_r]$. For the function

$$\psi_1(w_1) := \varphi_1(\overline{w}(0)) - \overline{v}(0) = \frac{\lambda}{2}\big[\sigma(w_1) - \sigma(0)\big] + \frac{1}{2}\Big(\int\limits_{w_r}^{w_1}\sqrt{\sigma'}\,ds + \int\limits_{w_r}^{0}\sqrt{\sigma'}\,ds\Big) + \int\limits_{\overline{w}(0)}^{w_r}\sqrt{\sigma'}\,ds,$$

we have $\psi_1(0) = 0$ and

$$\psi_1'(w_1) = \Big(\sqrt{\sigma'(w_1)} - \sqrt{\sigma'(\overline{w}(0))}\Big)\left(\frac{1}{2} + \frac{\lambda}{2}\sqrt{\sigma(w_1)}\right) \geqslant 0,$$

since from (3.9) it follows that $w_1 \geqslant \overline{w}(0) \geqslant 0$, for all $w_1 \in [0, w_r]$. Hence, 1^0.

2° P_1 lies above $\mathcal{S}_3 \cup \mathcal{S}_4$. First note that due to (3.3), the inequality

$$\overline{w}(w) - w = \underbrace{\frac{1}{2}w_1 + \frac{\lambda}{2}\int\limits_{0}^{w_1}\sqrt{\sigma'}\,ds}_{\geqslant 0} \quad \underbrace{-\frac{1}{2}w + \frac{\lambda}{2}\int\limits_{0}^{w}\sqrt{\sigma'}\,ds}_{\geqslant 0} \geqslant 0, \quad w \in [w_l, 0]$$

yields $\sqrt{\sigma'(\overline{w}(w))} \leqslant \sqrt{\sigma'(w)}$, whenever $\overline{w}(w) \leqslant 0$. Therefore, the part of the curve $\mathcal{K}$ which lies on the left side of the v − axis increases faster than $\mathcal{S}_2$.

1) case. $\overline{w}(w_l) < 0$, for all $w_1 \in [0, w_r]$. In order to verify 2°, it is necessary to prove that $\overline{v}(w_l) \geqslant \varphi_3(\overline{w}(w_l))$, for all $w_1 \in [0, w_r]$, where $\overline{w}(w_l) = (w_l + w_1)/2 + (\lambda/2)\int_{w_l}^{w_1}\sqrt{\sigma'}\,ds$.

Define the function

$$\psi_2(w_1) := \overline{v}(w_l) - \varphi_3(\overline{w}(w_l)) = \frac{1}{2}\int_{w_1}^{w_r} \sqrt{\sigma'}\,ds + \frac{\lambda}{2}\big[\sigma(w_l) - \sigma(w_1)\big] + \int_{w_l}^{\overline{w}(w_l)} \sqrt{\sigma'}\,ds.$$

From (3.9) one easily concludes that

$$\psi_2(0) \geqslant 0. \tag{3.11}$$

For the proof of the nontrivial inequality

$$\psi_2(w_r) = \frac{\lambda}{2}\big[\sigma(w_l) - \sigma(w_r)\big] + \int_{w_l}^{(w_l+w_r)/2} \sqrt{\sigma'}\,ds \geqslant 0, \tag{3.12}$$

see [10]. The equality

$$\psi_2'(w_1) = \left(\sqrt{\sigma'(\overline{w}(w_l))} - \sqrt{\sigma'(w_1)}\right)\left(\frac{1}{2} + \frac{\lambda}{2}\sqrt{\sigma(w_1)}\right)$$

implies that

$$\psi_2'(0) > 0. \tag{3.13}$$

If there existed some $w_1^* \in (0, w_r)$ such that $\psi_2(w_1^*) < 0$, then (3.11),(3.12) and (3.13) would imply that ψ_2' has at least two zeros. However, this is not possible, because

$$w_1 \mapsto \sqrt{\sigma'(\overline{w}(w_l))} - \sqrt{\sigma'(w_1)}$$

is a strictly decreasing function. Therefore, $\psi_2 \geqslant 0$ on $[0, w_r]$.

2) case. There exists $w_1^* \in (0, w_r]$, such that $\overline{w}(w_l) = 0$. Since

$$\psi_2(w_1^*) = \frac{1}{2}\int_{w_1^*}^{w_r} \sqrt{\sigma'}\,ds + \frac{\lambda}{2}\big[\sigma(w_l) - \sigma(w_1^*)\big] + \int_{w_l}^{0} \sqrt{\sigma'}\,ds =$$

$$\frac{1}{2}\int_{w_1^*}^{w_r} \sqrt{\sigma'}\,ds - \frac{\lambda}{2}\int_{w_l}^{w_1^*} \sigma'\,ds + \int_{w_l}^{0} \sqrt{\sigma'}\,ds \geqslant \frac{1}{2}\int_{w_1^*}^{w_r} \sqrt{\sigma'}\,ds + \int_{w_l}^{0} \big(\sqrt{\sigma'} - \lambda\sigma'\big)\,ds \geqslant 0,$$

then proceeding as in the 1) case, one obtains $\psi_2 \geqslant 0$ on $[0, w_1^*]$. Thus, the part of $\mathcal{K}$ to the left of v-axis lies above $\mathcal{S}_3$. In the end, it only remains to prove that the part of $\mathcal{K}$ to the right of the v-axis is located above $\mathcal{S}_4$. Introduce the function

$$\psi_3(w_1) := \overline{v}(w_l) - \varphi_4(\overline{w}(w_l)) = \frac{1}{2}\int_{w_1}^{w_r} \sqrt{\sigma'}\,ds + \frac{\lambda}{2}\big[\sigma(w_l) - \sigma(w_1)\big] + \int_{\overline{w}(w_l)}^{w_r} \sqrt{\sigma'}\,ds,$$

where $w_1 \in [w_1^*, w_r]$. Similarly to (3.12), one can prove that

$$\psi_3(w_r) = \frac{\lambda}{2}\big[\sigma(w_l) - \sigma(w_r)\big] + \int_{(w_l+w_r)/2}^{w_r} \sqrt{\sigma'}\,ds \geqslant 0.$$

Finally, the last inequality together with

$$\psi_3'(w_1) = -\Big(\sqrt{\sigma(\overline{w}(w_l))} + \sqrt{\sigma'(w_1)}\Big)\Big(\frac{1}{2} + \frac{\lambda}{2}\sqrt{\sigma'(w_1)}\Big) < 0$$

yields that $\psi_3 \geqslant 0$ on $[w_1^*, w_r]$. □

Proof of Proposition 3.1: *1) case.* Let $u_1 \in \partial S_K$ and $u = (w, v) \in S_K$ be arbitrary. If $u', u'' \in \partial S_K$ are the intersections of the line orthogonal in $(w, 0)$ to the w − axis, then Lemma 3.1 implies $F(u_1, u'), F(u_1, u'') \in S_K$. Thanks to the facts that $v \mapsto F(u_1, u)$ is a linear mapping (see (3.8)) and S_K is convex, one infers that $F(u_1, u) \in S_K$.

2) case. Applying the 1) case, one easily concludes that $F(u_1, u) \in S_K$, for arbitrary $u_1, u \in S_K$. □

Finally, we have

Theorem 3.1. *Suppose that the classical solution of* (3.1), (3.2) *meets the requirements from Assumption* 1.1, *with* $S = S_K$, $K > 0$ *from (3.4). Suppose further* $\mu \sup_{u\in S_K} \sqrt{\sigma'(w)} \leqslant 1$, $\lambda_i/\mu + \beta\Delta t \leqslant 1$, $r\lambda_i/\mu \leqslant 1$ $(i \in \mathbb{Z})$, $0 < \mu \leqslant \mu_0$, *where the constants* μ_0 *and* r *are from Proposition* 2.1. *We will also assume that the mesh satisfies* (2.1). *If* u_h *is the numerical solution of* (3.1) *given by* (1.5), (2.5), *then*

$$\int_0^T \int_{\mathbb{R}} e^{-\alpha t}|u - u_h|^2 \, dx \, dt \leqslant C(\Delta t + h), \tag{3.14}$$

where $\alpha > 0$ *is the constant from Theorem* 2.1.

Proof. For the numerical scheme (1.5), (2.5) applied to (3.1), we have

$$u_i^{n+1} = u_i^n - \frac{\lambda_i}{2}\big[G(u_{i+1}^n) - G(u_{i-1}^n)\big] + \frac{\lambda_i}{2\mu}\big(u_{i-1}^n - 2u_i^n + u_{i+1}^n\big) + \Delta t\, B(u_i^n) =$$

$$\Big(1 - \frac{\lambda_i}{\mu}\Big)\tilde{u_i^n} + \frac{\lambda_i}{\mu}\Big[u_i^n - \frac{\mu}{2}\big(G(u_{i+1}^n) - G(u_{i-1}^n)\big) + \frac{1}{2}\big(u_{i-1}^n - 2u_i^n + u_{i+1}^n\big)\Big],$$

where

$$\tilde{u_i^n} = \Big(w_i^n, \; \frac{1 - \lambda_i/\mu - \beta\Delta t}{1 - \lambda_i/\mu}\, v_i^n\Big).$$

If $u_i^n \in S_K$, then, due to the geometry of S_K, we conclude that $\tilde{u_i^n} \in S_K$. According to Proposition 3.1, the scheme (3.7) is invariant for S_K. Thus, convexity of S_K yields $u_i^{n+1} \in S_K$, provided $u_i^n \in S_K$, for all $i \in \mathbb{Z}$. Therefore, in view of Proposition 2.1, the conditions from Assumption 1.2 are fulfilled and Theorem 2.1 applies. □

4. Euler system with damping

The Cauchy problem for the polytropic perfect gas system with damping has the form

$$\partial_t\rho + \partial_x m = 0, \quad \partial_t m + \partial_x(\frac{m^2}{\rho} + p) = -\beta m, \tag{4.1}$$

$$\rho(x, 0) = \rho_0(x), \quad m(x, 0) = m_0(x) \quad (x \in \mathbb{R}). \tag{4.2}$$

where $m = \rho v$. Here $\rho > 0$ stands for the density, v for the velocity and $p(\rho) = A\rho^\gamma$ is the pressure of the gas. Thereby, $A > 0$, $\beta > 0$ and $\gamma \in (1,3)$ are constants. Let $u = (\rho, m)$, $G(u) = (m, m^2/\rho + p)$. The functions

$$\eta(u) = \frac{m^2}{2\rho} + \rho \int^{\rho} \frac{p(s)}{s^2}, \quad q(u) = \frac{m^2}{2\rho} + \rho \int^{\rho} \frac{p(s)}{s^2} + p(\rho)\frac{m}{\rho},$$

constitute a strictly convex entropy pair on $\mathfrak{U} = \{(\rho, m) : \rho > 0\}$ (see [2]). For arbitrary $K > 0$, introduce the set

$$\tilde{S}_K = \big\{(\rho, m) : \rho \geqslant 0, \quad \big|v \pm \rho^\theta\big| \leqslant K\big\}, \tag{4.3}$$

where $\theta = (\gamma - 1)/2 \in (0, 1)$. This set is convex and compact in the (ρ, m)-plane. It can be shown (see [6]), that $\tilde{S}$ is invariant with respect to (3.7), provided

$$\lambda_i \sup_{u \in \tilde{S}} \big|v \pm \rho^\theta\big| \leqslant 1. \tag{4.4}$$

Note that $\nabla^2 \eta$ is not bounded in $\tilde{S}$. Therefore, in order to pursue the programme of deriving error estimates described in the sections 1 and 2, it will be necessary to suppose that the exact and numerical solutions of (4.1) stay uniformly away from vacuum. Proceeding in the same way as in the proof of Theorem (3.1), we obtain

Theorem 4.1. *Let us assume that the classical solution u of* (4.1), (4.2) *satisfies the conditions from Assumption 1.1, with $S = \tilde{S}_K$, $K > 0$ from* (4.3). *Suppose also that*

$$\mu \sup_{u \in \tilde{S}} \big|v \pm \rho^\theta\big| \leqslant 1, \quad \frac{\lambda_i}{\mu} + \beta \Delta t \leqslant 1, \quad r\frac{\lambda_i}{\mu} \leqslant 1 \quad (i \in \mathbb{Z}), \quad 0 < \mu \leqslant \mu_0,$$

where the constants μ_0 and r are from Proposition 2.1. *Let u_h be given by* (1.5), (2.5), *provided* (2.1) *holds. If u and u_h take their values in $\{(\rho, m) : \rho \geqslant \rho_0\}$ for some $\rho_0 > 0$, then*

$$\int_0^T \int_{\mathbb{R}} e^{-\alpha t} |u - u_h|^2 \, dx \, dt \leqslant C(\Delta t + h), \tag{4.5}$$

where $\alpha > 0$ is the constant from Theorem 2.1.

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