COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 8 (2008), No. 2, pp. 155–170

NONLINEAR STABILITY OF THE DIFFERENCE SCHEMES FOR EQUATIONS OF ISENTROPIC GAS DYNAMICS

P. MATUS[1] AND A. KOLODYNSKA[2]

Abstract — For the difference scheme approximating the gas dynamics problem in Riemann invariants *a priory* estimates with respect to the initial data have been obtained. These estimates are proved without any assumptions about the solution of the differential problem using only limitations for the initial and boundary conditions. Estimates of stability in the general case have been obtained only for the finite instant of time $t < t_0$. The uniqueness and convergence of the difference solution are also considered. The results of the numerical experiment confirming theoretical results are given.

2000 Mathematics Subject Classification: 65N06, 65M12.

Keywords: stability, difference scheme, Riemann invariants, isentropic gas dynamics.

1. Introduction

Gas dynamics equations play a key role in the mathematical description of the gas processes. The nonlinearity of these equations can generate various physical effects such as shock waves or boundary layers independent of the smoothness of the input data. That is why the question arises whether we can give conditions which guarantee the absence of any irregularities of the solution.

In the present work we investigate the nonlinear stability of the difference scheme approximating the initial-boundary value problem for isentropic gas in Riemann invariants for $1 < \gamma < 3$. We consider the differential problem describing a supersonic flow.

By now, the most complete results in investigating the stability of the difference solution with respect to small perturbations of the input data have been obtained for computational methods approximating the linear problems of mathematical physics [13]. The main problem in studying the stability of nonlinear difference schemes is the necessity of estimating the difference derivatives. The linearized difference schemes constructed in our work for the approximated solution and for the difference derivatives are monotone [2, 8, 9] and using the maximum principle we are going to get all strong estimates imposing limitations only on the initial and boundary conditions. No other conditions need to be imposed on the properties of the behavior of the exact solution of the original differential problem. The first rigorous results in investigating the stability of the difference schemes for the nonlinear transfer

[1] *Institute of Mathematics, NAS of Belarus*, 11 Surganov Str., 220072 Minsk, Belarus; *Department of Mathematics, The John Paul II Catholic University of Lublin*, Al. Raclawickie 14, 20-950 Lublin, Poland. E-mail: matus@im.bas-net.by

[2] *Department of Mathematics, The John Paul II Catholic University of Lublin*, Al. Raclawickie 14, 20-950 Lublin, Poland. E-mail: agatak@kul.lublin.pl

equation, the Burgers equation, the quasi-linear parabolic equation, and the equations for a weakly compressible liquid were obtained in [7, 10, 11]. In [6], the $L^\infty - w^*$ stability for the isentropic gas dynamics for $\gamma \geqslant 3$ was proved.

The paper is organized as follows: Section 2 is devoted to the definition of the initial boundary value problem for a gas dynamics system. In the approximation of the system of equations in Riemann invariants, a linearized difference scheme is used. The conditions for the initial and boundary data that guarantee stability of the difference scheme are introduced. In Section 3, auxiliary results for the proof of the stability are obtained. In Section 4 the stability of the difference scheme is investigated. In Section 5, we consider the stability in the general case, where there are no fulfilled conditions for the first derivatives of the initial and boundary conditions. The uniqueness of the solution of the difference scheme is studied. Section 6 is devoted to the investigation of the convergence of the difference scheme. Section 7 presents the results of the numerical experiment, which confirm the theoretical results.

2. Problem statement

In the domain $\bar{Q}_T$, where $\bar{Q}_T = \{(x,t) : 0 \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T\}$, let us consider the initial boundary value problem for the gas dynamics written in Eulerian coordinates [5]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0, \quad \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(p + \rho v^2) = 0,$$

$$0 < x < l, \quad 0 < t < T, \quad \frac{p}{\rho^\gamma} = \hat{\kappa}, \quad \hat{\kappa} = \text{const} > 0, \tag{2.1}$$

$$\rho(x,0) = \rho_0(x), \quad v(x,0) = v_0(x), \quad 0 \leqslant x \leqslant l, \tag{2.2}$$

$$v(0,t) = \bar{\mu}_1(t), \quad \rho(0,t) = \bar{\mu}_2(t), \quad 0 < t \leqslant T. \tag{2.3}$$

Here ρ, v and p denote the density, the velocity, and the pressure respectively. The problem is well-posed in the case of supersonic flow if the Mach number v/c is more than 1, where $c = \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}$ is the sound speed [1]. Suppose that the initial and boundary conditions satisfy the inequalities

$$0 \leqslant v_0(x) + \frac{2}{\gamma - 1}\sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-1)/2} \leqslant c_1, \tag{2.4}$$

$$0 \leqslant v_0(x) - \frac{2}{\gamma - 1}\sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-1)/2} \leqslant c_2, \tag{2.5}$$

$$v_0'(x) + \sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-3)/2}\rho_0'(x) \geqslant 0, \tag{2.6}$$

$$v_0'(x) - \sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-3)/2}\rho_0'(x) \geqslant 0, \tag{2.7}$$

$$\rho_0(x) > 0, \tag{2.8}$$

$$1 < \gamma < 3, \tag{2.9}$$

$$0 \leqslant \mu_2(t) < \mu_1(t), \quad \mu_1(t) \leqslant c_3, \quad \mu_2(t) \leqslant c_4, \tag{2.10}$$

$$\mu_1'(t) \leqslant 0, \quad \mu_2'(t) \leqslant 0, \tag{2.11}$$

for $0 \leqslant x \leqslant l$, $0 < t \leqslant T$, where $\mu_{1,2} = \bar{\mu}_1 \pm \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\bar{\mu}_2^{(\gamma-1)/2}$. Hereinafter c_k, $k = 1, 2, \ldots, 15$ denote positive constants. In this paper, we are going to show that the fulfillment of conditions (2.4), (2.5), (2.8)–(2.10) guarantee the correct statement of the boundary conditions.

When the conditions for derivatives (2.6), (2.7), (2.11) are satisfied, the shock wave does not appear.

In the case of smooth flows, system (2.1) can be written in the form [3]

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \hat{\kappa}\gamma\rho^{\gamma-2}\frac{\partial \rho}{\partial x} = 0. \tag{2.12}$$

Now let us write the first-order system (2.12) in the canonical form [14]

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = \vec{0}, \tag{2.13}$$

where

$$\vec{u} = \begin{pmatrix} \rho \\ v \end{pmatrix}, \quad A = \begin{pmatrix} v & \rho \\ \gamma\hat{\kappa}\rho^{\gamma-2} & v \end{pmatrix}.$$

Since the characteristic equation is given by the formula $|A - \lambda E| = 0$, we obtain the real distinct eigenvalues $\lambda_{1,2} = v \pm \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}$ if $\rho(x,t) > 0$, $(x,t) \in \bar{Q}_T$. One of the main questions in solving problems for the gas dynamics is the correct statement of the boundary conditions. It should be emphasized that the system of equations is hyperbolic if there are two real distinct eigenvalues λ_1, λ_2. Moreover, if $\lambda_1, \lambda_2 > 0$, we should give both boundary conditions on the left bound.

Multiplying equation (2.13) by the left eigenvectors $\vec{l}^{(1,2)} = (\pm\sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-3)/2}, 1)^\top$ which fulfill the equations $\vec{l}^{(k)}A = \lambda_k\vec{l}^{(k)}$, $k = 1, 2$, we obtain a system of differential equations in the Riemann invariants r, s [12]

$$\frac{\partial r}{\partial t} + \lambda_1\frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + \lambda_2\frac{\partial s}{\partial x} = 0, \tag{2.14}$$

where

$$r = v + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}, \quad s = v - \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}$$

and

$$\lambda_1 = v + \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2} = (\alpha r + \beta s), \quad \lambda_2 = v - \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2} = (\alpha s + \beta r)$$

for $\alpha = 1/2 + (\gamma-1)/4$, $\beta = 1/2 - (\gamma-1)/4$, $\gamma \neq 1$. In a later investigation we will use the fact that for $1 < \gamma < 3$ we have $\alpha, \beta > 0$. The initial and boundary conditions for the new variables take the form

$$r_0(x) = v_0(x) + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-1)/2}, \tag{2.15}$$

$$s_0(x) = v_0(x) - \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\rho_0(x)^{(\gamma-1)/2}, \tag{2.16}$$

$$r(0,t) = \mu_1(t), \quad s(0,t) = \mu_2(t). \tag{2.17}$$

The inequalities for the initial conditions (2.4)–(2.7) have the following form:

$$0 \leqslant r_0(x) \leqslant c_1, \quad 0 \leqslant s_0(x) \leqslant c_2,$$
$$r_0'(x) \geqslant 0, \quad s_0'(x) \geqslant 0.$$

In the domain $\bar{Q}_T$ we introduce a uniform grid $\bar{\omega} = \bar{\omega}_h \times \bar{\omega}_\tau$

$$\bar{\omega}_h = \left\{x_i : x_i = ih,\, i = \overline{0, N},\, hN = l\right\}, \quad \bar{\omega}_\tau = \left\{t_n : t_n = n\tau,\, n = \overline{0, N_0},\, \tau N_0 = T\right\},$$

with constant space and time steps h and τ. Hereinafter we use standard notations of the difference schemes theory [13] $y = y_i^n = y(x_i, t_n)$, $\hat{y} = y_i^{n+1}$, $y_{t,i} = (y_i^{n+1} - y_i^n)/\tau$, $y_{\bar{x},i} = (y_i^n - y_{i-1}^n)/h$, $f_i = f(x_i)$.

On the grid $\bar{\omega}$ we approximate the differential problem in the Riemann invariants (2.14)–(2.17) by the linearized difference scheme

$$r_{ht,i} + \left(v_{h,i} + \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2}\right)\hat{r}_{h\bar{x},i} = 0, \tag{2.18}$$

$$s_{ht,i} + \left(v_{h,i} - \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2}\right)\hat{s}_{h\bar{x},i} = 0, \quad i = \overline{1, N}, \tag{2.19}$$

$$r_{h,i}^0 = r_0(x_i), \quad s_{h,i}^0 = s_0(x_i), \quad i = \overline{0, N}, \tag{2.20}$$

$$\hat{r}_{h,0} = \mu_1(t_{n+1}), \quad \hat{s}_{h,0} = \mu_2(t_{n+1}), \quad n = \overline{0, N_0 - 1}. \tag{2.21}$$

Using the above difference scheme, we can find

$$v_h = (r_h + s_h)/2, \quad \rho_h = \left((r_h - s_h)(\gamma - 1)/(4\sqrt{\gamma\hat{\kappa}})\right)^{2/(\gamma-1)}.$$

The difference schemes in the Riemann invariants (2.18)–(2.21) are monotone so we can get the estimate in the norm C.

3. Auxiliary results

In investigating the stability the difference problem (2.18)–(2.21), we will use the following canonical form of the two-point difference scheme for the first boundary value problem

$$C_i y_i = A_i y_{i-1} + F_i, \quad i = \overline{1, N}, \quad y_0 = \mu_1. \tag{3.1}$$

Lemma 3.1 [10]. *Let the conditions $A_i \geqslant 0$, $D_i = C_i - A_i > 0$ be met. Then for the solution of problem* (3.1) *the estimate*

$$\max_{0\leqslant i\leqslant N} |y_i| \leqslant \max\left\{|\mu_1|, \max_{0<i\leqslant N}\frac{|F_i|}{D_i}\right\}$$

is valid.

Lemma 3.2. *Let the conditions $A_i \geqslant 0$, $C_i > 0$ be met. Then the solution of problem* (3.1) *satisfies the inequalities*

$$y_i \geqslant 0, \quad i = \overline{0, N} \quad \textit{if} \quad F_i \geqslant 0, \quad i = \overline{1, N}, \quad \mu_1 \geqslant 0,$$

$$y_i \leqslant 0, \quad i = \overline{0, N} \quad \textit{if} \quad F_i \leqslant 0, \quad i = \overline{1, N}, \quad \mu_1 \leqslant 0.$$

Lemma 3.3 [7]. *Let the conditions*

$$1 - \lambda t_{n+1} a_0 > 0, \quad a_n \geqslant 0, \quad t_n = n\tau, \quad n = 0, 1, \ldots, N_0, \quad \tau N_0 = T$$

be met, where $\lambda = \text{const} > 0$. *Then from the inequality*

$$a_n \leqslant \frac{a_{n-1}}{1 - \lambda \tau a_{n-1}}, \quad n = 1, 2, \ldots, N_0$$

the estimates

$$a_n \leqslant \frac{a_0}{1 - \lambda t_n a_0}, \quad 1 - \lambda \tau a_n > 0, \quad n = 0, 1, \ldots, N_0$$

follow.

Lemma 3.3 about the nonlinear recursive relation will be used to get the estimations for the first difference derivatives.

4. Investigation of the nonlinear stability

It will be recalled that in investigating the stability of the difference scheme we are going to use limitations only for the input data. To get the difference scheme for perturbations, we consider the perturbed problem

$$\tilde{r}_{ht,i} + \left(\tilde{v}_{h,i} + \sqrt{\gamma\hat{\kappa}}\tilde{\rho}_{h,i}^{(\gamma-1)/2}\right)\hat{\tilde{r}}_{h\bar{x},i} = 0, \tag{4.1}$$

$$\tilde{s}_{ht,i} + \left(\tilde{v}_{h,i} - \sqrt{\gamma\hat{\kappa}}\tilde{\rho}_{h,i}^{(\gamma-1)/2}\right)\hat{\tilde{s}}_{h\bar{x},i} = 0, \quad i = \overline{1, N}, \tag{4.2}$$

$$\tilde{r}_{h,i}^0 = \tilde{r}_{0,i}, \quad \tilde{s}_{h,i}^0 = \tilde{s}_{0,i}, \quad i = \overline{0, N}, \tag{4.3}$$

$$\hat{\tilde{r}}_{h,0} = \mu_1(t_{n+1}), \quad \hat{\tilde{s}}_{h,0} = \mu_2(t_{n+1}), \quad n = \overline{0, N_0 - 1}. \tag{4.4}$$

Let for the perturbed initial data $\tilde{v}_0$, $\tilde{\rho}_0$ the inequalities analogous to (2.4)–(2.8)

$$0 \leqslant \tilde{v}_0(x) + \frac{2}{\gamma - 1}\sqrt{\gamma\hat{\kappa}}\tilde{\rho}_0(x)^{(\gamma-1)/2} \leqslant c_5, \tag{4.5}$$

$$0 \leqslant \tilde{v}_0(x) - \frac{2}{\gamma - 1}\sqrt{\gamma\hat{\kappa}}\tilde{\rho}_0(x)^{(\gamma-1)/2} \leqslant c_6, \tag{4.6}$$

$$\tilde{v}_0'(x) + \sqrt{\gamma\hat{\kappa}}\tilde{\rho}_0(x)^{(\gamma-3)/2}\tilde{\rho}_0'(x) \geqslant 0, \tag{4.7}$$

$$\tilde{v}_0'(x) - \sqrt{\gamma\hat{\kappa}}\tilde{\rho}_0(x)^{(\gamma-3)/2}\tilde{\rho}_0'(x) \geqslant 0, \tag{4.8}$$

$$\tilde{\rho}_0(x) > 0 \tag{4.9}$$

be satisfied. Subtracting the corresponding equations (2.18)–(2.21) from equations (4.1)–(4.4), we obtain the problem for the perturbations $\delta r_i = \tilde{r}_{h,i} - r_{h,i}$, $\delta s_i = \tilde{s}_{h,i} - s_{h,i}$

$$\delta r_{t,i} + (v_{h,i} + \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2})\delta\hat{r}_{\bar{x},i} + \hat{\tilde{r}}_{h\bar{x},i}\alpha\delta r_i + \hat{\tilde{r}}_{h\bar{x},i}\beta\delta s_i = 0, \tag{4.10}$$

$$\delta s_{t,i} + (v_{h,i} - \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2})\delta\hat{s}_{\bar{x},i} + \hat{\tilde{s}}_{h\bar{x},i}\alpha\delta s_i + \hat{\tilde{s}}_{h\bar{x},i}\beta\delta r_i = 0, \quad i = \overline{1, N}, \tag{4.11}$$

$$\delta r_i^0 = \delta r_{0,i}, \quad \delta s_i^0 = \delta s_{0,i}, \quad i = \overline{0, N}, \tag{4.12}$$

$$\delta\hat{r}_0 = 0, \quad \delta\hat{s}_0 = 0. \tag{4.13}$$

Hereinafter we will use the following grid norms:

$$\|y_h\|_{C^+} = \max_{1 \leqslant i \leqslant N} |y_{h,i}|, \quad \|y_h\|_{\bar{C}} = \max_{0 \leqslant i \leqslant N} |y_{h,i}|.$$

Definition 4.1. *The difference scheme* (2.18)–(2.21) *is stable in the uniform norm with respect to the initial data, if there exists a constant* $M > 0$ *independent of the grid steps* τ, h *and initial data* v_0, $\tilde{v}_0$, ρ_0, $\tilde{\rho}_0$ *that the following inequality*

$$\|\tilde{r}_h^n - r_h^n\|_{\bar{C}} + \|\tilde{s}_h^n - s_h^n\|_{\bar{C}} \leqslant M\left(\left\|\tilde{r}_h^0 - r_h^0\right\|_{\bar{C}} + \left\|\tilde{s}_h^0 - s_h^0\right\|_{\bar{C}}\right)$$

is satisfied for arbitrary τ, h *and for all* $t_n \in \bar{\omega}_\tau$.

From the form of the difference scheme (4.10), (4.11) we can see that to prove the nonlinear stability, we need to get the estimates for r_h, s_h and for the difference derivatives $\tilde{r}_{h\bar{x}}$, $\tilde{s}_{h\bar{x}}$ of the perturbed solution. Let us prove the following theorem.

Theorem 4.1. *Let conditions* (2.4), (2.5), (2.8)–(2.10) *be met. Then for the solution of the difference problem* (2.18)–(2.21) *the inequalities*

$$\|r_h^n\|_{\bar{C}} \leqslant c_7, \quad \|s_h^n\|_{\bar{C}} \leqslant c_8,$$

$$0 < \sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2} < v_{h,i}^n \leqslant \frac{c_7 + c_8}{2}, \quad i = \overline{0,N}, \quad n = \overline{0,N_0},$$

are valid.

Proof. We write the difference scheme (2.18)–(2.21) in the canonical form

$$C_{r,i}^n r_{h,i}^{n+1} = A_{r,i}^n r_{h,i-1}^{n+1} + F_{r,i}^n, \quad i = \overline{1,N}, \quad r_{h,0}^{n+1} = \mu_1^{n+1},$$

$$C_{s,i}^n s_{h,i}^{n+1} = A_{s,i}^n s_{h,i-1}^{n+1} + F_{s,i}^n, \quad i = \overline{1,N}, \quad s_{h,0}^{n+1} = \mu_2^{n+1},$$

where $A_{r,i}^n = h^{-1}\tau(v_{h,i}^n + \sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2}) = h^{-1}\tau\left(\alpha r_{h,i}^n + \beta s_{h,i}^n\right)$, $C_{r,i}^n = 1 + A_{r,i}^n$, $F_{r,i}^n = r_{h,i}^n$, $A_{s,i}^n = h^{-1}\tau(v_{h,i}^n - \sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2}) = h^{-1}\tau\left(\alpha s_{h,i}^n + \beta r_{h,i}^n\right)$, $C_{s,i}^n = 1 + A_{s,i}^n$, $F_{s,i}^n = s_{h,i}^n$.

We will use the method of mathematical induction. Taking into account conditions (2.4), (2.5) for $1 < \gamma < 3$, we have $A_{r,i}^0 \geqslant 0$, $A_{s,i}^0 \geqslant 0$, $C_{r,i}^0 - A_{r,i}^0 = C_{s,i}^0 - A_{s,i}^0 = 1$. Note that $A_{r,i}^0 = 0$, $A_{s,i}^0 = 0$ if and only if $r_{h,i}^0 = s_{h,i}^0 = 0$. In consequence of condition (2.8) such a situation does not appear. So we get $A_{r,i}^0 > 0$ $A_{s,i}^0 > 0$ and hence $\sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^0\right)^{(\gamma-1)/2} < v_{h,i}^0$. On the basis of Lemmas 3.1, 3.2 we get estimates $\|r_h^1\|_{\bar{C}} \leqslant \max\{|\mu_1^1|, \max\limits_{1\leqslant i\leqslant N}|r_h^0|\} \leqslant \max\{c_1, c_3\} = c_7$, $\|s_h^1\|_{\bar{C}} \leqslant \max\{|\mu_2^1|, \max\limits_{1\leqslant i\leqslant N}|s_h^0|\} \leqslant \max\{c_2, c_4\} = c_8$ and relations $r_{h,i}^1 \geqslant 0$, $s_{h,i}^1 \geqslant 0$, $i = \overline{0,N}$ (because we have $\mu_1^1 \geqslant 0$, $\mu_2^1 \geqslant 0$, $r_0(x) \geqslant 0$, $s_0(x) \geqslant 0$). Taking into account the considered canonical form of the difference scheme, we get the relations

$$r_{h,i}^{n+1} = \frac{r_{h,i}^n + A_{r,i}^n r_{h,i-1}^{n+1}}{C_{r,i}^n}, \quad s_{h,i}^{n+1} = \frac{s_{h,i}^n + A_{s,i}^n s_{h,i-1}^{n+1}}{C_{s,i}^n}. \tag{4.14}$$

Thus, if $r_{h,i}^n \neq 0$, $A_{r,i}^n > 0$ ($s_{h,i}^n \neq 0$, $A_{s,i}^n > 0$), then $r_{h,i}^{n+1} \neq 0$ ($s_{h,i}^{n+1} \neq 0$). Therefore, for $1 < \gamma < 3$ from the inequalities $\alpha r_{h,i}^0 + \beta s_{h,i}^0 > 0$, $\alpha s_{h,i}^0 + \beta r_{h,i}^0 > 0$ follow $\alpha r_{h,i}^1 + \beta s_{h,i}^1 > 0$ and $\alpha s_{h,i}^1 + \beta r_{h,i}^1 > 0$ and hence $\sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^1\right)^{(\gamma-1)/2} < v_{h,i}^1 = (r_{h,i}^1 + s_{h,i}^1)/2 \leqslant (c_7 + c_8)/2$, $i = \overline{0,N}$.

Now let us suppose that on the time layer n the inequalities $r_{h,i}^n \geqslant 0$, $s_{h,i}^n \geqslant 0$,

$$\alpha r_{h,i}^n + \beta s_{h,i}^n > 0, \quad \alpha s_{h,i}^n + \beta r_{h,i}^n > 0, \quad i = \overline{0,N}, \tag{4.15}$$

are fulfilled. Hence we have $A^n_{r,i} > 0$, $A^n_{s,i} > 0$, $C^n_{r,i} - A^n_{r,i} = C^n_{s,i} - A^n_{s,i} = 1$. Thus on the basis of Lemma 3.1 we get estimates $\|r^{n+1}_h\|_{\bar C} \leqslant \max\{|\mu^{n+1}_1|, \max_{1\leqslant i\leqslant N}|r^n_h|\}$, $\|s^{n+1}_h\|_{\bar C} \leqslant \{|\mu^{n+1}_2|, \max_{1\leqslant i\leqslant N}|s^n_h|\}$. Therefore,

$$\|r^{n+1}_h\|_{\bar C} \leqslant \max\{\max_{0\leqslant k\leqslant n}|\mu^{k+1}_1|, \max_{1\leqslant i\leqslant N}|r^0_h|\} \leqslant c_7,$$

$$\|s^{n+1}_h\|_{\bar C} \leqslant \max\{\max_{0\leqslant k\leqslant n}|\mu^{k+1}_2|, \max_{1\leqslant i\leqslant N}|s^0_h|\} \leqslant c_8.$$

Moreover, using Lemma 3.2 we get inequalities $r^{n+1}_{h,i} \geqslant 0$, $s^{n+1}_{h,i} \geqslant 0$, $i = \overline{0,N}$, because $\mu^{n+1}_1 \geqslant 0$, $\mu^{n+1}_2 \geqslant 0$. Therefore, taking into consideration relations (4.14), from inequalities (4.15) follow $\alpha r^{n+1}_{h,i} + \beta s^{n+1}_{h,i} > 0$ and $\alpha s^{n+1}_{h,i} + \beta r^{n+1}_{h,i} > 0$ and hence $\sqrt{\gamma\hat\kappa}\left(\rho^{n+1}_{h,i}\right)^{(\gamma-1)/2} < v^{n+1}_{h,i} = (r^{n+1}_{h,i} + s^{n+1}_{h,i})/2 \leqslant (c_7 + c_8)/2$, $i = \overline{0,N}$.

Let us show that $\sqrt{\gamma\hat\kappa}\left(\rho^{n+1}_{h,i}\right)^{(\gamma-1)/2} > 0$ for any $i = \overline{0,N}$, $n = \overline{0,N_0-1}$. We know that $\sqrt{\gamma\hat\kappa}\left(\rho^{n+1}_{h,i}\right)^{(\gamma-1)/2} = (r^{n+1}_{h,i} - s^{n+1}_{h,i})(\gamma-1)/4$, so we have to prove that $r^{n+1}_{h,i} - s^{n+1}_{h,i} > 0$. We are going to show that the inequalities $r^n_{h,i} - s^n_{h,i} > 0$, $r^{n+1}_{h,i-1} - s^{n+1}_{h,i-1} > 0$ imply the desired relation. From the conditions (2.8), (2.10) we have $\sqrt{\gamma\hat\kappa}\left(\rho^0_{h,i}\right)^{(\gamma-1)/2} > 0$, $\mu^{n+1}_1 - \mu^{n+1}_2 > 0$. From (4.14) we get

$$C^n_{r,i}C^n_{s,i}\left(r^{n+1}_{h,i} - s^{n+1}_{h,i}\right) = r^n_{h,i} - s^n_{h,i} + h^{-1}\tau\beta(\left(r^n_{h,i}\right)^2 - \left(s^n_{h,i}\right)^2) +$$
$$A^n_{r,i}r^{n+1}_{h,i-1} - A^n_{s,i}s^{n+1}_{h,i-1} + A^n_{r,i}A^n_{s,i}\left(r^{n+1}_{h,i-1} - s^{n+1}_{h,i-1}\right).$$

From here it follows that $r^{n+1}_{h,i} - s^{n+1}_{h,i} > 0$ according to the mathematical induction assumption $r^n_{h,i} - s^n_{h,i} > 0$, $r^{n+1}_{h,i-1} - s^{n+1}_{h,i-1} > 0$ and $A^n_{r,i} > A^n_{s,i}$. Therefore $\sqrt{\gamma\hat\kappa}\left(\rho^{n+1}_{h,i}\right)^{(\gamma-1)/2} > 0$, $i = \overline{0,N}$, $n = \overline{0,N_0-1}$. □

We also need to estimate the difference derivatives $\tilde r_{h\bar x}$, $\tilde s_{h\bar x}$ of the perturbed solution. Differentiating equations (4.1), (4.2) with respect to $\bar x$ and introducing notations $\xi = \tilde r_{h\bar x}$, $\eta = \tilde s_{h\bar x}$, we get the following problem:

$$\xi_{t,i} + \left(\left(\tilde v_{h,i} + \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2}_{h,i}\right)\hat\xi\right)_{\bar x,i} = 0, \tag{4.16}$$

$$\eta_{t,i} + \left(\left(\tilde v_{h,i} - \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2}_{h,i}\right)\hat\eta\right)_{\bar x,i} = 0, \quad i = \overline{1,N}, \tag{4.17}$$

$$\xi^0_i = \tilde r^0_{h\bar x,i}, \quad \eta^0_i = \tilde s^0_{h\bar x,i}, \quad i = \overline{1,N}, \tag{4.18}$$

$$\hat\xi_0 = \frac{-\mu_{1t}}{\tilde v_{h,0} + \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2}_{h,0}}, \quad \hat\eta_0 = \frac{-\mu_{2t}}{\tilde v_{h,0} - \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2}_{h,0}}. \tag{4.19}$$

When the functions $\tilde r$, $\tilde s$ are continuous, we can use the equations

$$\frac{\partial\tilde r}{\partial x} = -\frac{\partial\tilde r}{\partial t}/(\tilde v + \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2}), \quad \frac{\partial\tilde s}{\partial x} = -\frac{\partial\tilde s}{\partial t}/(\tilde v - \sqrt{\gamma\hat\kappa}\tilde\rho^{(\gamma-1)/2})$$

on the left bound. Then ξ_0, η_0 are the approximation of the differential equations for $x = 0$ and we do not use the values of $\tilde r_{h,-1}$, $\tilde s_{h,-1}$ to calculate them. Let us rewrite the difference scheme (4.16)–(4.19) in the canonical form

$$C^n_{\xi,i}\xi^{n+1}_i = A^n_{\xi,i}\xi^{n+1}_{i-1} + F^n_{\xi,i}, \quad i = \overline{1,N}, \tag{4.20}$$

$$\xi_0^{n+1} = \frac{-\mu_{1t}^n}{\tilde{v}_{h,0}^n + \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,0}^n\right)^{(\gamma-1)/2}}, \tag{4.21}$$

$$C_{\eta,i}^n \eta_i^{n+1} = A_{\eta,i}^n \eta_{i-1}^{n+1} + F_{\eta,i}^n, \quad i = \overline{1,N}, \tag{4.22}$$

$$\eta_0^{n+1} = \frac{-\mu_{2t}^n}{\tilde{v}_{h,0}^n - \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,0}^n\right)^{(\gamma-1)/2}}, \tag{4.23}$$

where

$$A_{\xi,i}^n = \frac{\tau}{h}\left(\tilde{v}_{h,i-1}^n + \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,i-1}^n\right)^{(\gamma-1)/2}\right), \quad C_{\xi,i}^n = 1 + \frac{\tau}{h}\left(\tilde{v}_{h,i}^n + \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,i}^n\right)^{(\gamma-1)/2}\right), \quad F_{\xi,i}^n = \xi_i^n,$$

$$A_{\eta,i}^n = \frac{\tau}{h}\left(\tilde{v}_{h,i-1}^n - \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,i-1}^n\right)^{(\gamma-1)/2}\right), \quad C_{\eta,i}^n = 1 + \frac{\tau}{h}\left(\tilde{v}_{h,i}^n - \sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,i}^n\right)^{(\gamma-1)/2}\right), \quad F_{\eta,i}^n = \eta_i^n.$$

Hence we get $D_{\xi,i}^n = C_{\xi,i}^n - A_{\xi,i}^n = 1 + \tau\left(\tilde{v}_{h\bar{x},i}^n + \sqrt{\gamma\hat{\kappa}}\left(\left(\tilde{\rho}_{h,i}^n\right)^{(\gamma-1)/2}\right)_{\bar{x}}\right) = 1 + \tau\left(\alpha\xi_i^n + \beta\eta_i^n\right)$ and $D_{\eta,i}^n = C_{\eta,i}^n - A_{\eta,i}^n = 1 + \tau\left(\tilde{v}_{h\bar{x},i}^n - \sqrt{\gamma\hat{\kappa}}\left(\left(\tilde{\rho}_{h,i}^n\right)^{(\gamma-1)/2}\right)_{\bar{x}}\right) = 1 + \tau\left(\alpha\eta_i^n + \beta\xi_i^n\right).$

Theorem 4.2. *Let conditions* (2.9)–(2.11), (4.5)–(4.9) *be met. Then for the difference derivatives of the solution of the difference problem* (4.1)–(4.4) *the estimates*

$$\|\tilde{r}_{h\bar{x}}^n\|_{C^+} \leqslant c_9, \quad \|\tilde{s}_{h\bar{x}}^n\|_{C^+} \leqslant c_{10}, \quad n = \overline{0, N_0} \tag{4.24}$$

are valid.

Proof. In proving Theorem 4.1, it was shown that $\sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2} < v_{h,i}^n$, $i = \overline{0,N}$, $n = \overline{0,N_0}$. Similarly, by mathematical induction it is possible to prove that $\sqrt{\gamma\hat{\kappa}}\left(\tilde{\rho}_{h,i}^n\right)^{(\gamma-1)/2} < \tilde{v}_{h,i}^n$, $i = \overline{0,N}$, $n = \overline{0,N_0}$. Thus, for the difference scheme (4.20)–(4.23) the inequalities $A_{\xi,i}^n > 0$, $A_{\eta,i}^n > 0$ are satisfied.

On the first time layer $t = \tau$ we have

$$D_{\xi,i}^0 = 1 + \tau\left(\alpha\tilde{r}_{h\bar{x},i}^0 + \beta\tilde{s}_{h\bar{x},i}^0\right) = 1 + \frac{\tau}{h}\int\limits_{x_{i-1}}^{x_i}\left(\alpha\tilde{r}_0'(x) + \beta\tilde{s}_0'(x)\right)\mathrm{d}x \geqslant 1,$$

$$D_{\eta,i}^0 = 1 + \tau\left(\alpha\tilde{s}_{h\bar{x},i}^0 + \beta\tilde{r}_{h\bar{x},i}^0\right) = 1 + \frac{\tau}{h}\int\limits_{x_{i-1}}^{x_i}\left(\alpha\tilde{s}_0'(x) + \beta\tilde{r}_0'(x)\right)\mathrm{d}x \geqslant 1.$$

Thus, on the basis of Lemma 3.1 we get the estimates

$$\left\|\xi^1\right\|_{\bar{C}} \leqslant \max\{\left|\xi_0^1\right|, \left\|\xi^0\right\|_{C_+}\}, \quad \left\|\eta^1\right\|_{\bar{C}} \leqslant \max\{\left|\eta_0^1\right|, \left\|\eta^0\right\|_{C_+}\}.$$

From conditions (4.7), (4.8) we get

$$\xi_i^0 = \tilde{r}_{h\bar{x},i}^0 = \frac{1}{h}\int\limits_{x_{i-1}}^{x_i}\tilde{r}_0'(x)\,\mathrm{d}x \geqslant 0, \quad \eta_i^0 = \tilde{s}_{h\bar{x},i}^0 = \frac{1}{h}\int\limits_{x_{i-1}}^{x_i}\tilde{s}_0'(x)\,\mathrm{d}x \geqslant 0.$$

From assumptions (2.11) it follows that $\xi_0^1 \geqslant 0$, $\eta_0^1 \geqslant 0$. Therefore, from Lemma 3.2 we get $\xi_i^1 \geqslant 0$, $\eta_i^1 \geqslant 0$, $i = \overline{0,N}$.

Now let us suppose that on the time layer n the conditions $\xi_i^n \geqslant 0$, $\eta_i^n \geqslant 0$, $i = \overline{0,N}$ are satisfied. So we get $D_{\xi,i}^n \geqslant 1$, $D_{\eta,i}^n \geqslant 1$. Taking into account that for (2.11) we have $\xi_0^{n+1} \geqslant 0$, $\eta_0^{n+1} \geqslant 0$ and using Lemma 3.2 we obtain $\xi_i^{n+1} \geqslant 0$, $\eta_i^{n+1} \geqslant 0$, $i = \overline{0,N}$. On the basis of Lemma 3.1 we get the relations $\|\xi^{n+1}\|_{\bar{C}} \leqslant \max\{|\xi_0^{n+1}|, \|\xi^n\|_{C_+}\}$, $\|\eta^{n+1}\|_{\bar{C}} \leqslant \max\{|\eta_0^{n+1}|, \|\eta^n\|_{C_+}\}$, from where follow the estimates

$$\left\|\tilde{r}_{h\bar{x}}^{n+1}\right\|_{C^+} \leqslant \max\{\max_{0\leqslant k\leqslant n} |\xi_0^{k+1}|, \left\|\tilde{r}_{h\bar{x}}^0\right\|_{C_+}\} \leqslant c_9,$$

$$\left\|\tilde{s}_{h\bar{x}}^{n+1}\right\|_{C^+} \leqslant \max\{\max_{0\leqslant k\leqslant n} |\eta_0^{k+1}|, \left\|\tilde{s}_{h\bar{x}}^0\right\|_{C_+}\} \leqslant c_{10}.$$

This completes the proof of the theorem. □

Using the obtained estimates we can formulate the main theorem about the nonlinear stability.

Theorem 4.3. *Let the conditions for the input data* (2.4)–(2.11), (4.5)–(4.9) *be met. Then the difference scheme* (2.18)–(2.21) *is ρ-stable with respect to the initial data and for the solution the estimate*

$$\left\|\tilde{r}_h^{n+1} - r_h^{n+1}\right\|_{\bar{C}} + \left\|\tilde{s}_h^{n+1} - s_h^{n+1}\right\|_{\bar{C}} \leqslant$$
$$M\left(\left\|\delta v_0 + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}} + \left\|\delta v_0 - \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}}\right)$$

is valid, where $M = e^{T\alpha(c_9+c_{10})}$.

Proof. We rewrite the difference scheme (4.10)–(4.13) in the canonical form

$$C_{r,i}^n \delta r_i^{n+1} = A_{r,i}^n \delta r_{i-1}^{n+1} + F_{r,i}^n, \quad i = \overline{1,N}, \quad \delta r_0^{n+1} = 0, \tag{4.25}$$

$$C_{s,i}^n \delta s_i^{n+1} = A_{s,i}^n \delta s_{i-1}^{n+1} + F_{s,i}^n, \quad i = \overline{1,N}, \quad \delta s_0^{n+1} = 0, \tag{4.26}$$

where

$$A_{r,i}^n = \frac{\tau}{h}\left(v_{h,i}^n + \sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2}\right), \quad C_{r,i}^n = 1 + A_{r,i}^n, \quad F_{r,i}^n = \left(1 - \tau\alpha\tilde{r}_{h\bar{x}}^{n+1}\right)\delta r_i^n - \tau\beta\tilde{r}_{h\bar{x}}^{n+1}\delta s_i^n,$$

$$A_{s,i}^n = \frac{\tau}{h}\left(v_{h,i}^n - \sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2}\right), \quad C_{s,i}^n = 1 + A_{s,i}^n, \quad F_{s,i}^n = \left(1 - \tau\alpha\tilde{s}_{h\bar{x}}^{n+1}\right)\delta s_i^n - \tau\beta\tilde{s}_{h\bar{x}}^{n+1}\delta r_i^n.$$

Since for the coefficients of the difference scheme for perturbations the assumptions of Lemma 3.1 hold, then due to the obtained estimates (4.24) for the derivatives of the perturbed solution we have

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} = \left\|\delta r^{n+1}\right\|_{C^+} \leqslant \left(1 + \tau\alpha\left\|\tilde{r}_{h\bar{x}}^{n+1}\right\|_{C^+}\right)\|\delta r^n\|_{C^+} + \tau\beta\left\|\tilde{r}_{h\bar{x}}^{n+1}\right\|_{C^+}\|\delta s^n\|_{C^+} \leqslant$$
$$\|\delta r^n\|_{\bar{C}} + \tau\alpha\left\|\tilde{r}_{h\bar{x}}^{n+1}\right\|_{C^+}\left(\|\delta r^n\|_{\bar{C}} + \|\delta s^n\|_{\bar{C}}\right)$$

and analogously

$$\left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \|\delta s^n\|_{\bar{C}} + \tau\alpha\left\|\tilde{s}_{h\bar{x}}^{n+1}\right\|_{C^+}\left(\|\delta s^n\|_{\bar{C}} + \|\delta r^n\|_{\bar{C}}\right).$$

Summing the obtained estimates, we get

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} + \left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \left(1 + \tau\alpha\left(\left\|\tilde{r}_{h\bar{x}}^{n+1}\right\|_{C^+} + \left\|\tilde{s}_{hx}^{n+1}\right\|_{C^+}\right)\right)\left(\|\delta r^n\|_{\bar{C}} + \|\delta s^n\|_{\bar{C}}\right) \leqslant$$
$$e^{\tau\alpha(c_9+c_{10})}\left(\|\delta r^n\|_{\bar{C}} + \|\delta s^n\|_{\bar{C}}\right).$$

Applying this inequality sequentially, we obtain

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} + \left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \ldots \leqslant M\left(\left\|\delta r^0\right\|_{\bar{C}} + \left\|\delta s^0\right\|_{\bar{C}}\right).$$

□

5. Investigation of the nonlinear stability in the general case

Regardless of the smoothness of the input data the nonlinearity of the system of equations can generate various physical effects such as "gradient catastrophe" or a shock wave [4]. To get the previous estimates, we needed fulfillment of conditions (2.6), (2.7), (2.11), (4.7), (4.8), which implied the absence of large gradients. Now we are going to consider a problem without any restrictions for derivatives. For simplicity of further investigations, we assume that the boundary conditions ξ_0^{n+1}, η_0^{n+1} are homogeneous, i.e., $\xi_0^{n+1} = \eta_0^{n+1} = 0$, which means $\mu_1, \mu_2 = \text{const}$.

Theorem 5.1. *Let conditions* (2.4), (2.5), (2.8)–(2.10), (4.5), (4.6), (4.9) *be met. Then the difference scheme* (2.18) – (2.21) *is ρ-stable with respect to the initial data and for its solution the estimate*

$$\left\|\tilde{r}_h^{n+1} - r_h^{n+1}\right\|_{\bar{C}} + \left\|\tilde{s}_h^{n+1} - s_h^{n+1}\right\|_{\bar{C}} \leqslant$$
$$M\left(\left\|\delta v_0 + \frac{2}{\gamma-1}\sqrt{\gamma\tilde{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}} + \left\|\delta v_0 - \frac{2}{\gamma-1}\sqrt{\gamma\tilde{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}}\right),$$

is valid for $T < t_0$, where $M = e^{T\alpha c_{11}}$ and

$$t_0 = \frac{1}{2\alpha\left(\|\tilde{v}_{0\bar{x}}\|_{C^+} + \frac{2}{\gamma-1}\sqrt{\gamma\tilde{\kappa}}\left\|\left(\tilde{\rho}_0^{(\gamma-1)/2}\right)_{\bar{x}}\right\|_{C^+}\right)}.$$

Proof. Let us define the grid norm $Q^n = \|\xi^n\|_{C^+} + \|\eta^n\|_{C^+}$. Let $1 - \alpha t_n Q^0 > 0$, $n = 0, 2, \ldots, N_0$. Then on the first time layer $t = \tau$ we have $D_{\xi,i}^0 = 1 + \tau\left(\alpha\xi_i^0 + \beta\eta_i^0\right) \geqslant 1 - \tau\left(\alpha\|\xi^0\|_{C^+} + \beta\|\eta^0\|_{C^+}\right) \geqslant 1 - \alpha\tau Q^0 > 0$, $i = \overline{1, N}$. Similarly we can show that $D_{\eta,i}^0 \geqslant 1 - \alpha\tau Q^0 > 0$, $i = \overline{1, N}$. On the basis of Theorem 4.1 we have $A_{\xi,i}^0 > 0$, $A_{\eta,i}^0 > 0$, $i = \overline{1, N}$. Thus, using Lemma 3.1, we get the following estimates for the difference derivatives $\tilde{r}_{h\bar{x}}^1$, $\tilde{s}_{h\bar{x}}^1$:

$$\left\|\xi^1\right\|_{C^+} = \left\|\xi^1\right\|_{\bar{C}} \leqslant \max_{1\leqslant i\leqslant N} \frac{|\xi_i^0|}{1 + \tau\left(\alpha\xi_i^0 + \beta\eta_i^0\right)} \leqslant \frac{\|\xi_i^0\|_{C^+}}{1 - \alpha\tau Q^0},$$

$$\left\|\eta^1\right\|_{C^+} = \left\|\eta^1\right\|_{\bar{C}} \leqslant \max_{1\leqslant i\leqslant N} \frac{|\eta_i^0|}{1 + \tau\left(\alpha\eta_i^0 + \beta\xi_i^0\right)} \leqslant \frac{\|\eta_i^0\|_{C^+}}{1 - \alpha\tau Q^0}$$

Summing the obtained estimates, we get

$$Q^1 \leqslant \frac{Q^0}{1 - \alpha\tau Q^0}.$$

Hence

$$D_{\xi,i}^1, D_{\eta,i}^1 \geqslant 1 - \alpha\tau Q^1 \geqslant 1 - \frac{\alpha\tau Q^0}{1 - \alpha\tau Q^0} = \frac{1 - \alpha t_2 Q^0}{1 - \alpha t_1 Q^0} > 0.$$

By the mathematical induction method it is possible to show the fulfillment of the inequalities

$$Q^n \leqslant \frac{Q^{n-1}}{1 - \alpha\tau Q^{n-1}}, \quad n = 1, 2, \ldots, N_0,$$

so on the basis of Lemma 3.3 we obtain the following estimate:

$$\left\|\xi^{n+1}\right\|_{C^+} + \left\|\eta^{n+1}\right\|_{C^+} = Q^{n+1} \leqslant \frac{Q^0}{1 - \alpha t_{n+1} Q^0} \leqslant$$

$$\frac{2\left\|\tilde{v}_{0\bar{x}}\right\|_{C^+} + \frac{4}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\left\|\left(\tilde{\rho}_0^{(\gamma-1)/2}\right)_{\bar{x}}\right\|_{C^+}}{1 - t_{n+1}2\alpha\left(\left\|\tilde{v}_{0\bar{x}}\right\|_{C^+} + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\left\|\left(\tilde{\rho}_0^{(\gamma-1)/2}\right)_{\bar{x}}\right\|_{C^+}\right)} = c_{11}.$$

Therefore, the derivatives $\tilde{r}^n_{h\bar{x}}$, $\tilde{s}^n_{h\bar{x}}$ are bounded in the norm Q for the time instant $t_n < t_0$, where

$$t_0 = \frac{1}{2\alpha\left(\left\|\tilde{v}_{0\bar{x}}\right\|_{C^+} + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\left\|\left(\tilde{\rho}_0^{(\gamma-1)/2}\right)_{\bar{x}}\right\|_{C^+}\right)}.$$

For the coefficients of the difference scheme (4.25), (4.26) the assumptions of Lemma 3.1 hold, so due to the obtained estimates for the derivatives of the perturbed solution we have

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} = \left\|\delta r^{n+1}\right\|_{C^+} \leqslant \left(1 + \tau\alpha\left\|\tilde{r}^{n+1}_{h\bar{x}}\right\|_{C^+}\right)\left\|\delta r^n\right\|_{C^+} + \tau\beta\left\|\tilde{r}^{n+1}_{h\bar{x}}\right\|_{C^+}\left\|\delta s^n\right\|_{C^+} \leqslant$$
$$\left\|\delta r^n\right\|_{\bar{C}} + \tau\alpha\left\|\tilde{r}^{n+1}_{h\bar{x}}\right\|_{C^+}\left(\left\|\delta r^n\right\|_{\bar{C}} + \left\|\delta s^n\right\|_{\bar{C}}\right)$$

and analogously

$$\left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \left\|\delta s^n\right\|_{\bar{C}} + \tau\alpha\left\|\tilde{s}^{n+1}_{h\bar{x}}\right\|_{C^+}\left(\left\|\delta s^n\right\|_{\bar{C}} + \left\|\delta r^n\right\|_{\bar{C}}\right).$$

Summing the obtained estimates we get

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} + \left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \left(1 + \tau\alpha Q_{n+1}\right)\left(\left\|\delta r^n\right\|_{\bar{C}} + \left\|\delta s^n\right\|_{\bar{C}}\right) \leqslant$$
$$e^{\tau\alpha c_{11}}\left(\left\|\delta r^n\right\|_{\bar{C}} + \left\|\delta s^n\right\|_{\bar{C}}\right).$$

Applying this inequality sequentially, we obtain

$$\left\|\delta r^{n+1}\right\|_{\bar{C}} + \left\|\delta s^{n+1}\right\|_{\bar{C}} \leqslant \ldots \leqslant M\left(\left\|\delta r^0\right\|_{\bar{C}} + \left\|\delta s^0\right\|_{\bar{C}}\right) =$$
$$M\left(\left\|\delta v_0 + \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}} + \left\|\delta v_0 - \frac{2}{\gamma-1}\sqrt{\gamma\hat{\kappa}}\delta\left(\rho_0^{(\gamma-1)/2}\right)\right\|_{\bar{C}}\right). \qquad \square$$

Remark 5.1. In the case of $\gamma = 3$ ($\alpha = 1$, $\beta = 0$), the differential problem for the Riemann invariants (2.14) takes the form of two independent transfer equations

$$\frac{\partial r}{\partial t} + r\frac{\partial r}{\partial x} = 0, \quad \frac{\partial s}{\partial t} + s\frac{\partial s}{\partial x} = 0.$$

The stability of the difference schemes for nonlinear transfer equations is considered in [7].

Remark 5.2. Much in the same manner, the uniqueness of the solution of the difference scheme (2.18)–(2.21) is proved. The proof is carried out by contradiction. We assume that there exist two solutions (r_{1h}, s_{1h}), (r_{2h}, s_{2h}) that satisfy the difference scheme with the same initial and boundary conditions. Then for the differences $R = r_{2h} - r_{1h}$, $S = s_{2h} - s_{1h}$ we get the following difference equations with homogeneous initial and boundary conditions:

$$R_{t,i} + \left(v_{1h,i} + \sqrt{\gamma\hat{\kappa}}\rho_{1h,i}^{(\gamma-1)/2}\right)\hat{R}_{\bar{x},i} + \hat{r}_{2h\bar{x},i}\alpha R_i + \hat{r}_{2h\bar{x},i}\beta S_i = 0,$$

$$S_{t,i} + \left(v_{1h,i} - \sqrt{\gamma\hat{\kappa}}\rho_{1h,i}^{(\gamma-1)/2}\right)\hat{S}_{\bar{x},i} + \hat{s}_{2h\bar{x},i}\alpha S_i + \hat{s}_{2h\bar{x},i}\beta R_i = 0, \quad i = \overline{1, N},$$

$$R_i^0 = 0, \quad S_i^0 = 0, \quad i = \overline{0, N}, \quad \hat{R}_0 = 0, \quad \hat{S}_0 = 0.$$

Using the introduced investigation technique, we can get the estimate

$$\left\|r_{2h}^{n+1} - r_{1h}^{n+1}\right\|_{\bar{C}} + \left\|s_{2h}^{n+1} - s_{1h}^{n+1}\right\|_{\bar{C}} \leqslant 0.$$

Therefore, the difference scheme has a unique solution.

6. Investigation of convergence

In investigating the convergence of the difference scheme (2.18)–(2.21), we will suppose that for the solution $r(x,t)$, $s(x,t)$ of the differential problem (2.14)–(2.17) there exist continuous second-order derivatives with respect to the space and time, i.e., $r(x,t)$, $s(x,t) \in C^{2,2}(\bar{Q}_T)$.

Theorem 6.1. *Let the solution* $r(x,t)$, $s(x,t) \in C^{2,2}(\bar{Q}_T)$ *of the differential problem* (2.14)–(2.17) *exist and let conditions* (2.4), (2.5), (2.8), (2.9) *be met. Then the solution of the difference scheme* (2.18)–(2.21) *converges to the solution of the differential problem* (2.14)–(2.17) *and the following a priory estimate holds:*

$$\left\|r_h^{n+1} - r^{n+1}\right\|_{\bar{C}} + \left\|s_h^{n+1} - s^{n+1}\right\|_{\bar{C}} \leqslant c_{15}(h+\tau), \quad n = \overline{0, N_0 - 1}.$$

Proof. Let us introduce the errors of the method $\Delta r = r_h - r$, $\Delta s = s_h - s$. Substituting $r_h = \Delta r + r$, $s_h = \Delta s + s$ into the difference problem (2.18)–(2.21), we get the relations

$$\Delta r_{t,i} + \left(v_{h,i} + \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2}\right)\Delta\hat{r}_{\bar{x},i} + \hat{r}_{\bar{x},i}\left(\alpha\Delta r_i + \beta\Delta s_i\right) = \psi_{1,i}, \tag{6.1}$$

$$\Delta s_{t,i} + \left(v_{h,i} - \sqrt{\gamma\hat{\kappa}}\rho_{h,i}^{(\gamma-1)/2}\right)\Delta\hat{s}_{\bar{x},i} + \hat{s}_{\bar{x},i}\left(\alpha\Delta s_i + \beta\Delta r_i\right) = \psi_{2,i}, \quad i = \overline{1, N}, \tag{6.2}$$

$$\Delta r_i^0 = 0, \quad \Delta s_i^0 = 0, \quad i = \overline{0, N}, \quad \Delta\hat{r}_0 = 0, \quad \Delta\hat{s}_0 = 0, \tag{6.3}$$

$$\psi_1 = -r_t - \left(v + \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}\right)\hat{r}_{\bar{x}} = \left(\frac{\partial r}{\partial t} - r_t\right) + \left(v + \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}\right)\left(\frac{\partial r}{\partial x} - \hat{r}_{\bar{x}}\right),$$

$$\psi_2 = -s_t - \left(v - \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}\right)\hat{s}_{\bar{x}} = \left(\frac{\partial s}{\partial t} - s_t\right) + \left(v - \sqrt{\gamma\hat{\kappa}}\rho^{(\gamma-1)/2}\right)\left(\frac{\partial s}{\partial x} - \hat{s}_{\bar{x}}\right).$$

Using the Taylor series expansion, it is easy to show that for $\psi_{1,i}$, $\psi_{2,i}$ the following inequality is valid:

$$\max_{t\in\omega_\tau}\|\psi_1(t)\|_{\bar{C}} + \max_{t\in\omega_\tau}\|\psi_2(t)\|_{\bar{C}} \leqslant c_{12}(h+\tau).$$

We rewrite the difference scheme (6.1)–(6.3) in the canonical form

$$C_{r,i}^n \Delta r_i^{n+1} = A_{r,i}^n \Delta r_{i-1}^{n+1} + F_{r,i}^n, \quad i = \overline{1, N}, \quad \Delta r_0^{n+1} = 0,$$

$$C_{s,i}^n \Delta s_i^{n+1} = A_{s,i}^n \Delta s_{i-1}^{n+1} + F_{s,i}^n, \quad i = \overline{1, N}, \quad \Delta s_0^{n+1} = 0,$$

where

$$A_{r,i}^n = \frac{\tau}{h}(v_{h,i}^n + \sqrt{\gamma\hat{\kappa}}(\rho_{h,i}^n)^{(\gamma-1)/2}), \quad C_{r,i}^n = 1 + A_{r,i}^n,$$

$$F_{r,i}^n = (1 - \tau\alpha r_{\bar{x}}^{n+1})\Delta r_i^n - \tau\beta r_{\bar{x}}^{n+1}\Delta s_i^n + \tau\psi_{1,i}^n,$$

$$A_{s,i}^n = \frac{\tau}{h}(v_{h,i}^n - \sqrt{\gamma\hat{\kappa}}(\rho_{h,i}^n)^{(\gamma-1)/2}), \quad C_{s,i}^n = 1 + A_{s,i}^n,$$

$$F_{s,i}^n = (1 - \tau\alpha s_{\bar{x}}^{n+1})\Delta s_i^n - \tau\beta s_{\bar{x}}^{n+1}\Delta r_i^n + \tau\psi_{2,i}^n.$$

Due to the fulfillment of conditions (2.4), (2.5), (2.8), (2.9), the inequality $\sqrt{\gamma\hat{\kappa}}\left(\rho_{h,i}^n\right)^{(\gamma-1)/2} < v_{h,i}^n$ holds, as was shown in Section 4. Then all conditions of Lemma 3.1 are satisfied and hence we get the estimates

$$\left\|\Delta r^{n+1}\right\|_{\bar{C}} = \left\|\Delta r^{n+1}\right\|_{C^+} \leqslant (1 + \tau\alpha\left\|r_{\bar{x}}^{n+1}\right\|_{C^+})\left\|\Delta r^n\right\|_{C^+} + \tau\beta\left\|r_{\bar{x}}^{n+1}\right\|_{C^+}\left\|\Delta s^n\right\|_{C^+} + \tau\left\|\psi_{1,i}^n\right\|_{\bar{C}} \leqslant$$

$$1+\tau\alpha\|r_{\bar{x}}^{n+1}\|_{C^+}(\|\Delta r^n\|_{C^+}+\|\Delta s^n\|_{C^+})+\tau\left\|\psi_{1,i}^n\right\|_{\bar{C}}\leqslant 1+\tau\alpha c_{13}(\|\Delta r^n\|_{\bar{C}}+\|\Delta s^n\|_{\bar{C}})+\tau\|\psi_{1,i}^n\|_{\bar{C}}$$

and analogously

$$\left\|\Delta s^{n+1}\right\|_{\bar{C}}\leqslant 1+\tau\alpha c_{14}\left(\|\Delta s^n\|_{\bar{C}}+\|\Delta r^n\|_{\bar{C}}\right)+\tau\left\|\psi_{2,i}^n\right\|_{\bar{C}},$$

where $c_{13}=\max\limits_{(x,t)\in\bar{Q}_T}|\partial r(x,t)/\partial x|$, $c_{14}=\max\limits_{(x,t)\in\bar{Q}_T}|\frac{\partial s}{(}x,t)/\partial x|$. Summing the above inequalities, we get

$$\|\Delta r^{n+1}\|_{\bar{C}}+\|\Delta s^{n+1}\|_{\bar{C}}\leqslant(1+\tau\alpha\left(c_{13}+c_{14}\right))\left(\|\Delta r^n\|_{\bar{C}}+\|\Delta s^n\|_{\bar{C}}\right)+\tau(\left\|\psi_{1,i}^n\right\|_{\bar{C}}+\left\|\psi_{2,i}^n\right\|_{\bar{C}})\leqslant$$

$$e^{\tau\alpha(c_{13}+c_{14})}\left(\|\Delta r^n\|_{\bar{C}}+\|\Delta s^n\|_{\bar{C}}\right)+\tau\left(\left\|\psi_{1,i}^n\right\|_{\bar{C}}+\left\|\psi_{2,i}^n\right\|_{\bar{C}}\right)\leqslant$$

$$\sum_{k=0}^{n}\tau e^{\alpha t_{n-k}(c_{13}+c_{14})}\left(\left\|\psi_{1,i}^k\right\|_{\bar{C}}+\left\|\psi_{2,i}^k\right\|_{\bar{C}}\right)\leqslant c_{15}(h+\tau).$$

This completes the proof of the theorem. □

7. Numerical experiment

The results of the numerical experiment presented below confirm the theoretical results. In both experiments we used the following values of the parameters: $T=4$, $l=4$, $\gamma=5/3$, $h=\tau=0.02$. We first consider the approximate solution of the differential problem (2.1) with the initial conditions

$$v_0(x)=2\sin\left(\frac{5\pi x}{l}\right)+\frac{9}{4},\quad \rho_0(x)=\left(\frac{\gamma-1}{8\sqrt{\gamma}}\right)^{2/(\gamma-1)},\quad 0\leqslant x\leqslant l,\tag{7.1}$$

and the boundary conditions

$$\mu_1(t)=r_0(0),\quad \mu_2(t)=s_0(0),\quad 0<t\leqslant T.\tag{7.2}$$

It should be noted that the initial and boundary conditions satisfy the obtained relations (2.4), (2.5), (2.8)–(2.11) but do not satisfy the conditions for derivatives (2.6), (2.7). In this case, a shock wave arises. The results of the first experiment are shown in Figs. 7.1, 7.2.

Now we are going to consider the approximate solution of the differential problem with the following initial and boundary conditions:

$$v_0(x)=-2\cos\left(\frac{\pi x}{l}\right)+\frac{9}{4},\quad \rho_0(x)=\left(\frac{\gamma-1}{8\sqrt{\gamma}}\right)^{2/(\gamma-1)},\quad 0\leqslant x\leqslant l,\tag{7.3}$$

$$\mu_1(t)=10t+r_0(0),\quad \mu_2(t)=10t+s_0(0),\quad 0<t\leqslant T.\tag{7.4}$$

In this case, the input data fulfill all inequalities for the initial conditions but do not satisfy the relations for the derivatives of the boundary conditions (2.11) and a shock wave arises as well. The corresponding results are given in Figs. 7.3, 7.4.

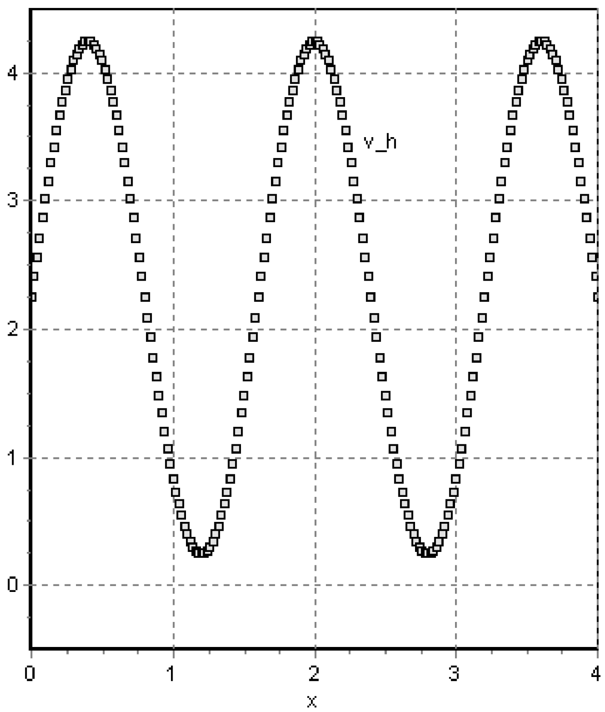

Fig. 7.1. Velocity for problem (2.18), (2.19), (7.1), (7.2) at time $t = 0$

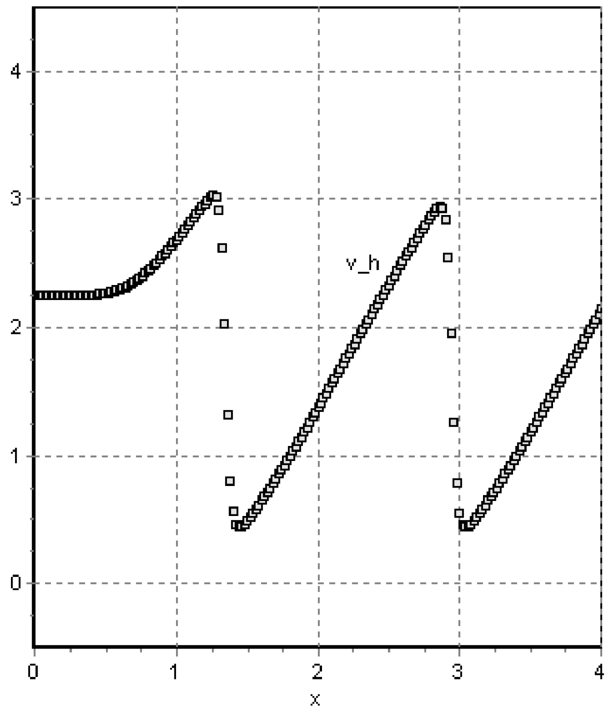

Fig. 7.2. Velocity for problem (2.18), (2.19), (7.1), (7.2) at time $t = 0.34$

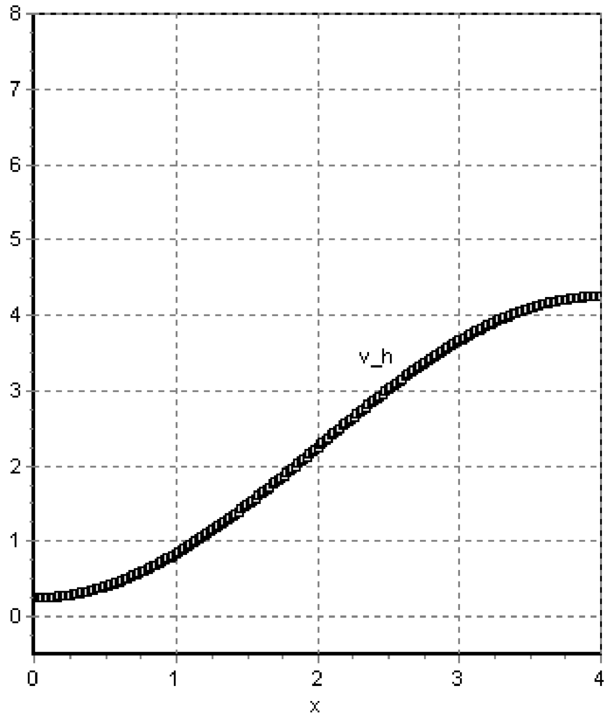

Fig. 7.3. Velocity for problem (2.18), (2.19), (7.3), (7.4) at time $t = 0$

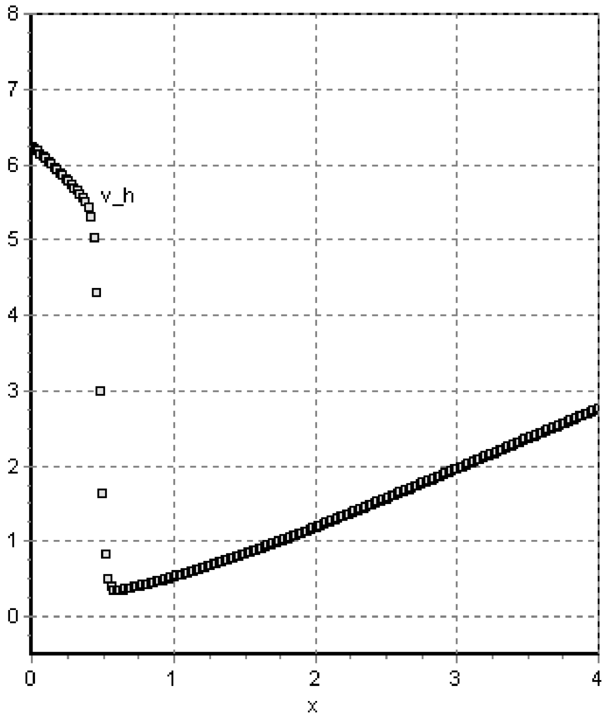

Fig. 7.4. Velocity for problem (2.18), (2.19), (7.3), (7.4) at time $t = 0.6$

8. Conclusions

In this paper, we have obtained an *a priory* estimates of the nonlinear stability for monotone difference schemes approximating the system of equations for isentropic gas in the case of a supersonic flow. In investigating the stability, we have used limitations only for the input data (initial and boundary conditions). On the basis of the investigations performed we can draw the following conclusions.

1. To get proper *a priory* estimates expressing stability of the difference scheme or its continuous dependence on the input data, first we need to prove the existence of a solution of the difference problem in strong norms. In our case, it was necessary to get estimates in the C norm not only for the difference solution r_h, s_h but also for the difference derivatives $\tilde{r}_{h\bar{x}}$, $\tilde{s}_{h\bar{x}}$.

2. The uniqueness of the solution of the difference scheme follows from the stability.

3. Once *a priory* estimates of the stability have been obtained, the investigation of the convergence of the difference solution to the differential one becomes much simpler since the problem for the error of the method can be written in linear form. To use the Lax theorem (from the approximation and stability convergence follows), it is necessary to prove the stability with respect to the right-hand side. In our paper, we have restricted ourselves to considering only homogeneous equations of the gas dynamics.

4. For first-order nonlinear hyperbolic systems great care must be taken that the differential and the difference problems are stated correctly taking into account the signs of the eigenvalues depending on the unknown solution. In our case, we clearly define the conditions only for the input data where the eigenvalues of λ_1, λ_2 are strictly positive. This means that we have considered only a supersonic flow.

5. In this paper, we have obtained conditions that guarantee the absence of a shock waves. As was shown in the numerical experiment, the appearance of a shock wave is connected not only with behavior of the initial but also the boundary conditions. That is why investigation of the stability with respect to small perturbations of the boundary conditions is important.

6. It is essential that constructed difference scheme be monotone not only with respect to the approximated solution, but also to its derivatives. In that case, the maximum principle can be used to prove nonlinear stability of the difference scheme.

7. The nonlinearity of the problem and the inhomogeneity of the boundary conditions does not allow us to use the method of energy inequalities [13] to get corresponding *a priory* estimates.

Acknowledgements. This work was supported by the Belarusian Republic Basic Research Foundation under grant F08R-049.

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