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APPROXIMATE SOLUTION OF A SINGULAR INTEGRAL EQUATION WITH A MULTIPLICATIVE CAUCHY KERNEL IN THE HALF-PLANE

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Abstract — In this paper, Jacobi and trigonometric polynomials are used to construct the approximate solution of a singular integral equation with multiplicative Cauchy kernel in the half-plane.

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1. Introduction

Let us consider the equation of the form

$$\frac{1}{(\pi i)^2}\iint\limits_D \frac{\varphi(\sigma_1,\sigma_2)}{(\sigma_1-x)(\sigma_2-y)}\,d\sigma_1 d\sigma_2 = f(x,y), \tag{1.1}$$

where $(x,y)\in\{(x,\,y):0<x<\infty,\,-\infty<y<\infty\}$, $f(x,y)$ is a given function and $\varphi(x,y)$ is an unknown function. Let us shortly recall the explicit solution of (1.1) presented in [3]. For this purpose we introduce function classes that will be used throughout the paper.

Definition 1.1. *We say that the function $\varphi(x)$, defined on the real half-line $(x>0)$, belongs to class $h(\infty)$, if the function $\varphi^*(t)=\varphi\left(\frac{1+t}{1-t}\right)$, $t\in(-1,1)$ satisfies the inequality*

$$|\varphi^*(t')-\varphi^*(t'')|\leqslant K\,|t'-t''|^{\mu}, \tag{1.2}$$

where $K>0$, $0<\mu\leqslant 1$ are constants independent of the arrangement of the points t', t'' in each closed interval contained in $(-1,1)$, and if the function

$$\varphi^{**}(t)=\varphi^*(t)\,|t+1|^{\alpha},\quad 0\leqslant \operatorname{Re}\alpha<1,$$

satisfies the Hölder condition in a neighbourhood of the point $t=-1$, and if

$$\lim_{t\to 1-}\varphi^*(t)=\lim_{x\to\infty}\varphi(x)=0. \tag{1.3}$$

Definition 1.2. *We say that the function $\varphi(x)$, defined on the real line $(-\infty<x<\infty)$, belongs to class $h(\infty)$, if the function $\varphi^*(t)=\varphi\left(i\frac{1+t}{1-t}\right)$ of the complex variable t, $|t|=1$ satisfies inequality* (1.2).

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Definition 1.3. *We write* $\varphi(x,y) \in h(\infty) \times h(\infty)$, $x > 0$, $-\infty < y < \infty$, *if the function* $\varphi^*(t_1,t_2) = \varphi\left(\frac{1+t_1}{1-t_1}, i\frac{1+t_2}{1-t_2}\right)$, $(t_1,t_2) \in (-1,1) \times L$, $L = \{t_2 : |t_2| = 1\}$ *satisfies the inequality*

$$|\varphi^*(t_1',t_2') - \varphi^*(t_1'',t_2'')| \leqslant K_1 |t_1' - t_1''|^{\mu_1} + K_2 |t_2' - t_2''|^{\mu_2}, \quad K_1, K_2 > 0, \quad 0 < \mu_1, \mu_2 \leqslant 1, \tag{1.4}$$

in each closed domain contained in $(-1,1) \times L$, *and if the function*

$$\varphi^{**}(t_1,t_2) = \varphi^*(t_1,t_2)|t_1+1|^{\alpha}, \quad 0 \leqslant \operatorname{Re}\alpha < 1,$$

satisfies the Hölder condition with respect to both variables on $[-1,1) \times L$, *and moreover, if*

$$\lim_{t_1 \to 1-} \varphi^*(t_1,t_2) = \lim_{x\to\infty} \varphi(x,y) = 0, \quad \text{for} \quad t_2 \in L \quad \text{and} \quad y \in (-\infty,\infty). \tag{1.5}$$

Definition 1.4. *We write* $\varphi(x,y) \in h(0,\infty) \times h(\infty)$, $0 \leqslant x < \infty$, $-\infty < y < \infty$, *if the function* $\varphi^*(t_1,t_2) = \varphi\left(\frac{1+t_1}{1-t_1}, i\frac{1+t_2}{1-t_2}\right)$, $(t_1,t_2) \in [-1,1) \times L$, $L = \{t_2 : |t_2| = 1\}$, *satisfies conditions* (1.4) *and* (1.5).

Theorem 1.1 [3]. *Let* $f(x,y) \in h(0,\infty) \times h(\infty)$ *and let*

$$\lim_{|y|\to\infty} f(x,y) = 0, \quad \forall x \in [0,\infty). \tag{1.6}$$

Then each solution $\varphi(x,y)$ *of* (1.1) *in the function class* $h(\infty) \times h(\infty)$ *is given by the formula*

$$\varphi(x,y) = R(f;x,y) + C_1(x) + \frac{C_2(y)\,i}{\sqrt{x}},$$

where

$$R(f;x,y) = \frac{(x+1)(y+i)}{\sqrt{x}(\pi i)^2} \int_0^{\infty}\int_{-\infty}^{\infty} \frac{\sqrt{\sigma_1} f(\sigma_1,\sigma_2)}{(\sigma_1+1)(\sigma_2+i)(\sigma_1-x)(\sigma_2-y)} d\sigma_1 d\sigma_2$$

and $C_1(x)$, $x > 0$, *and* $C_2(y)$, $-\infty < y < \infty$ *are arbitrary functions of class* $h(\infty)$. *Additionally, the solution* $\varphi(x,y)$ *in the class of functions satisfying the conditions*

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(x,\sigma_2)}{\sigma_2+i} d\sigma_2 = p(x), \tag{1.7}$$

$$\frac{1}{\pi i}\int_{0}^{\infty} \frac{\varphi(\sigma_1,y)}{\sigma_1+1} d\sigma_1 = q(y), \tag{1.8}$$

where $p(x) \in h(\infty)$, $x > 0$, $q(y) \in h(\infty)$, $-\infty < y < \infty$ *are the functions fulfilling the relation*

$$\frac{1}{\pi i}\int_{0}^{\infty} \frac{p(\sigma_1)}{\sigma_1+1} d\sigma_1 = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{q(\sigma_2)}{\sigma_2+i} d\sigma_2 = \omega, \tag{1.9}$$

is given by the formula

$$\varphi(x,y) = R(f;x,y) - p(x) + \frac{q(y)\,i}{\sqrt{x}} + \frac{\omega i}{\sqrt{x}}.$$

Theorem 1.2 [3]. *Let $f(x,y) \in h(0,\infty) \times h(\infty)$ satisfies condition (1.6). Then the solution $\varphi(x,y)$ of (1.1) in the function class $h(0,\infty) \times h(\infty)$, satisfying the relations*

$$\varphi(x,\infty) = 0, \quad x \in [0,\infty), \tag{1.10}$$

and

$$\frac{1}{\pi i}\int_0^\infty \frac{\varphi(\sigma_1,y)}{\sigma_1+1}d\sigma_1 = \frac{i}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}(\sigma_1+1)(\sigma_2-y)}, \tag{1.11}$$

is given by the following formula

$$\varphi(x,y) = \frac{\sqrt{x}}{(\pi i)^2}\int_0^\infty\int_{-\infty}^\infty \frac{f(\sigma_1,\sigma_2)\,d\sigma_1 d\sigma_2}{\sqrt{\sigma_1}(\sigma_1-x)(\sigma_2-y)}.$$

In the literature [1,2,4,6] the methods of approximate solution of equation (1.1) are well-known in the case where D is bounded. In this paper, we present a method of approximate solution of (1.1) in the function classes $h(\infty) \times h(\infty)$ and $h(0,\infty) \times h(\infty)$ based on Jacobi and trigonometric polynomials.

2. Approximate solution in the function class $h(\infty) \times h(\infty)$

Using the identities

$$\frac{1}{\sigma_1-x} = \frac{x+1}{\sigma_1+1}\frac{1}{\sigma_1-x} + \frac{1}{\sigma_1+1}, \qquad \frac{1}{\sigma_2-y} = \frac{y+i}{\sigma_2+i}\frac{1}{\sigma_2-y} + \frac{1}{\sigma_2+i}$$

and substitutions

$$\sigma_1 = \frac{1+\tau_1}{1-\tau_1}, \quad x = \frac{1+t_1}{1-t_1}, \quad \sigma_2 = i\frac{1+\tau_2}{1-\tau_2}, \quad y = i\frac{1+t_2}{1-t_2}, \quad \tau_1, t_1 \in (-1,1), \quad \tau_2, t_2 \in L,$$

we can rewrite equation (1.1) in the form

$$\frac{1}{(\pi i)^2}\int_{-1}^1\int_L \frac{\varphi^*(\tau_1,\tau_2)}{(\tau_1-t_1)(\tau_2-t_2)}\,d\tau_1 d\tau_2 - \frac{1}{(\pi i)^2}\int_{-1}^1\int_L \frac{\varphi^*(\tau_1,\tau_2)}{(\tau_1-t_1)(\tau_2-1)}\,d\tau_1 d\tau_2 -$$

$$\frac{1}{(\pi i)^2}\int_{-1}^1\int_L \frac{\varphi^*(\tau_1,\tau_2)}{(\tau_1-1)(\tau_2-t_2)}\,d\tau_1 d\tau_2 + \frac{1}{(\pi i)^2}\int_{-1}^1\int_L \frac{\varphi^*(\tau_1,\tau_2)}{(\tau_1-1)(\tau_2-1)}\,d\tau_1 d\tau_2 = f^*(t_1,t_2), \tag{2.1}$$

where $\varphi^*(\tau_1,\tau_2) = \varphi\left(\frac{1+\tau_1}{1-\tau_1}, i\frac{1+\tau_2}{1-\tau_2}\right)$, $f^*(t_1,t_2) = f\left(\frac{1+t_1}{1-t_1}, i\frac{1+t_2}{1-t_2}\right)$. In a similar way, conditions (1.7)–(1.9) can be rewritten as

$$\frac{1}{\pi i}\int_L \frac{\varphi^*(t_1,\tau_2)}{1-\tau_2}d\tau_2 = p^*(t_1), \tag{2.2}$$

$$\frac{1}{\pi i}\int_{-1}^1 \frac{\varphi^*(\tau_1,t_2)}{1-\tau_1}d\tau_1 = q^*(y), \tag{2.3}$$

$$\frac{1}{\pi i}\int_{-1}^{1}\frac{p^*(\tau_1)}{1-\tau_1}d\tau_1=\frac{1}{\pi i}\int_{L}\frac{q^*(\tau_2)}{1-\tau_2}d\tau_2=\omega, \tag{2.4}$$

where $p^*(t_1)=p\left(\frac{1+t_1}{1-t_1}\right)$, $q^*(t_2)=q\left(i\frac{1+t_2}{1-t_2}\right)$.

Let us introduce a new unknown function $u(t_1,t_2)$ by the relation

$$\varphi^*(t_1,t_2)=\sqrt{\frac{1-t_1}{1+t_1}}u(t_1,t_2),\quad t_1\in(-1,1),\quad t_2\in L=\{t_2:|t_2|=1\}. \tag{2.5}$$

By using (2.5), equation (1.1) has the form

$$\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{L}\sqrt{\frac{1-t_1}{1+t_1}}\frac{u(\tau_1,\tau_2)\,d\tau_1d\tau_2}{(\tau_1-t_1)(\tau_2-t_2)}-\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{L}\sqrt{\frac{1-t_1}{1+t_1}}\frac{u(\tau_1,\tau_2)\,d\tau_1d\tau_2}{(\tau_1-t_1)(\tau_2-1)}-$$

$$\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{L}\sqrt{\frac{1-t_1}{1+t_1}}\frac{u(\tau_1,\tau_2)\,d\tau_1d\tau_2}{(\tau_1-1)(\tau_2-t_2)}+\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{L}\sqrt{\frac{1-t_1}{1+t_1}}\frac{u(\tau_1,\tau_2)\,d\tau_1d\tau_2}{(\tau_1-1)(\tau_2-1)}=f^*(t_1,t_2). \tag{2.6}$$

Moreover, conditions (2.2) and (2.3) can be expressed as follows:

$$\frac{1}{\pi i}\int_{L}\sqrt{\frac{1-t_1}{1+t_1}}\frac{u(t_1,\tau_2)}{1-\tau_2}d\tau_2=p^*(t_1), \tag{2.7}$$

$$\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{u(\tau_1,t_2)}{1-\tau_1}d\tau_1=q^*(t_2). \tag{2.8}$$

By using (2.7) and (2.8), from (2.4) we obtain

$$\frac{1}{(\pi i)^2}\int_{-1}^{1}\int_{L}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{u(\tau_1,\tau_2)}{(1-\tau_1)(1-\tau_2)}d\tau_1d\tau_2=\omega. \tag{2.9}$$

Let us introduce a new function $p^{**}(t_1)$ by the relation

$$p^*(t_1)=\sqrt{\frac{1-t_1}{1+t_1}}\,p^{**}(t_1).$$

Then (2.7) can be transformed into the equality

$$\frac{1}{\pi i}\int_{L}\frac{u(t_1,\tau_2)}{1-\tau_2}d\tau_2=p^{**}(t_1).$$

Now we approximate the given function $f^*(t_1,t_2)$ by the interpolating polynomial $f^*_{n,n}(t_1,t_2)$ of the form (cf. [5])

$$f^*(t_1,t_2)\approx\frac{2}{n+1}\sum_{i=0}^{n}{}'\left(\sum_{p=0}^{n}T_i(t_{1,p})\,f(t_{1,p},t_2)\right)T_i(t_1)\approx$$

$$\frac{2}{n+1}\sum_{i=0}^{n}{}'\left(\sum_{p=0}^{n}T_i\left(t_{1,p}\right)\sum_{j=-n}^{n}\left(\frac{1}{2n+1}\sum_{r=0}^{2n}t_{2,r}^{-j}f\left(t_{1,p},t_{2,r}\right)t_2^j\right)\right)T_i\left(t_1\right)=f_{n,n}^*\left(t_1,t_2\right), \quad (2.10)$$

where $T_i\left(t_1\right)$, $i=0,\ 1,\ \ldots,\ n$ are Chebyshev polynomials of the first kind, the points $t_{1,p}=\cos\frac{(2p+1)\pi}{2(n+1)}$, $p=0,1,\ldots,n$ are Chebyshev nodes, and $t_{2,r}=e^{is_r}$, where $s_r=(2\pi r)/(2n+1)$, $r=0,\ 1,\ \ldots,\ 2n$.

Now we express the Chebyshev polynomial $T_i\left(t_1\right)$ in terms of Jacobi polynomials [7], i.e.

$$T_i\left(t_1\right)=\sum_{l=0}^{i}\rho_{i,l}P_l^{(-\alpha,-\beta)}\left(t_1\right). \quad (2.11)$$

Let us denote by $p\left(t_1\right)=\left(1-t_1\right)^{\alpha}\left(1+t_1\right)^{\beta}$ the weight function of the Jacobi polynomial $P_n^{(\alpha,\beta)}\left(t_1\right)$ and by $q\left(t_1\right)=\left(1-t_1\right)^{-\alpha}\left(1+t_1\right)^{-\beta}$ the weight function of the Jacobi polynomial $P_n^{(-\alpha,-\beta)}\left(t_1\right)$.

Lemma 2.1. *The following identities hold:*

$$\frac{1}{\pi}\int_{-1}^{1}q(\tau_1)P_l^{(-\alpha,-\beta)}(\tau_1)T_i(\tau_1)d\tau_1=\frac{-1}{\sin\alpha\pi}\operatorname*{Res}_{z=\infty}\left\{(z-1)^{-\alpha}(z+1)^{-\beta}P_l^{(-\alpha,-\beta)}(z)T_i(z)\right\}, \quad (2.12)$$

$$\frac{1}{\pi}\int_{-1}^{1}q\left(\tau_1\right)[P_l^{(-\alpha,-\beta)}(\tau_1)]^2d\tau_1=\frac{-1}{\sin\alpha\pi}\operatorname*{Res}_{z=\infty}\left\{(z-1)^{-\alpha}(z+1)^{-\beta}[P_l^{(-\alpha,-\beta)}(z)]^2\right\}, \quad (2.13)$$

$$\frac{1}{\pi}\int_{-1}^{1}p(\tau_1)P_l^{(\alpha,\beta)}(\tau_1)T_i(\tau_1)d\tau_1=\frac{1}{\sin\alpha\pi}\operatorname*{Res}_{z=\infty}\left\{(z-1)^{\alpha}\left(z+1\right)^{\beta}P_l^{(\alpha,\beta)}\left(z\right)T_i\left(z\right)\right\}, \quad (2.14)$$

$$\frac{1}{\pi}\int_{-1}^{1}p\left(\tau_1\right)[P_l^{(\alpha,\beta)}\left(\tau_1\right)]^2d\tau_1=\frac{1}{\sin\alpha\pi}\operatorname*{Res}_{z=\infty}\left\{(z-1)^{\alpha}\left(z+1\right)^{\beta}[P_l^{(\alpha,\beta)}\left(z\right)]^2\right\}. \quad (2.15)$$

Proof. First we prove (2.12). For this purpose, we apply the following formula:

$$\frac{1}{2\pi i}\int_{\Lambda}\left(\zeta-1\right)^{-\alpha}\left(\zeta+1\right)^{-\beta}P_l^{(-\alpha,-\beta)}\left(\zeta\right)T_i\left(\zeta\right)d\zeta=$$

$$\operatorname*{Res}_{z=\infty}\left\{\left(z-1\right)^{-\alpha}\left(z+1\right)^{-\beta}P_l^{(-\alpha,-\beta)}\left(z\right)T_i\left(z\right)\right\}, \quad (2.16)$$

where Λ is a closed contour surrounding the interval $[-1,\ 1]$. Deforming Λ into the interval, equation (2.16) can be rewritten in the following form:

$$\frac{1}{2\pi i}\int_{-1}^{1}\left(e^{-\alpha\pi i}-e^{\alpha\pi i}\right)\left(1-\tau_1\right)^{-\alpha}\left(1+\tau_1\right)^{-\beta}P_l^{(-\alpha,-\beta)}\left(\tau_1\right)T_i\left(\tau_1\right)d\tau_1=$$

$$\operatorname*{Res}_{z=\infty}\left\{\left(z-1\right)^{-\alpha}\left(z+1\right)^{-\beta}P_l^{(-\alpha,-\beta)}\left(z\right)T_i\left(z\right)\right\}, \quad (2.17)$$

from which we get (2.12). The proofs of (2.13)–(2.15) are similar to the proof of (2.12). $\square$

Multiplying both sides of (2.11) by $(1/\pi)q(t_1)P_s^{(-\alpha,-\beta)}(t_1)$ and integrating over the interval $[-1,1]$, we obtain

$$\frac{1}{\pi}\int_{-1}^{1} q(\tau_1)P_s^{(-\alpha,-\beta)}(\tau_1)T_i(\tau_1)\,d\tau_1=\sum_{l=0}^{i}\rho_{i,l}\frac{1}{\pi}\int_{-1}^{1} q(\tau_1)P_l^{(-\alpha,-\beta)}(\tau_1)P_s^{(-\alpha,-\beta)}(\tau_1)\,d\tau_1.$$

Since (cf. [7]), we have

$$\frac{1}{\pi}\int_{-1}^{1} q(\tau_1)P_n^{(-\alpha,-\beta)}(\tau_1)P_m^{(-\alpha,-\beta)}(\tau_1)\,d\tau_1=0,\quad n\neq m,$$

it follows that

$$\rho_{i,l}=\frac{\frac{1}{\pi}\int_{-1}^{1} q(\tau_1)P_l^{(-\alpha,-\beta)}(\tau_1)T_i(\tau_1)\,d\tau_1}{\frac{1}{\pi}\int_{-1}^{1} q(\tau_1)[P_l^{(-\alpha,-\beta)}(\tau_1)]^2d\tau_1}=\frac{\operatorname*{Res}_{z=\infty}\left\{(z-1)^{-\alpha}(z+1)^{-\beta}P_l^{(-\alpha,-\beta)}(z)T_i(z)\right\}}{\operatorname*{Res}_{z=\infty}\left\{(z-1)^{-\alpha}(z+1)^{-\beta}[P_l^{(-\alpha,-\beta)}(z)]^2\right\}}.$$

Using (2.11), we can rewrite the polynomial (2.10) in the form

$$f_{n,n}^{*}(t_1,t_2)=\sum_{k=0}^{n}\sum_{j=-n}^{n} f_{k,j}P_k^{(-\alpha,-\beta)}(t_1)\,t_2^{j},\tag{2.18}$$

where

$$f_{0,j}=\frac{2}{(n+1)(2n+1)}\sum_{i=0}^{n}{}'\sum_{p=0}^{n}T_i(t_{1,p})\sum_{r=0}^{2n}t_{2,r}^{-j}f(t_{1,p},t_{2,r})\,\rho_{i,0},$$

$$f_{k,j}=\frac{2}{(n+1)(2n+1)}\sum_{i=k}^{n}\sum_{p=0}^{n}T_i(t_{1,p})\sum_{r=0}^{2n}t_{2,r}^{-j}f(t_{1,p},t_{2,r})\,\rho_{i,k},\quad k\neq 0.$$

Next we approximate the function $p^{**}(t_1)$ by the interpolating polynomial $p_n^{*}(t_1)$ of the form

$$p^{**}(t_1)\approx p_n^{*}(t_1)=\frac{2}{n+1}\sum_{i=0}^{n}{}'\left(\sum_{p=0}^{n}T_i(t_{1,p})\,p^{**}(t_{1,p})\right)T_i(t_1),\tag{2.19}$$

and, as before, we express the Chebyshev polynomial $T_i(t_1)$ in the terms of Jacobi polynomials in the following way:

$$T_i(t_1)=\sum_{l=0}^{i}\rho'_{i,l}P_l^{(\alpha,\beta)}(t_1),\tag{2.20}$$

where

$$\rho'_{i,l}=\frac{\frac{1}{\pi}\int_{-1}^{1} p(\tau_1)P_l^{(\alpha,\beta)}(\tau_1)T_i(\tau_1)\,d\tau_1}{\frac{1}{\pi}\int_{-1}^{1} p(\tau_1)[P_l^{(\alpha,\beta)}(\tau_1)]^2d\tau_1}=\frac{\operatorname*{Res}_{z=\infty}\left\{(z-1)^{\alpha}(z+1)^{\beta}P_l^{(\alpha,\beta)}(z)T_i(z)\right\}}{\operatorname*{Res}_{z=\infty}\left\{(z-1)^{\alpha}(z+1)^{\beta}[P_l^{(\alpha,\beta)}(z)]^2\right\}}.$$

The polynomial (2.19) can be rewritten as

$$p_n^{**}(t_1) = \sum_{k=0}^{n} p_k P_k^{(\alpha,\beta)}(t_1), \tag{2.21}$$

where

$$p_0 = \frac{2}{n+1} \sum_{i=0}^{n}{}' \left(\sum_{p=0}^{n} T_i(t_{1,p})\, p^{**}(t_{1,p}) \right) \rho'_{i0},$$

$$p_k = \frac{2}{n+1} \sum_{i=k}^{n} \left(\sum_{p=0}^{n} T_i(t_{1,p})\, p^{**}(t_{1,p}) \right) \rho'_{ik}, \quad k = 1, 2, \ldots, n.$$

Now we approximate the function $q^*(t_2)$ by a trigonometric polynomial $q_n^*(t_2)$ of the form

$$q^*(t_2) \approx q_n^*(t_2) = \sum_{j=-n}^{n} q_j t_2^j,$$

where

$$q_j = \frac{1}{2n+1} \sum_{r=0}^{2n} t_{2,r}^{-j} q^*(t_{2,r}), \quad j = -n, \ldots, 0, \ldots, n.$$

The approximate solution

$$u(t_1, t_2) \approx u_{n,n}(t_1, t_2) = \sum_{k=0}^{n} \sum_{j=-n}^{n} c_{kj} P_k^{(\alpha,\beta)}(t_1)\, t_2^j, \tag{2.22}$$

of (2.6)–(2.9) is defined as a solution of the following problem:

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-t_1}{1+t_1}} \frac{u_{n,n}(\tau_1,\tau_2)\, d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - t_2)} - \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-t_1}{1+t_1}} \frac{u_{n,n}(\tau_1,\tau_2)\, d\tau_1 d\tau_2}{(\tau_1 - t_1)(\tau_2 - 1)} -$$

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-t_1}{1+t_1}} \frac{u_{n,n}(\tau_1,\tau_2)\, d\tau_1 d\tau_2}{(\tau_1 - 1)(\tau_2 - t_2)} + \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-t_1}{1+t_1}} \frac{u_{n,n}(\tau_1,\tau_2)\, d\tau_1 d\tau_2}{(\tau_1 - 1)(\tau_2 - 1)} =$$

$$f_{n,n}^*(t_1, t_2) - f_{n,n}^*(t_1, 1) - f_{n,n}^*(1, t_2) + f_{n,n}^*(1, 1), \tag{2.23}$$

$$\frac{1}{\pi i} \int_{L} \sqrt{\frac{1-t_1}{1+t_1}} \frac{u_{n,n}(t_1,\tau_2)}{1-\tau_2} d\tau_2 = p_n^*(t_1), \tag{2.24}$$

$$\frac{1}{\pi i} \int_{-1}^{1} \sqrt{\frac{1-\tau_1}{1+\tau_1}} \frac{u_{n,n}(\tau_1,t_2)}{1-\tau_1} d\tau_1 = q_n^*(t_2), \tag{2.25}$$

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-\tau_1}{1+\tau_1}} \frac{u_{n,n}(\tau_1,\tau_2)}{(1-\tau_1)(1-\tau_2)} d\tau_1 d\tau_2 = \omega. \tag{2.26}$$

Putting $\alpha=-\beta=1/2$ into (2.11), (2.20) and (2.22) and substituting these formulae into (2.23)–(2.26), we obtain the following equalities

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}c_{kj}\Bigg\{\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\tau_1)}{\tau_1-t_1}d\tau_1\frac{1}{\pi i}\int_L\frac{\tau_2^j}{\tau_2-t_2}d\tau_2-$$

$$\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{(1/2,-1/2)}(\tau_1)}{\tau_1-t_1}d\tau_1\frac{1}{\pi i}\int_L\frac{\tau_2^j}{\tau_2-1}d\tau_2-\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{(1/2,-1/2)}(\tau_1)}{\tau_1-1}d\tau_1\times$$

$$\frac{1}{\pi i}\int_L\frac{\tau_2^j}{\tau_2-t_2}d\tau_2+\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{(1/2,-1/2)}(\tau_1)}{\tau_1-1}d\tau_1\frac{1}{\pi i}\int_L\frac{\tau_2^j}{\tau_2-1}d\tau_2\Bigg\}=$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}f_{kj}\Bigg\{P_k^{(-1/2,1/2)}(t_1)\,t_2^j-P_k^{(-1/2,1/2)}(t_1)-P_k^{(-1/2,1/2)}(1)\,t_2^j+P_k^{(-1/2,1/2)}(1)\Bigg\},\quad(2.27)$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}c_{kj}P_k^{(1/2,-1/2)}(t_1)\frac{1}{\pi i}\int_L\frac{\tau_2^j}{1-\tau_2}=\sum_{k=0}^{n}p_kP_k^{(1/2,-1/2)}(t_1),\quad(2.28)$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}c_{kj}\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{(1/2,-1/2)}(\tau_1)}{1-\tau_1}d\tau_1t_2^j=\sum_{j=-n}^{n}q_jt_2^j,\quad(2.29)$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}c_{kj}\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{P_k^{(1/2,-1/2)}(\tau_1)}{\tau_1-1}d\tau_1\frac{1}{\pi i}\int_L\frac{\tau_2^j}{(\tau_2-1)}d\tau_2=\omega,\quad(2.30)$$

from which one can determine the coefficients c_{kj}, $k=0,1,\dots,n$, $j=-n,\dots,0,\dots,n$.

Since (cf. [6])

$$\frac{1}{\pi}\int_{-1}^{1}p(\tau_1)\frac{P_k^{(\alpha,\beta)}(\tau_1)}{\tau_1-t_1}d\tau_1=\cot\alpha\pi P_k^{(\alpha,\beta)}(t_1)\,p(t_1)-\frac{1}{\sin\alpha\pi}2^{\alpha+\beta}P_{k+\alpha+\beta}^{(-\alpha,-\beta)}(t_1),\quad(2.31)$$

the formula (2.27) has the form

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}c_{kj}\frac{-1}{i}\operatorname{sgn}(j)\Big\{P_k^{(-1/2,1/2)}(t_1)t_2^j-P_k^{(-1/2,1/2)}(t_1)-P_k^{(-1/2,1/2)}(1)t_2^j+P_k^{(-1/2,1/2)}(1)\Big\}=$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}f_{kj}\Big\{P_k^{(-1/2,1/2)}(t_1)\,t_2^j-P_k^{(-1/2,1/2)}(t_1)-P_k^{(-1/2,1/2)}(1)\,t_2^j+P_k^{(-1/2,1/2)}(1)\Big\}.\quad(2.32)$$

From (2.32) we get

$$c_{kj}=-i\operatorname{sgn}(j)f_{kj},\quad k=1,\dots,n,\quad j=-n,\dots,-1,1,\dots,n.$$

Next, using (2.28), (2.29) and (2.30), we obtain

$$c_{k0}=-p_k-\sum_{j=-n}^{-1}\operatorname{sgn}(j)\,c_{kj}-\sum_{j=1}^{n}\operatorname{sgn}(j)\,c_{kj},\quad k=1,2,\dots,n,$$

$$c_{0j} = iq_j - \sum_{k=1}^{n} c_{kj} P_k^{(-1/2,1/2)}(1) = iq_j - \sum_{k=1}^{n} c_{kj} \frac{\Gamma(k+1/2)}{k!\Gamma(1/2)}, \quad j = -n, \dots, -1, 1, \dots, n.$$

$$c_{00} = -i\omega - \sum_{k=1}^{n} \sum_{j=-n}^{n} c_{kj} P_k^{(-1/2,1/2)}(1) \operatorname{sgn}(j) + \sum_{j=-n}^{-1} c_{0j} - \sum_{j=1}^{n} c_{0j}.$$

3. Approximate solution in the function class $h(0,\infty) \times h(\infty)$

As in the previous section, equation (1.1) can be rewritten in the form (2.1). If we introduce a new unknown function $u(t_1, t_2)$ by the relation

$$\varphi^*(t_1, t_2) = \sqrt{1 - t_1^2}\, u(t_1, t_2). \tag{3.1}$$

then substituting (3.1) into (2.1), (1.10) and (1.11), we get

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 d\tau_2 - \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - 1)} d\tau_1 d\tau_2 -$$

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u(\tau_1, \tau_2)}{(\tau_1 - 1)(\tau_2 - t_2)} d\tau_1 d\tau_2 + \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u(\tau_1, \tau_2)}{(\tau_1 - 1)(\tau_2 - 1)} d\tau_1 d\tau_2 = f^*(t_1, t_2), \tag{3.2}$$

$$u(t_1, 1) = 0, \tag{3.3}$$

$$\frac{1}{\pi i} \int_{-1}^{1} \frac{\sqrt{1-\tau_1^2} u(\tau_1, t_2)}{1 - \tau_1} d\tau_1 = \frac{i(1 - t_2)}{(\pi i)^2} \int_{-1}^{1} \int_{L} \sqrt{\frac{1-\tau_1}{1+\tau_1}} \frac{f^*(\tau_1, \tau_2)}{(1-\tau_1)(1-\tau_2)(\tau_2 - t_2)} d\tau_1 d\tau_2. \tag{3.4}$$

Now let us construct a numerical problem corresponding to (3.2)–(3.4). We approximate the given function $f^*(t_1, t_2)$ by the interpolating polynomial $f^*_{n,n}(t_1, t_2)$ of form (2.10), and next we transform it into (2.18).

The approximate solution

$$u(t_1, t_2) \approx u_{n-1,n}(t_1, t_2) = \sum_{k=0}^{n-1} \sum_{j=-n}^{n} c_{kj} P_k^{(\alpha,\beta)}(t_1)\, t_2^j \tag{3.5}$$

of (3.2)–(3.4) is defined as a solution of the following problem:

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u_{n-1,n}(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - t_2)} d\tau_1 d\tau_2 - \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u_{n-1,n}(\tau_1, \tau_2)}{(\tau_1 - t_1)(\tau_2 - 1)} d\tau_1 d\tau_2 -$$

$$\frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u_{n-1,n}(\tau_1, \tau_2)}{(\tau_1 - 1)(\tau_2 - t_2)} d\tau_1 d\tau_2 + \frac{1}{(\pi i)^2} \int_{-1}^{1} \int_{L} \frac{\sqrt{1-\tau_1^2} u_{n-1,n}(\tau_1, \tau_2)}{(\tau_1 - 1)(\tau_2 - 1)} d\tau_1 d\tau_2 =$$

$$G^*_{n,n}(t_1, t_2) - G^*_{n,n}(t_1, 1) - G^*_{n,n}(1, t_2) + G^*_{n,n}(1, 1), \tag{3.6}$$

$$u_{n-1,n}(t_1, 1) = 0, \tag{3.7}$$

$$\frac{1}{\pi i}\int\limits_{-1}^{1}\frac{\sqrt{1-\tau_1^2}u_{n-1,n}(\tau_1,t_2)}{1-\tau_1}d\tau_1=\frac{i(1-t_2)}{(\pi i)^2}\int\limits_{-1}^{1}\int\limits_{L}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{f^*_{n,n}(\tau_1,\tau_2)+Q^*(\tau_2)}{(1-\tau_1)(1-\tau_2)(\tau_2-t_2)}d\tau_1 d\tau_2, \quad (3.8)$$

where

$$G^*_{n,n}(t_1,t_2)=f^*_{n,n}(t_1,t_2)+Q^*_n(t_2).$$

Here the polynomial

$$Q^*_n(t_2)=\sum_{j=-n}^{n}q^*_j t_2^j$$

should satisfy (1.11). Note that the unknown coefficients c_{kj} are independent of $Q^*_n(t_2)$.

Putting $\alpha=\beta=1/2$ into (2.18) and (3.5) and substituting these formulae into (3.6)–(3.8), we get

$$\sum_{k=0}^{n-1}\sum_{j=-n}^{n}c_{kj}\Bigg\{\frac{1}{\pi i}\int\limits_{-1}^{1}\sqrt{1-\tau_1^2}\frac{P_k^{(1/2,1/2)}(\tau_1)}{\tau_1-t_1}d\tau_1\frac{1}{\pi i}\int\limits_{L}\frac{\tau_2^j}{\tau_2-t_2}d\tau_2-$$

$$\frac{1}{\pi i}\int\limits_{-1}^{1}\sqrt{1-\tau_1^2}\frac{P_k^{(1/2,1/2)}(\tau_1)}{\tau_1-t_1}d\tau_1\frac{1}{\pi i}\int\limits_{L}\frac{\tau_2^j}{\tau_2-1}d\tau_2-\frac{1}{\pi i}\int\limits_{-1}^{1}\sqrt{1-\tau_1^2}\frac{P_k^{(1/2,1/2)}(\tau_1)}{\tau_1-1}d\tau_1\frac{1}{\pi i}\int\limits_{L}\frac{\tau_2^j}{\tau_2-t_2}d\tau_2+$$

$$\frac{1}{\pi i}\int\limits_{-1}^{1}\sqrt{1-\tau_1^2}\frac{P_k^{(1/2,1/2)}(\tau_1)}{\tau_1-1}d\tau_1\frac{1}{\pi i}\int\limits_{L}\frac{\tau_2^j}{\tau_2-1}d\tau_2\Bigg\}=$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}f_{kj}\Big\{P_k^{(-1/2,-1/2)}(\tau_1)t_2^j-P_k^{(-1/2,-1/2)}(\tau_1)-P_k^{(-1/2,-1/2)}(1)t_2^j+P_k^{(-1/2,-1/2)}(1)\Big\}, \quad (3.9)$$

$$\sum_{k=0}^{n-1}\sum_{j=-n}^{n}c_{kj}P_k^{(1/2,1/2)}(t_1)=0, \quad (3.10)$$

$$\sum_{k=0}^{n-1}\sum_{j=-n}^{n}\frac{c_{kj}}{\pi i}\int\limits_{-1}^{1}\frac{\sqrt{1-\tau_1^2}P_k^{(1/2,1/2)}(\tau_1)\,t_2^j}{1-\tau_1}d\tau_1=$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n}\frac{i(1-t_2)}{(\pi i)^2}\int\limits_{-1}^{1}\int\limits_{L}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{f_{kj}P_k^{(-1/2,-1/2)}(\tau_1)\,\tau_2^j}{(1-\tau_1)(1-\tau_2)(\tau_2-t_2)}d\tau_1 d\tau_2+$$

$$\sum_{j=-n}^{n}\frac{i(1-t_2)}{(\pi i)^2}\int\limits_{-1}^{1}\int\limits_{L}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{q_j^*\tau_2^j}{(1-\tau_1)(1-\tau_2)(\tau_2-t_2)}d\tau_1 d\tau_2. \quad (3.11)$$

Taking into account (2.31), it follows from from (3.9) that

$$\sum_{k=0}^{n-1}\sum_{j=-n}^{n}c_{kj}\frac{-2}{i}\operatorname{sgn}(j)\Big\{P_k^{(-1/2,-1/2)}(\tau_1)t_2^j-P_k^{(-1/2,-1/2)}(\tau_1)-P_k^{(-1/2,-1/2)}(1)t_2^j+P_k^{(-1/2,-1/2)}(1)\Big\}=$$

$$\sum_{k=0}^{n}\sum_{j=-n}^{n} f_{kj}\left\{P_k^{(-1/2,-1/2)}(\tau_1)\,t_2^j - P_k^{(-1/2,-1/2)}(\tau_1) - P_k^{(-1/2,-1/2)}(1)\,t_2^j + P_k^{(-1/2,-1/2)}(1)\right\}.$$

Hence we get

$$c_{kj} = \frac{-i}{2}\operatorname{sgn}(j)\, f_{k+1,j}, \quad k = 0, 1, \ldots, n-1, \quad j = -n, \ldots, -1, 1, \ldots, n.$$

Next we find the coefficients c_{k0} from condition (3.10):

$$c_{k0} = -\sum_{\substack{j=-n \\ j\neq 0}}^{n} c_{kj}, \quad k = 0, 1, \ldots, n-1.$$

Now let us find the coefficients of $Q_n^*(t_2)$. Using the formula

$$\frac{1}{\pi i}\int_{-1}^{1}\sqrt{\frac{1-\tau_1}{1+\tau_1}}\frac{\tau_1^k}{\tau_1 - t_1}d\tau_1 = i\left(t_1^k + p_1 t_1^{k-1} + \ldots + p_k\right), \quad k = 0, 1, \ldots,$$

where $p_1, p_2, \ldots, p_k$ are the coefficients in the identity

$$\sqrt{\frac{z-1}{z+1}}z^k = z^k\left(1 + \frac{p_1}{z} + \frac{p_2}{z^2} + \ldots\right),$$

we conclude from (3.11) that

$$q_j^* = -\operatorname{sgn}(j)\sum_{k=0}^{n-1} 2ic_{kj}P_{k+1}^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1) - \sum_{k=0}^{n} f_{kj}\sum_{r=0}^{n} d_r^{(k)}\sum_{l=0}^{r} p_l,\ j = -n, \ldots, n,$$

where $d_r^{(k)}$ are the coefficients of the Jacobi polynomial $P_k^{(-1/2,-1/2)}(t_1)$ written in the form

$$P_k^{(-1/2,-1/2)}(t_1) = \sum_{r=0}^{k} d_r^{(k)} t_1^r.$$

4. Numerical experiments

Let $f(x,y)$, $p(x)$, $q(y)$ be given by the following formulae:

$$f(x,y) = \frac{1}{x+2}\frac{1}{y+1+i}, \quad p(x) = 0, \quad q(y) = \frac{1}{y+1+i}.$$

Then the solution of (1.1) in the function class $h(\infty)\times h(\infty)$ is given by the formula

$$\varphi(x,y) = \frac{i\sqrt{2}\,(x+1)}{\sqrt{x}\,(x+2)\,(y+1+i)}.$$

The values of the exact and the approximate solution of (1.1), for $n = 20$, are presented in Table 4.1.

Moreover, the solution of (1.1) in the function class $h(0,\infty)\times h(\infty)$ has the following form:

$$\varphi(x,y) = \frac{i\sqrt{2}}{2}\frac{\sqrt{x}}{x+2}\frac{1}{y+i+1},$$

and the values of the exact and the approximate solution of (1.1), for $n = 20$, are presented in Table 4.2.

Table 4.1. **Comparison of the values of the exact and the approximate solution of (1.1) in the function class $h(\infty) \times h(\infty)$**

x	y	$\varphi(x,y)$	$\varphi_{n,n}(x,y)$
0.001	0.001	$11.1747469295 + 11.1859216764i$	$11.1747922809 + 11.1859670483i$
1	1	$0.1885618083 + 0.3771236166i$	$0.1885621474 + 0.3771242961i$
100	−100	$0.0000142864 - 0.0014143494i$	$0.0000142866 - 0.0014143579i$
0.1	1000	$0.0000023379 + 0.0023402049i$	$0.0000023408 + 0.0023402303i$
5000	−0.1	$0.0110475147 + 0.0099427632i$	$0.0110465183 + 0.0099418664i$

Table 4.2. **Comparison of the values of the exact and the approximate solution of (1.1) in the function class $h(0,\infty) \times h(\infty)$**

x	y	$\varphi(x,y)$	$\varphi_{n-1,n}(x,y)$
715	−5	$0,0015541485 - 0,006216594i$	$0,0015563272 - 0,0062252961i$
500	−100	$3,2197161076\text{E-}6 - 0,0003187519i$	$3,2263672814\text{E-}6 - 0,0003193894i$
10000	−500	$2,8394795125\text{E-}8 - 1,4169002768\text{E-}5i$	$2,8398883607\text{E-}8 - 1,4170420639\text{E-}5i$
25	1000	$1,3067271496\text{E-}7 + 0,0001308034i$	$1,3571507026\text{E-}7 + 0,0001358328i$
50	125	$6,0564746646\text{E-}6 + 0,0007631158i$	$6,1756795953\text{E-}6 + 0,0007780731i$

5. Conclusions

In this paper, numerical solutions of equation (1.1) in the function classes $h(\infty) \times h(\infty)$ and $h(0,\infty) \times h(\infty)$ are presented. Numerical experiments show that the method gives very accurate results and may be useful in practice. An estimation of the error of the approximate solution will be presented elsewhere.

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