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# BIFURCATION ANALYSIS OF NONLINEAR PARAMETERIZED TWO-POINT BVPS WITH LIAPUNOV — SCHMIDT REDUCED FUNCTIONS

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**Abstract** — In this paper, we study nonlinear two-point boundary value problems (BVPs) which depend on an external control parameter. In order to determine numerically the singular points (turning or bifurcation points) of such a problem with so-called extended systems and to realize branch switching, some information on the type of the singularity is required. In this paper, we propose a strategy to gain numerically this information. It is based on strongly equivalent approximations of the corresponding Liapunov — Schmidt reduced function which are generated by a simplified Newton method. The graph of the reduced function makes it possible to determine the type of singularity. The efficiency of our numerical-graphical technique is demonstrated for two BVPs.

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**Keywords:** boundary value problem, bifurcation points, Liapunov — Schmidt reduction, extended systems, simplified Newton method.

## 1. Introduction

In this paper, we consider a two-point boundary value problem (BVP) for a nonlinear system of $n$ ordinary differential equations

$$x'(t) = f(t, x(t); \lambda), \quad a < t < b, \tag{1.1}$$

subject to $n$ linear homogeneous boundary conditions

$$B_a x(a) + B_b x(b) = 0, \tag{1.2}$$

where $f : \Omega \to \mathbb{R}^n$, $\Omega \subset (a, b) \times \mathbb{R}^n \times \mathbb{R}$, and $B_a, B_b \in \mathbb{R}^{n\times n}$. The BVP (1.1), (1.2) depends on the external control parameter $\lambda \in \mathbb{R}$ which is usually called the *bifurcation parameter*. Let us assume for convenience that the function $f$ is as smooth as desired. In addition, we suppose that the extended matrix $[B_a|B_b] \in \mathbb{R}^{n\times 2n}$ has the rank $n$. Defining the Banach spaces $X$ and $Y$ as

$$X := BC^1_n([a, b]) := \{x(t) \in C^1_n([a, b]) : B_a x(a) + B_b x(b) = 0\},$$

$$\|x\|_X = \|x\|_{C^1_n([a,b])} := \sup_{t\in[a,b]} \{|x(t)| + |x'(t)|\},$$

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$$Y := C_n([a,b]), \quad \|y\|_Y := \sup_{t\in[a,b]} \{|y(t)|\},$$

where $|v| := (v^\top v)^{1/2}$ for each $v \in \mathbb{R}^n$, the BVP (1.1), (1.2) may be formulated as an operator equation

$$T(x,\lambda) = 0, \tag{1.3}$$

where

$$T : X \times \mathbb{R} \to Y, \quad \text{with} \;\; T(x,\lambda) := x' - f(\cdot, x, \lambda), \quad (x,\lambda) \in X \times \mathbb{R}.$$

The smoothness of $f$ implies that the operator $T$ is Fréchet differentiable. The partial Fréchet derivative of $T$ with respect to the first argument is

$$T_x(x,\lambda) : X \to Y, \quad \text{with} \;\; T_x(x,\lambda)\varphi = \varphi' - f_x(\cdot, x, \lambda)\varphi, \quad \varphi \in X. \tag{1.4}$$

The essential property of the operator $T_x(x,\lambda)$ is claimed in the following theorem.

**Theorem 1.1.** *The operator $T_x(x,\lambda)$ is a Fredholm operator of index null for each $(x,\lambda) \in X \times \mathbb{R}$.*

*Proof.* See, e.g., [5]. □

The solution $(x_0, \lambda_0)$ of (1.3), for which the linear homogeneous equation $T_x(x_0,\lambda_0)\varphi_0 = 0$ has a nontrivial solution $\varphi_0$, is called the *singular point* of $T$. The *simple* singular point of $T$ is a singular point $(x_0,\lambda_0)$ with the property that the kernel $\mathcal{N}(T_x(x_0,\lambda_0)$ is one-dimensional.

Let $(x_0,\lambda_0) \in X \times \mathbb{R}$ be a simple singular point of $T$ and let us choose an element $\varphi_0 \in X$ such that $\operatorname{span}\{\varphi_0\} = \mathcal{N}(T_x^0)$, where we have used the abbreviation $T_x^0 = T_x(x_0,\lambda_0)$. Our assumptions on $T$ imply that the adjoint operator $T_x^{0*}$ has a one-dimensional kernel too, i.e., $\dim \mathcal{N}(T_x^{0*}) = 1$. If $Y^*$ denotes the space dual to $Y$, we can find $\psi_0^* \in Y^*$ which satisfies the $\operatorname{span}\{\psi_0^*\} = \mathcal{N}(T_x^{0*})$. Then, the Banach spaces $X$ and $Y$ can be decomposed in the form

$$X = \mathcal{N}(T_x^0) \oplus M, \quad Y = N \oplus \mathcal{R}(T_x^0). \tag{1.5}$$

Furthermore, from the theorem on bi-orthogonal bases it follows that there exist elements $\varphi_0^*$ and $\psi_0$ which satisfy $\varphi_0^*(\varphi_0) = \psi_0^*(\psi_0) = 1$. We can now write (1.5) in the form

$$X = \operatorname{span}\{\varphi_0\} \oplus \{x \in X : \varphi_0^*(x) = 0\}, \quad Y = \operatorname{span}\{\psi_0\} \oplus \{y \in Y : \psi_0^*(y) = 0\}. \tag{1.6}$$

Comparing (1.5) and (1.6), we get

$$M = \{x \in X : \varphi_0^*(x) = 0\}, \quad N = \operatorname{span}\{\psi_0\} \;\; \text{and} \;\; \mathcal{R}(T_x^0) = \{y \in Y : \psi_0^*(y) = 0\}.$$

Using the decomposition of the Banach space $X$ we write $x \in X$ in the form

$$x = s\varphi_0 + w, \quad w \in M. \tag{1.7}$$

The element $x_0$ can be formulated as

$$x_0 = \varphi_0^*(x_0)\varphi_0 + (x_0 - \varphi_0^*(x_0)\varphi_0) =: s_0\varphi_0 + w_0.$$

From this equation we obtain the starting value $w_0 := x_0 - s_0\varphi_0$, with $s_0 := \varphi_0^*(x_0)$, which is needed to compute $w \in M$ iteratively as an isolated solution of the problem

$$G(w; s, \lambda) := \begin{bmatrix} T(s\varphi_0 + w, \lambda) - \psi_0^*(T(s\varphi_0 + w, \lambda))\psi_0 \\ \varphi_0^*(w) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{1.8}$$

where the parameter vector $(s,\lambda)^\top$ must be given from a neighborhood of $(s_0,\lambda_0)^\top$. As we will see later, system (1.8) is used to gain successively information about the Liapunov — Schmidt reduced function $g(s,\lambda) := \psi_0^*(T(s\varphi_0 + w, \lambda))$. This information is required to determine the special type of the singularity (turning point, bifurcation point). If the type of the singularity is known, then the the corresponding simple singular point can be determined numerically by an adequate extended system. In these extended systems, the same $\varphi_0$, $\varphi_0^*$, $\psi_0$ and $\psi_0^*$ can be used.

## 2. A simplified Newton method, strong equivalence and finite codimension

The solution of problem (1.8) requires a method to solve nonlinear operator equations acting in Banach spaces. In this paper, we show that the following simplified Newton method should be used:

$$G_w(w_0; s_0, \lambda_0)\, z_k = -G(w_k; s, \lambda), \quad w_{k+1} = w_k + z_k, \qquad k = 0, 1, \ldots, \tag{2.1}$$

where $G_w$ denotes the partial Fréchet derivative of $G$ with respect to $w$. Since

$$G_w(w_0; s_0, \lambda_0) = \begin{bmatrix} T_x(x_0, \lambda_0) \\ \varphi_0^* \end{bmatrix},$$

the iteration (2.1) takes the form

$$T_x(x_0, \lambda_0)\, z_k = -T(s\varphi_0 + w_k, \lambda) + \psi_0^*(T(s\varphi_0 + w_k, \lambda))\psi_0, \tag{2.2}$$

$$\varphi_0^*(z_k) = -\varphi_0^*(w_k) = 0, \quad w_{k+1} = w_k + z_k, \quad k = 0, 1, \ldots \tag{2.3}$$

As a by-product of this iteration, we get successively approximations $g_k$ of the value of the Liapunov — Schmidt reduced function $g(s,\lambda)$ by means of

$$g_k := g_k(s,\lambda) = \psi_0^*(T(s\varphi_0 + w_k, \lambda)). \tag{2.4}$$

Our focus is not directed to the convergence behavior of the sequence $\{w_k\}_{k=0}^\infty$. Rather we are interested to compute values of a function which is strongly equivalent to $g(s,\lambda)$. As we will see, such a strongly equivalent function is given by $g_k(s,\lambda)$, when $k$ is sufficiently large but finite.

To simplify the representation, let us now assume that the singular point of the operator equation (1.3) is in the origin, i.e., $(x_0,\lambda_0) = (0,0)$.

As usual, two smooth functions $g, h : \mathbb{R}^2 \to \mathbb{R}$ defined in the neighborhood of the origin are called strongly equivalent if there exist two smooth functions $\tilde{T}(s,\lambda)$ and $S(s,\lambda)$ such that for $(s,\lambda)$ close to $(0,0)$ the following relation holds:

$$g(s,\lambda) = S(s,\lambda) h(\tilde{T}(s,\lambda), \lambda). \tag{2.5}$$

In addition, $S$ and $\tilde{T}$ must satisfy

$$\tilde{T}(0,0) = 0, \quad \tilde{T}_s(s,\lambda) > 0, \quad S(s,\lambda) > 0 \tag{2.6}$$

in the neighborhood of the origin. Due to the smoothness of the functions, condition (2.5) is fulfilled if it can be proved at $(0,0)$. Obviously, strong equivalence preserves the qualitative behavior of the zero-set and thus of the type of bifurcation in $(0,0)$ (see, e.g., [1,4]).

Let us denote by $\mathcal{E}_{s,\lambda}$ the space of all smooth functions $g : \mathbb{R}^2 \to \mathbb{R}$ which are defined in the neighborhood of the origin. In $\mathcal{E}_{s,\lambda}$ we identify two elements $g_1$ and $g_2$ if there is a disc containing the origin, where the two functions coincide. The elements of $\mathcal{E}_{s,\lambda}$ are called *germs*. If we want to consider the germs $g$ with more than two variables, we write $g \in \mathcal{E}_n$, if $g$ has $n$ variables. The restricted tangent space $RT(g)$ of the germ $g \in \mathcal{E}_{s,\lambda}$ consists of all germs $p \in \mathcal{E}_{s,\lambda}$, which can be written in the form

$$p(s,\lambda) = a(s,\lambda)g(s,\lambda) + b(s,\lambda)g_s(s,\lambda), \tag{2.7}$$

with $a, b \in \mathcal{E}_{s,\lambda}$ and $b(0,0) = 0$.

We now have the following theorem.

**Theorem 2.1.** *Let $g, p \in \mathcal{E}_{s,\lambda}$. If*

$$RT(g + tp) = RT(g) \quad \textit{for all} \quad t \in [0,1], \tag{2.8}$$

*then $g + tp$ is strongly equivalent to $g$ for all $t \in [0,1]$.*

*Proof.* See [1]. □

Let $\mathfrak{I} \subset \mathcal{E}_n$ be a subspace. If there exists a finite dimensional subspace $V \subset \mathcal{E}_n$ with

$$\mathfrak{I} + V = \mathcal{E}_n,$$

we will say that $\mathfrak{I}$ has a finite codimension. Otherwise we will say that $\mathfrak{I}$ has an infinite codimension.

Let us state the main result.

**Theorem 2.2.** *Let $g$ be the Liapunov — Schmidt reduced function defined above and let $g_k$ be the $k$-th approximation of $g$ generated with the simplified Newton method. Furthermore, let the subspace $RT(g)$ of $\mathcal{E}_{s,\lambda}$ have a finite codimension. Then there exists a $k \in \mathbb{N}$, such that $g$ is strongly equivalent to $g_k$.*

*Proof.* See [6]. □

In [1], one can find a theorem, where the bifurcation problems are classified by their codimension (i.e., by the codimension of their restricted tangent spaces) and normal forms. It turns out, that the codimension of the most common bifurcation phenomena is less than five. Therefore we have to execute in general at most five steps of the simplified Newton method to obtain values of the function $g_k$ which is strongly equivalent to the function $g$ in request.

## 3. The simplified Newton method for BVPs

In this section, we will show how the simplified Newton method reads in the case of our BVP (1.1),(1.2) and how numerical standard techniques can be used to solve the resulting extended BVPs. First, we need the functions $\varphi_0$, $\varphi_0^*$, $\psi_0$ and $\psi_0^*$. The functions $\varphi_0$ and $\psi_0$ can be computed by solving the following BVP:

$$x_0'(t) = f(t, x_0(t); \lambda_0), \quad B_a x_0(a) + B_b x_0(b) = 0;$$

$$\begin{aligned} \varphi_0'(t) &= f_x(t, x_0(t); \lambda_0)\varphi_0(t), \quad B_a\varphi_0(a) + B_b\varphi_0(b) = 0;\\ \psi_0'(t) &= -f_x(t, x_0(t); \lambda_0)^\top\psi_0(t), \quad C_a\psi_0(a) + C_b\psi_0(b) = 0;\\ \xi_1'(t) &= \varphi_0(t)^\top\varphi_0(t), \quad \xi_1(a) = 0;\\ \xi_2'(t) &= \psi_0(t)^\top\psi_0(t), \quad \xi_2(b) - \xi_2(a) = 1;\\ \lambda_0'(t) &= 0, \quad \xi_1(b) = 1; \end{aligned} \tag{3.1}$$

where the matrices $C_a, C_b \in \mathbb{R}^{n\times n}$ have to satisfy the relations $B_aC_a^\top - B_bC_b^\top = 0$ and the $\text{rank}[C_a|C_b] = n$. Note that the first line in (3.1) represents the given BVP (1.1), (1.2). The second line is the linearized BVP and the third line is the corresponding adjoint BVP. If we define

$$\varphi_0^*(w) := \int_a^b \varphi_0(t)^\top w(t)\,dt \quad \text{and} \quad \psi_0^*(w) := \int_a^b \psi_0(t)^\top w(t)\,dt, \tag{3.2}$$

then the requirements $\varphi_0^*(\varphi_0) = 1$ and $\psi_0^*(\psi_0) = 1$ can be formulated in the ODE representation which is given in lines 4 and 5. In order that the number of ODEs matches with the number of boundary conditions the trivial ODE $\lambda_0'(t) = 0$ is added. The boundary condition $\xi_2(b) - \xi_2(a) = 1$ ensures that the integral, representing $\psi_0^*(\psi_0)$, is equal to one.

The solution of the nonlinear BVP (3.1) must be realized by an iterative technique. For this purpose good starting values are required. The function $(t-a)/(b-a)$ can be used as an approximation of $\xi_1(t)$ and $\xi_2(t)$. Furthermore, approximations of $\varphi_0$ and $\psi_0$ can be generated by the strategy proposed in [3]. The desired functions $\varphi_0^*$ and $\psi_0^*$ are represented by $\varphi_0^\top$ and $\psi_0^\top$, respectively (see formula (3.2)). Thus we have starting values to solve (3.1) by standard shooting techniques like the multiple shooting code RWPM [2].

To execute the simplified Newton iterations (2.2) and (2.3) for the determination of $w$, we need the starting function $w_0(t)$ which is defined by $w_0 := x_0 - \varphi_0^*(x_0)\,\varphi_0$. Given $x_0$ and $\varphi_0$, the computation of $w_0$ can be realized by the following BVP:

$$\begin{aligned} w_0'(t) &= x_0'(t) - \varphi_0'(t)\xi_3(t), \quad w_0(a) = x_0(a) - \varphi_0(a)\xi_3(a);\\ \xi_3'(t) &= 0, \quad \xi_4(a) = 0;\\ \xi_4'(t) &= \varphi_0^\top(t)x_0(t), \quad \xi_4(b) = \xi_3(b). \end{aligned} \tag{3.3}$$

$\xi_4(t)$ is the integral of the function $\varphi_0^\top(\tau)x_0(\tau)$ on the interval $[a, t]$.

If the BVP (3.1) is solved by a multiple shooting code, say, e.g., by RWPM, we have tabulated the values for $\varphi_0$ and $x_0$ in the interval $[a, b]$. These values can be used to compute an approximation of $\xi_4(t)$ on the same grid of shooting points with a numerical integration formula. Let us assume that the values of $\varphi_0$ and $x_0$ are known at the points $a = t_0 < t_1 < \cdots < t_m = b$. Then, we can compute a (first-order) approximation of the values $\xi_4(t_i)$, $i = 0, 1, \ldots, m$, by discretizing the integral

$$\xi_4(t_i) = \int_a^{t_i} \varphi_0^\top(\tau)x_0(\tau)d\tau$$

via the simple quadrature formula

$$\xi_4(t_0) = 0, \quad \xi_4(t_i) \approx \frac{1}{i}\sum_{k=1}^{i} \varphi_0^\top(t_i)x_0(t_i), \quad i > 0.$$

The computed value of $\xi_4(b)$ can be used as an approximation for the constant $\xi_3(t)$. Now the starting values for $w_0(t)$ are generated by $w_0(t) := x_0(t) - \xi_3(b)\varphi_0(t)$.

After the separate computation of the BVP (3.1), we enlarge (3.1) by the BVP (3.3) and obtain an extended BVP for the computation of $w_0(t)$. The derivatives on the right-hand-side of the ODE in (3.3) can be taken from the right-hand-side of the corresponding ODEs for $x_0$ and $\varphi_0$ in (3.1). Another advantage of this enlarged problem is that the integration of the associated initial value problems can be executed by numerical methods with an automatic step-size control.

Let us suppose that all unknown functions in (3.1) and (3.3) have been computed by a multiple shooting method on a predefined grid. Obviously, we can define $s_0 := \xi_3(a)$. We now choose an odd number $k \in \mathbb{N}$, set $(s_0, \lambda_0) = (s_{(k+1)/2}, \lambda_{(k+1)/2})$ and define an equidistant grid $G \subset \mathbb{R}^2$ as follows:

$$G := \{(s_1, \lambda_1), (s_2, \lambda_1), \ldots, (s_k, \lambda_1), (s_1, \lambda_2), \ldots, (s_{k-1}, \lambda_k), (s_k, \lambda_k)\}.$$

To make sure that the value of the reduced function can be computed for all $(s_i, \lambda_j) \in G$, the diameter of $G$ must be sufficiently small. Therefore, $G$ can only be chosen in an experimental way and the following strategy is more or less heuristic.
Our aim is to compute the function $g_0(s, \lambda)$ which is defined in (2.4) at a given point $(s, \lambda) = (s_{\kappa_1}, \lambda_{\kappa_2}) \in G$. Due to formula (2.2), $z_0$ is the unique solution of the problem

$$T_x^0 z_0 = -T(s\varphi_0 + w_0, \lambda) + g_0(s, \lambda)\,\psi_0, \quad \varphi_0^* z_0 = 0. \tag{3.4}$$

We will now show that the two operator equations in (3.4) can be written in the following BVP formulation:

$$\begin{gathered}
\eta_0'(t) = 0, \quad \xi_{0,1}'(t) = \psi_0^\top(t)[s\varphi_0'(t) + w_0'(t) - f(t, s\varphi_0(t) + w_0(t), \lambda)], \\
z_0'(t) = f_x(t, x_0(t), \lambda_0) z_0(t) - [s\varphi_0'(t) + w_0'(t) - f(t, s\varphi_0(t) + w_0(t), \lambda)] + \eta_0(t)\,\psi_0(t), \\
\xi_{0,2}'(t) = \varphi_0^\top(t) z_0(t),
\end{gathered} \tag{3.5}$$

$$\xi_{0,1}(a) = 0, \quad \xi_{0,1}(b) = \eta_0(b), \quad B_a z_0(a) + B_b z_0(b) = 0, \quad \xi_{0,2}(b) = \xi_{0,2}(a). \tag{3.6}$$

Here $\eta_0(t)$ is a constant function which coincides with the value of $g_0(s, \lambda)$. The expression for $g_0(s, \lambda)$ results from (2.4) for $k = 0$. Using the integral representation (3.2) for the functional $\psi_0^*$, this expression can be formulated as an initial value problem for the function $\xi_{0,1}(t)$. $\xi_{0,1}(a) = 0$ and $\xi_{0,1}(b) = g_0(s, \lambda)$ hold. Now the third equation in (3.5) is the ODE formulation of the first operator equation in (3.4). If the functional $\varphi_0^*$ is represented in the integral form (3.2), then the second operator equation in (3.4) can be written as an ODE for the function $\xi_{0,2}(t)$ which satisfies $\xi_{0,2}(b) - \xi_{0,2}(a) = 0$.

For the numerical solution of the BVP (3.5), (3.6) it is appropriate to enlarge it by the previously solved BVPs (3.1) and (3.3). The required starting values can be determined as follows. The technique described for the functions in (3.3) can also be applied to $\eta_0(t)$ and $\xi_{0,1}(t)$. Suitable starting functions for $z_0(t)$ are $w_0(t)$ or $z_0(t) \equiv 0$. The function $\xi_{0,2}(t)$ can be approximated by $[(t-a)/(b-a)] - [(t-a)/(b-a)]^2$.

If one has determined a solution of the enlarged BVPs (3.1), (3.3), (3.5), (3.6), the next iteration step of the simplified Newton method must be executed. The corresponding operator equations read

$$w_1 = w_0 + z_0, \quad g_1(s, \lambda) = \psi_0^* T(s\varphi_0 + w_1, \lambda), \quad T_x^0 z_1 = -T(s\varphi_0 + w_1, \lambda) + g_1(s, \lambda)\psi_0, \quad \varphi_0^* z_1 = 0.$$

The realization of these equations for our BVP (1.1), (1.2) is

$$w_1'(t) = w_0'(t) + z_0'(t), \quad \eta_1'(t)=0, \quad \xi_{1,1}'(t) = \psi_0^\top(t)[s\varphi_0'(t) + w_1'(t) - f(t, s\varphi_0(t) + w_1(t), \lambda)],$$

$$z_1'(t) = f_x(t, x_0(t), \lambda_0)z_1(t) - [s\varphi_0'(t) + w_1'(t) - f(t, s\varphi_0(t) + w_1(t), \lambda)] + \eta_1(t)\,\psi_0(t),$$

$$\xi_{1,2}'(t) = \varphi_0^\top(t)z_1(t), \tag{3.7}$$

$$w_1(a) = w_0(a) + z_0(a), \quad \xi_{1,1}(a) = 0, \quad \xi_{1,1}(b) = \eta_1(b), \quad B_a z_1(a) + B_b z_1(b) = 0,$$

$$\xi_{1,2}(b) = \xi_{1,2}(a). \tag{3.8}$$

As a starting function for $w_1$, we can use $w_0 + z_0$. A good starting function for $z_1$ will be $z_0$. For the remaining variables we should proceed as described in the previous iteration step.

We can compute $w_j$ and $g_j$, $j = 2, 3, 4, 5$, in the same way as we have determined $w_1$ and $g_1$. Thus, in order to compute successively approximations $g_j$ of the reduced function and to get starting functions for the next iteration step of the simplified Newton method, we have to construct and solve a series of enlarged BVPs.

An alternative way would be to construct only one large BVP in each iteration step of the simplified Newton method. The disadvantage of this strategy is the lack of suitable starting values. Therefore, our approach described above is more sophisticated and we prefer the stepwise construction of the corresponding BVPs until $w_5$, $g_5$ and $z_5$ are computed.

Each iterate $g_0(s,\lambda), \ldots, g_5(s,\lambda)$ should be saved during the computational process. This makes it possible to draw the corresponding surfaces in $\mathbb{R}^3$ step by step and thus observe visually the changes in each iteration step. If no further changes can be recognized in the $k$-th iteration step, we conclude that the corresponding function $g_k(s,\lambda)$ is strongly equivalent to $g(s,\lambda)$. A comparison of the generated picture with the 3D-graphics of the normal forms presented in [1, Table 1.2] and the contour plot of the zero level makes it possible to conclude on the type of the singularity.

Now, the extended system associated with the special type of singularity (see, e.g., [7]) can be chosen to determine numerically the singular point $(x_0, \lambda_0)$. And, finally, the solution curves branching at $(x_0, \lambda_0)$ can be determined by the suitable transformation technique described in [7].

Summarizing our results, for the nonlinear parametrized BVP (1.1), (1.2) we have proposed a strategy to gain numerically the information which enables the determination of the type of singularity. The type of singularity must be known if the associated (simple) singular point is to be computed with an extended system.

## 4. Numerical results

The following two examples illustrate our method. The first is a parametrized BVP which describes the buckling of the so-called Euler — Bernoulli rod (see, e.g., [7]):

$$x_1'(t) = x_2(t), \quad x_1(0) = 0;$$

$$x_2'(t) = -\lambda \sin(x_1(t)), \quad x_1(1) = 0. \tag{4.1}$$

For each $\lambda \in \mathbb{R}$ the solution of (4.1) is known, namely $x_1(t) = x_2(t) = 0$. Thus $s_0 = 0$. We focus on the simple bifurcation point at the critical value $\lambda = \pi^2$. To simplify the presentation, we transformed the problem so that $\lambda_0 = s_0 = 0$ is satisfied. Then, we created a grid $G$ with the center $(s_0, \lambda_0)$ by the MATLAB command

$$[s, \lambda] = \text{meshgrid}(-0.2 : 0.02 : 0.2, -0.1 : 0.02 : 0.2).$$

In the first step of our algorithm, we selected $s = -0.2$ and $\lambda = -0.1$, i.e. the lower left corner of $G$. Now we have successively constructed a BVP defining $g_1(s, \lambda), \ldots, g_5(s, \lambda)$. Let us call the solution of this BVP $S_{(s,\lambda)}(t)$. Obviously, $g_5(s, \lambda)$ is only one component of $S_{(s,\lambda)}(t)$. If the value of $s$ is modified to $s = -0.18$, then the BVP and the corresponding solution $S_{(s,\lambda)}$ change only a little. Therefore, we can use the computed solution $S_{(-0.2,-0.1)}(t)$ as a starting point for the computation of $S_{(-0.18,-0.2)}(t)$ which contains the required $g_5(-0.18, -0.2)$ as a component. If this strategy is used for neighbouring grid points, then we can successively compute $g_5$ on the entire grid. Figure 4.1 shows a picture of $g_5$ for the problem (4.1).

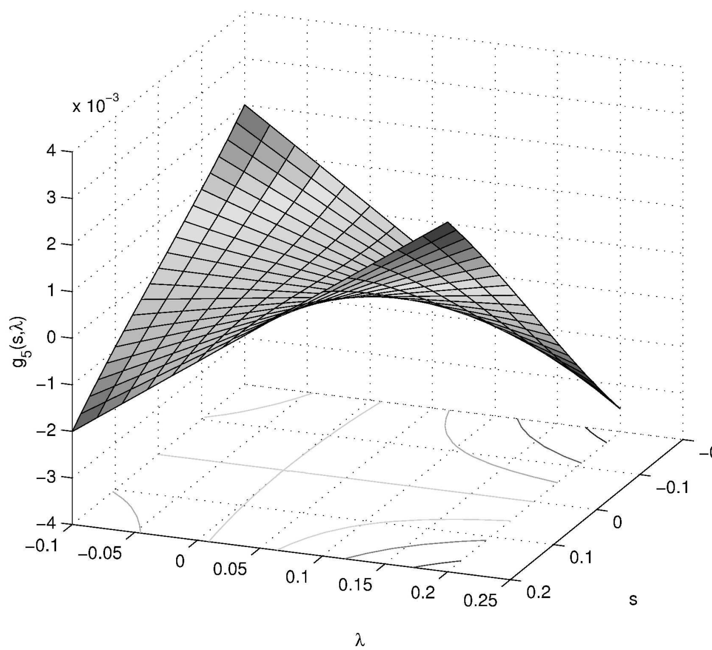


Fig. 4.1. Approximation $g_5$ of the reduced function $g(s, \lambda)$ for the critical value $\lambda = \pi^2$

In Fig. 4.1, we have additionally plotted the zero-set of $g_5$. There, we can see two curves which intersect in the shape of a pitchfork bifurcation. Comparison of Fig. 4.1 to the plot of the normal form of a pitchfork bifurcation (see, e.g., Fig. 4.2) shows that a pitchfork bifurcation at $\lambda = \pi^2$ is actually present.

The second example is the following parametrized BVP:

$$x_1'(t) = x_2(t), \quad x_2'(t) = \lambda(2 + x_1(t) + x_1(t)^2) + \lambda^2 t(1 - t + 2x_1(t)(1 - t)) + \lambda^3(t^2 - t)^2,$$
$$x_1(0) = 0, \quad x_1(1) = 0. \tag{4.2}$$

Here, we have a singular point at $\lambda = -\pi^2$. With the same steps as in the previous example we computed an approximation of the corresponding reduced function $g$ on the grid $G$. This grid has been generated by the MATLAB command

$$[s, \lambda] = \text{meshgrid}(-0.1 : 0.005 : 0.1, -0.05 : 0.01 : 0.06).$$

The plot of the computed function values together with the zero-set can be seen in Fig. 4.3. Obviously, there are two intersecting lines in the zero-set. Comparison of Fig. 4.3 to the plot of a normal form of a secondary bifurcation point (see, e.g., Fig. 4.4) shows that a secondary bifurcation at $\lambda = -\pi^2$ is actually present.

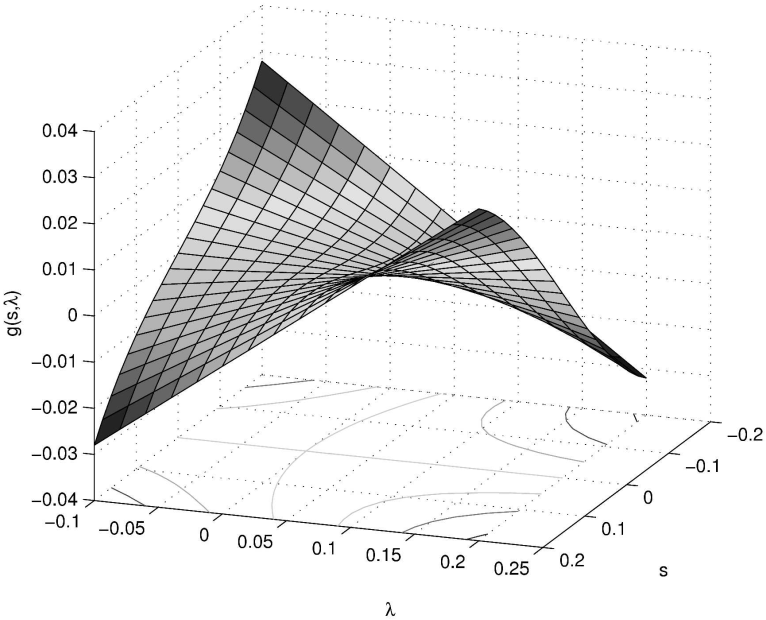


Fig. 4.2. Normal form $g(s,\lambda) = -s^3 + \lambda s$ of a pitchfork bifurcation

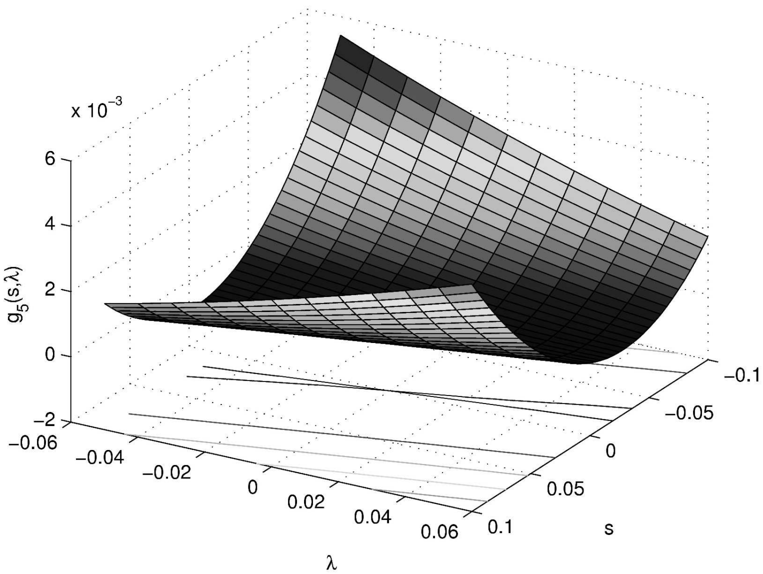


Fig. 4.3. Approximation $g_5$ of the reduced function $g(s,\lambda)$ for the critical value $\lambda = -\pi^2$

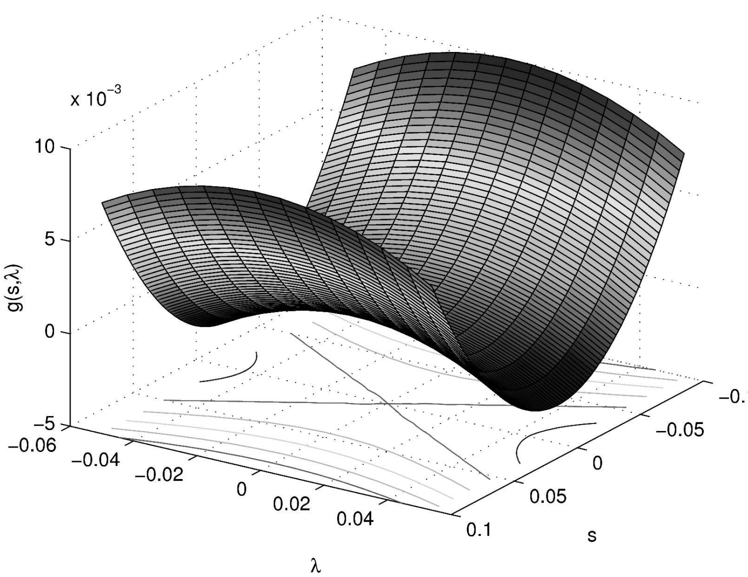


Fig. 4.4. Normal form $g(s,\lambda) = s^2 - \lambda^2$ of a secondary bifurcation

## 5. Conclusions

In this paper, we propose a new method for analyzing simple singular points in parameterized two-point BVPs. It is based on the numerical approximation of a function which is equivalent to the corresponding Liapunov — Schmidt reduced function. Our method is not restricted to BVPs. It is applicable to all problems which can be formulated as an operator equation with nonlinear Fredholm operators acting in Banach spaces. The only restriction is the presence of a simple singular point.

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