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ON THE STABILITY OF FINITE-DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS SUBJECT TO INTEGRAL CONDITIONS WITH APPLICATIONS TO THERMOELASTICITY

Ž. JESEVIČIŪTĖ[1] AND M. SAPAGOVAS[1]

Abstract — The stability of implicit difference scheme for parabolic equations subject to integral conditions, which correspond to the quasi-static flexure of a thermoelastic rod is considered. The stability analysis is based on the spectral structure of matrix of the difference scheme. The stability conditions obtained here differ from those presented in the articles of other authors.

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1. Introduction

This article investigates the stability of difference schemes for parabolic equations subject to nonlocal integral conditions

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad a < x < b, \quad 0 \leqslant t \leqslant T, \tag{1.1}$$

$$u(a,t) = \int_a^b \alpha(x) u(x,t) dx + \mu_1(t), \tag{1.2}$$

$$u(b,t) = \int_a^b \beta(x) u(x,t) dx + \mu_2(t), \tag{1.3}$$

$$u(x,0) = \varphi(x). \tag{1.4}$$

The investigation of parabolic equations subject to nonlocal conditions was started in articles [3, 11], where the heat equation subject to the specification of energy is analyzed. There in the formulation of the problem one of the boundary conditions is replaced by a nonlocal one of the form

$$\int_a^b g(x,t) u(x,t) dx = E(t), \quad 0 \leqslant t \leqslant T,$$

[1] *Institute of Mathematics and Informatics*, Akademijos str. 4, LT-08663, Vilnius, Lithuania.

where a and b are the functions of variable t, in general. Article [10] deals with the nonlocal condition of the different type, which relates the boundary values of the unknown function

$$u'(a) = u'(b).$$

For the first time nonlocal conditions of the type (1.2), (1.3) were introduced in [5]. In investigating the quasi-static flexure of a thermoelastic rod, this article proves that entropy $\eta(x,t)$ is a solution of the parabolic equation

$$\frac{\partial \eta}{\partial t} = a^2 \frac{\partial^2 \eta}{\partial x^2}, \quad -l < x < l, \tag{1.5}$$

subject to the integral conditions

$$\eta(-l,t) = -w \int_{-l}^{l} (l - 3x)\eta(x,t)dx, \tag{1.6}$$

$$\eta(l,t) = -w \int_{-l}^{l} (l + 3x)\eta(x,t)dx, \tag{1.7}$$

and some initial condition. There a and w are constants describing the physical and geometrical properties. The stability of finite difference schemes subject to various types of nonlocal conditions has been investigated in numerous articles. Article [6] proves the stability of finite difference schemes for parabolic equations subject to integral conditions of the type (1.2), (1.3) provided the following conditions hold:

$$\int_a^b |\alpha(x)|\, dx < 1, \quad \int_a^b |\beta(x)|\, dx < 1. \tag{1.8}$$

Sufficient stability conditions similar to conditions (1.8) have been obtained in a lot of articles, e.g., the stability conditions in [12] are as follows:

$$\int_a^b |\alpha(x)|^2\, dx + \int_a^b |\beta(x)|^2\, dx < 2. \tag{1.9}$$

Article [7] proves the stability of finite difference schemes for the nonlinear parabolic equation subject to the integral conditions (1.2), (1.3) under the conditions

$$\int_a^b |\alpha(x)|^2 dx < 1, \quad \int_a^b |\beta(x)|^2 dx < 1. \tag{1.10}$$

The stability of the finite difference schemes for parabolic equations subject to various other types of nonlocal conditions is dealt in the articles [2, 4, 8] (also, see [9] and the references therein). Article [13] uses the method of lines to solve the quasi-linear parabolic equation.

In the present paper, we investigate the stability of implicit difference schemes for parabolic equations subject to the integral conditions (1.2), (1.3), where the values of $\alpha(x)$,

$\beta(x)$ correspond to the quasi-static flexure of a thermoelastic rod [5]. The stability conditions are formulated in terms of the spectral properties of the difference operator, which enabled us to get stability conditions of difference schemes that differ radically from those presented in [6, 7, 12]. Such a method of difference scheme investigation in the case $\alpha(x) = \text{const}$, $\beta(x) = \text{const}$ was used in [16], and in [17] the same method is used for some other concrete expressions of $\alpha(x)$, $\beta(x)$.

The paper is organized as follows. In Section 2 differential and difference problems are stated. In Section 3 the stability of finite difference schemes in terms of the spectrum of the nonsymmetric difference operator is discussed. Section 4 contains analysis of the spectrum of the operator, as well as the theoretical results of the investigation of the spectrum. This is followed by the results of computer modeling in Section 5.

2. Problem formulation

We investigate a parabolic equation subject to the integral conditions (1.2), (1.3). We give these conditions in a different form, so the differential problem takes the following form:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad -l < x < l, \quad 0 \leqslant t \leqslant T \tag{2.1}$$

$$u(-l,t) = \frac{\gamma_1}{l^2} \int_{-l}^{l} \alpha(x) u(x,t) dx + \mu_1(t), \tag{2.2}$$

$$u(l,t) = \frac{\gamma_2}{l^2} \int_{-l}^{l} \beta(x) u(x,t) dx + \mu_2(t), \tag{2.3}$$

$$u(x,0) = \varphi(x), \tag{2.4}$$

where

$$\alpha(x) = l - 3x, \quad \beta(x) = l + 3x, \tag{2.5}$$

γ_1, γ_2 are the given parameters.

We construct a difference scheme for this problem

$$\frac{U_i^{j+1} - U_i^j}{\tau} = \frac{U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}}{h^2} + f_i^{j+1}, \quad i = \overline{-N+1, N-1}, \tag{2.6}$$

$$U_{-N}^{j+1} = l_1(U), \tag{2.7}$$

$$U_N^{j+1} = l_2(U), \tag{2.8}$$

$$U_i^0 = \varphi_i, \quad i = \overline{-N, N}, \tag{2.9}$$

where $h = l/N$, $\tau = T/M$,

$$l_1(U) = \frac{\gamma_1 h}{l^2} \left(\frac{\alpha_{-N} U_{-N}^{j+1} + \alpha_N U_N^{j+1}}{2} + \sum_{i=-N+1}^{N-1} \alpha_i U_i^{j+1} \right) + \mu_1^{j+1}, \tag{2.10}$$

$$l_2(U) = \frac{\gamma_2 h}{l^2} \left(\frac{\beta_{-N} U_{-N}^{j+1} + \beta_N U_N^{j+1}}{2} + \sum_{i=-N+1}^{N-1} \beta_i U_i^{j+1} \right) + \mu_2^{j+1}. \tag{2.11}$$

3. Stability of the difference scheme

Firstly, to investigate the stability of the difference scheme (2.6)–(2.9), we give it in the standard form

$$U^{j+1} = SU^j + \bar{f}^j, \tag{3.1}$$

where $U^j = (U^j_{-N+1}, U^j_{-N+2}, \dots, U^j_{N-1})'$ is the solution of the difference scheme on the $(j+1)$-th time layer in vector form.

To achieve this, we give (2.7), (2.8) in the form of the system of two equations in two unknowns U^{j+1}_{-N}, U^{j+1}_N

$$\begin{cases} \left(1 - \dfrac{\gamma_1 h \alpha_{-N}}{2l^2}\right) U^{j+1}_{-N} - \dfrac{\gamma_1 h \alpha_N}{2l^2} U^{j+1}_N = \dfrac{\gamma_1 h}{l^2} \displaystyle\sum_{i=-N+1}^{N-1} \alpha_i U^{j+1}_i + \mu^{j+1}_1, \\ -\dfrac{\gamma_2 h \beta_{-N}}{2l^2} U^{j+1}_{-N} + \left(1 - \dfrac{\gamma_2 h \beta_N}{2}\right) U^{j+1}_N = \dfrac{\gamma_2 h}{l^2} \displaystyle\sum_{i=-N+1}^{N-1} \beta_i U^{j+1}_i + \mu^{j+1}_2. \end{cases} \tag{3.2}$$

Next, we express the unknowns U^{j+1}_{-N}, U^{j+1}_N in terms of the remaining unknowns U^{j+1}_i, $i = \overline{-N+1, N-1}$,

$$U^{j+1}_{-N} = \sum_{i=-N+1}^{N-1} \bar{\alpha}_i U^{j+1}_i + \bar{\mu}^{j+1}_1, \tag{3.3}$$

$$U^{j+1}_N = \sum_{i=-N+1}^{N-1} \bar{\beta}_i U^{j+1}_i + \bar{\mu}^{j+1}_2. \tag{3.4}$$

We do not give the expressions of the coefficients $\bar{\alpha}_i, \bar{\beta}_i, \bar{\mu}^{j+1}_1, \bar{\mu}^{j+1}_2$ because they will not be used further. Equalities (3.3), (3.4) hold, provided the determinant of system (3.2) is nonzero, i.e., if

$$\begin{vmatrix} 1 - \dfrac{\gamma_1 h \alpha_{-N}}{2l^2} & -\dfrac{\gamma_1 h \alpha_N}{2l^2} \\ -\dfrac{\gamma_2 h \beta_{-N}}{2l^2} & 1 - \dfrac{\gamma_2 h \beta_N}{2} \end{vmatrix} \neq 0.$$

This inequality reduces to

$$1 - \frac{2h(\gamma_1 + \gamma_2)}{l} + \frac{3h^2 \gamma_1 \gamma_2}{l^2} \neq 0. \tag{3.5}$$

The left-hand side of (3.5), being a quadratic trinomial in h, is equal to zero for two values of h_1, h_2. For the rest values (3.5) is true. In the thermoelasticity problems [5] $\gamma_1 = \gamma_2 < 0$. For $\gamma_1 = \gamma_2 < 0$, inequality (3.5) imposes no restrictions on the mesh step h, thus, it holds for any value $h > 0$. In the case $\gamma = \gamma_1 = \gamma_2 > 0$, the roots of the quadratic trinomial are $h_1 = 1/3\gamma l, h_2 = 1/\gamma l$, therefore, inequality (3.5) is true for any sufficiently small values of h: $0 < h < 1/3\gamma l$. In general, if $\gamma_1 \neq \gamma_2$, then one can argue that inequality (3.5) is true, as h is sufficiently small.

Next, we substitute expression (3.3) for U^{j+1}_{-N} into equation (2.6), as $i = -N+1$, and the expression U^{j+1}_N into equation (2.6), as $i = N-1$. The difference scheme (2.1)–(2.3) in the $(j+1)$-th time layer takes the form

$$U^{j+1} = U^j + \tau(-AU^{j+1} + f^{j+1}), \tag{3.6}$$

where the matrix A of order $(2N-1)$ is defined by

$$A = \frac{1}{h^2}\begin{pmatrix} 2-\bar{\alpha}_{-N+1} & -1-\bar{\alpha}_{-N+2} & \dots & \dots & -\bar{\alpha}_{N-1} \\ -1 & 2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ & & & 2 & -1 \\ -\bar{\beta}_{N+1} & -\bar{\beta}_{N+2} & \dots & -1-\bar{\beta}_{N-2} & 2-\bar{\beta}_{N-1} \end{pmatrix}. \tag{3.7}$$

The vector f^{j+1} is defined by the values of f_i^{j+1} and $\bar{\mu}_l^{j+1}, l = 1, 2$. Now it is possible to give the difference scheme (3.6) in the form (3.1), where

$$S = (E + \tau A)^{-1}, \quad \bar{f}^j = \tau (E + \tau A)^{-1} f^{j+1}. \tag{3.8}$$

The necessary stability condition of the difference scheme (3.1) can be given in the form [14]

$$\|S\| \leqslant 1 + C_0 \tau, \tag{3.9}$$

involving a constant C_0 not depending on τ or h. If S is a symmetric matrix, we can use a different sufficient condition instead of (3.9)

$$\rho(S) < 1, \tag{3.10}$$

where $\rho(S)$ is the spectral radius of the matrix S. However, in the case of nonlocal conditions, the matrix S is always nonsymmetric. For such matrices S, the inequality $\rho(S) < 1$ is a necessary and sufficient condition for defining the matrix norm $\|S\|_*$, such that the inequality $\|S\|_* < 1$ holds (see [1]).

Note that if S is a simple structured matrix, then $\|S\|_*$ can be defined as follows [15]:

$$\|S\|_* = \rho(S).$$

Then the vector norm compatible with the matrix norm is defined by

$$\|U\|_* = \|P^{-1}U\|_3, \tag{3.11}$$

where P is a matrix whose columns are linearly independnt eigenvectors of the matrix S, and $\|U\|_3 = (u, u)^{1/2}$.

Therefore, we use the sufficient stability condition (3.10) of the difference scheme for the nonsymmetric matrix S as well. For a more detailed analysis of the issue see [2, 9, 17].

Since $S = (E + \tau A)^{-1}$, then the sufficient stability condition $\rho(S) < 1$ of the difference scheme can be given in the following way [17]:

$$\operatorname{Re} \lambda_i(A) > 0, \quad i = \overline{-N+1, N-1}. \tag{3.12}$$

4. Spectral structure of the matrix A

This section is devoted to the investigation of the spectrum of the matrix A.

Let us analyze the following differential eigenvalue problem:

$$\frac{d^2u}{dx^2} + \lambda u = 0, \quad -l < x < l,$$

$$u(-l) = \frac{\gamma_1}{l^2}\int_{-l}^{l} \alpha(x)u(x)dx, \quad u(l) = \frac{\gamma_2}{l^2}\int_{-l}^{l} \beta(x)u(x)dx,$$

We construct a difference analog for the differential problem

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + \lambda U_i = 0, \quad i = \overline{-N+1, N-1}, \tag{4.1}$$

$$U_{-N} = l_1(U), \tag{4.2}$$

$$U_N = l_2(U), \tag{4.3}$$

where $l_1(U)$, $l_2(U)$ are defined above by (2.10), (2.11) with $\mu_1^{j+1} = \mu_2^{j+1} = 0$.

Lemma 4.1. *If $h > 0$ is a sufficiently small positive number, then the difference eigenvalue problem* (4.1–4.3) *is equivalent to the eigenvalue problem for the matrix A given by* (3.7).

Proof. In the same manner as it was done in Section 3 for the difference scheme (2.6)–(2.8), we rearrange system (4.1)–(4.3) into a form analogous to (3.6)

$$AU = \lambda U. \tag{4.4}$$

Defining two unknows U_{-N}, U_N in terms of (4.2), (4.3), we give system (4.4) in the form (4.1)–(4.3). Therefore, problems (4.1)–(4.3) and (4.4) are equivalent. This completes the proof of the lemma. □

Consequently, the spectrum of the matrix A can be investigated by means of the eigenvalues of problem (4.1)–(4.3). For this purpose we analyze 3 separate cases.

Case 1: $\boldsymbol{\lambda = 0}$. We determine whether the number $\lambda = 0$ is an eigenvalue of problem (4.1)–(4.3). In other words, we determine if there exists a nontrivial solution U_i of problem (4.1)–(4.3) for $\lambda = 0$.

Theorem 4.1. *$\lambda = 0$ is an eigenvalue of the matrix A if and only if the following equality holds:*

$$2(2l^2 + h^2)\gamma_1\gamma_2 - \left(2l^2 + \frac{h^2}{2}\right)(\gamma_1 + \gamma_2) + l^2 = 0. \tag{4.5}$$

Proof. For $\lambda = 0$, the general solution of (4.1) is of the form

$$U_i = c_1 ih + c_2. \tag{4.6}$$

Substituting it into nonlocal conditions (4.2), (4.3) we obtain

$$\left\{(2l^3 + lh^2)\frac{\gamma_1}{l^2} - l\right\}c_1 + (1 - 2\gamma_1)c_2 = 0, \quad \left\{l - \frac{\gamma_2}{l^2}(2l^3 + lh^2)\right\}c_1 + (1 - 2\gamma_2)c_2 = 0. \tag{4.7}$$

Solution (4.6) is nontrivial if and only if the determinant of system (4.7) equals zero. This directly implies (4.5). □

Equation (4.5) on the γ_1, γ_2 coordinate plane represents a hyperbola (Fig. 4.1).

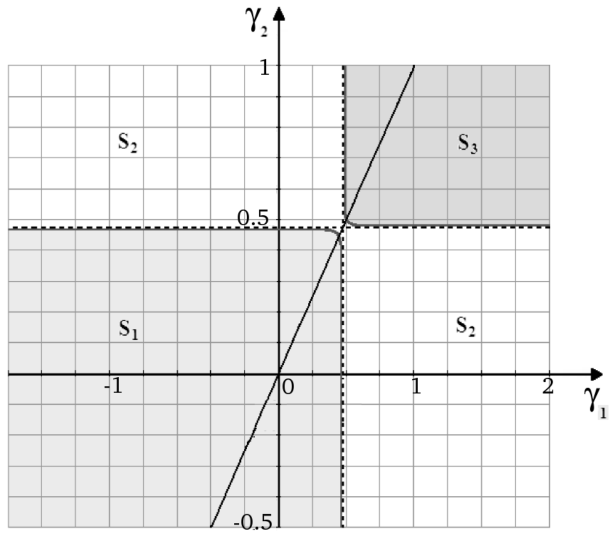

Fig. 4.1. Hyperbola and domains S_1, S_2, S_3 ($l = 1$, $N = 10$).

The asymptotes of the hyperbola are given by

$$\gamma_1 = a, \quad \gamma_2 = a, \quad a = \frac{1}{2} + O(h^2) = \frac{1}{2} - \frac{h^2}{4(2l^2 + h^2)},$$

with center in (a, a). As $h \to 0$, the branches of the hyperbola approach the asymptotes.

Remark 4.1. If the nonlocal conditions in the system of difference equations were approximated by the Simpson rule rather than by the trapezoidal one, i.e.,

$$l_1(U) = \frac{\gamma_1 h}{3l^2}\left(\alpha_{-N}U_{-N} + \alpha_N U_N + 2\sum_{k=0}^{N-1} \alpha_{-N+2k+1}U_{-N+2k+1} + 4\sum_{k=1}^{N-1} \alpha_{-N+2k}U_{-N+2k}\right), \quad (4.8)$$

$$l_2(U) = \frac{\gamma_2 h}{3l^2}\left(\beta_{-N}U_N + \beta_N U_N + 2\sum_{k=0}^{N-1} \beta_{-N+2k+1}U_{-N+2k+1} + 4\sum_{k=1}^{N-1} \beta_{-N+2k}U_{-N+2k}\right), \quad (4.9)$$

then (4.5) would become

$$4l^2\gamma_1\gamma_2 - 2l^2(\gamma_1 + \gamma_2) + l^2 = 0. \quad (4.10)$$

Therefore, in this case the hyperbola degenerates into two straight lines parallel to the coordinate axes intersecting at the point $\gamma_1 = 1/2$, $\gamma_2 = 1/2$.

Case 2: $\boldsymbol{\lambda > 0}$. In this case, we begin with determining the positive eigenvalues of the matrix A. We give equation (4.1) in the form

$$U_{i-1} - 2\left(1 - \frac{h^2\lambda}{2}\right)U_i + U_{i+1} = 0. \quad (4.11)$$

If $\lambda > 0$, then $1 - h^2\lambda/2 < 1$. We will look for positive eigenvalues satisfying the inequality

$$\left|1 - \frac{h^2\lambda}{2}\right| < 1. \quad (4.12)$$

In other words, we will find positive eigenvalues from the interval $(0, 4/h^2)$.

Theorem 4.2. *Positive eigenvalues of the matrix* A *over the interval* $(0, 4/h^2)$ *are given by*

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\alpha_k h}{2}, \tag{4.13}$$

where α_k *are the positive roots of the equation*

$$\frac{24\sin\alpha l}{B\alpha l}\left(\frac{\sin\alpha l}{A\alpha^2} - \frac{l\cos\alpha l}{B\alpha}\right)\gamma_1\gamma_2 - \left(\frac{2l\sin^2\alpha l}{B\alpha} + 6\cos\alpha l\left(\frac{\sin\alpha l}{A\alpha^2} - \frac{l\cos\alpha l}{B\alpha}\right)\right)\times$$

$$(\gamma_1+\gamma_2) + 2l^2\cos\alpha l\sin\alpha l = 0, \tag{4.14}$$

$$A = \frac{4}{\alpha^2h^2}\sin^2\frac{\alpha h}{2}, \quad B = \frac{2}{\alpha h}\tan\frac{\alpha h}{2}. \tag{4.15}$$

Proof. In the case where inequality (4.12) holds, denote $1-(h^2\lambda/2) = \cos\alpha h$. It implies

$$\lambda = \frac{4}{h^2}\sin^2\frac{\alpha h}{2}.$$

In this case, the general solution of equation (4.11) is given by

$$U_i = C_1\cos\alpha ih + C_2\sin\alpha ih. \tag{4.16}$$

Substituting this form of the solution into the nonlocal conditions (4.2), (4.3) and making elementary rearrangements, we get

$$C_1\cos\alpha l - C_2\sin\alpha l = \frac{2\sin\alpha l}{B\alpha l}\gamma_1 C_1 - 6\left(\frac{\sin\alpha l}{A\alpha^2} - \frac{l\cos\alpha l}{B\alpha}\right)\frac{\gamma_1}{l^2}C_2,$$

$$C_1\cos\alpha l + C_2\sin\alpha l = \frac{2\sin\alpha l}{B\alpha l}\gamma_2 C_1 + 6\left(\frac{\sin\alpha l}{A\alpha^2} - \frac{l\cos\alpha l}{B\alpha}\right)\frac{\gamma_2}{l^2}C_2. \tag{4.17}$$

Solution (4.16) is not a trivial one if and only if the determinant of system (4.17) is zero. Evaluation of the determinant yields (4.14). This completes the proof of the theorem. □

In general, equation (4.14) can have an infinite number of roots. However, we are interested in the number of distinct eigenvalues λ_k, defined by (4.13) of the matrix A corresponding to these roots. For this purpose denote the left-hand side of equation (4.14) as $\Phi(\alpha, \gamma_1, \gamma_2)$ and then investigate the properties of this function.

Lemma 4.2. *The function* $\Phi(\alpha, \gamma_1, \gamma_2)$ *is a periodic function of the variable* α *with period* $2\pi N/l$, *i.e.*,

$$\Phi\left(\alpha + \frac{2\pi N}{l}, \gamma_1, \gamma_2\right) = \Phi(\alpha, \gamma_1, \gamma_2).$$

The implication of the lemma is based upon the evident fact that all summands in $\Phi(\alpha, \gamma_1, \gamma_2)$ are periodic (this clearly seen when using A, B given by (4.15)). The least period is defined by the functions $\sin^2(\alpha h/2)$, $\tan(\alpha h/2)$.

Lemma 4.3. *The following equality is true*:

$$\Phi\left(\frac{2\pi N}{l} - \alpha, \gamma_1, \gamma_2\right) = \Phi(\alpha, \gamma_1, \gamma_2). \tag{4.18}$$

Proof. The validity of equality (4.18) is confirmed directly by the formulas

$$\sin\left(\frac{2\pi N}{l}-\alpha\right)l=-\sin\alpha l,\quad \cos\left(\frac{2\pi N}{l}-\alpha\right)l=\cos\alpha l,$$

$$\sin\frac{\left(\frac{2\pi N}{l}-\alpha\right)h}{2}=\sin\frac{\alpha h}{2},\quad \cos\frac{\left(\frac{2\pi N}{l}-\alpha\right)h}{2}=-\cos\frac{\alpha h}{2}.$$

□

Corolary 4.1. Lemmas 4.2 and 4.3 directly imply that eigenvalues λ_k given by (4.13) are distinct only for $\alpha\in(0,\pi N/l)$. For the roots of the function $\Phi(\alpha,\gamma_1,\gamma_2)$ not belonging to this interval, other eigenvalues λ_k do not appear (Fig. 4.2). If $\gamma_1=\gamma_2=2, N=10, l=1$, then there are 35 different roots of the function $\Phi(\alpha,\gamma_1,\gamma_2)$ in the interval $(0,20\pi)$. It is easy to calculate that $\alpha_1=20\pi-\alpha_{35}\approx 3.9238, \alpha_2=20\pi-\alpha_{34}\approx 4.9158, \alpha_3=20\pi-\alpha_{33}=7.3765$ and s.o. So, there are 17 different roots in the interval $(0,10\pi)$.

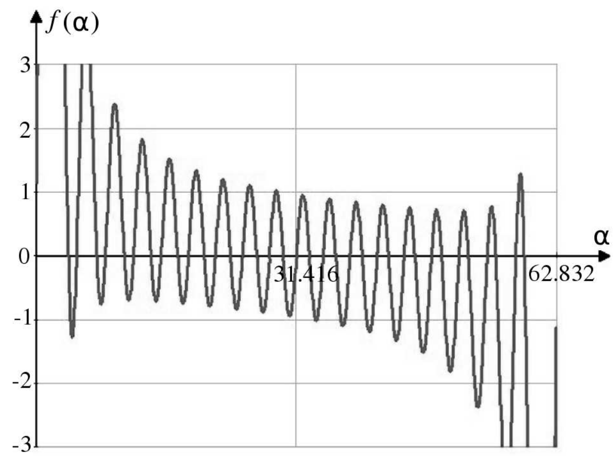

Fig. 4.2. Roots of the function $\Phi(\alpha,\gamma_1,\gamma_2)$ in the first period $\alpha\in(0,2\pi N/l)$, $\gamma_1=\gamma_2=2$, $N=10$, $l=1$.

Case 3: $\boldsymbol{\lambda<0}$. If $\lambda<0$, then $1-h^2\lambda/2>1$. Let us denote

$$1-\frac{h^2\lambda}{2}=\cosh\beta h.$$

This implies

$$\lambda=-\frac{4}{h^2}\sinh^2\frac{\beta h}{2}.$$

The general solution of equation (4.11) is given by

$$U_i=C_1\cosh\beta ih+C_2\sinh\beta ih. \tag{4.19}$$

Theorem 4.3. *If the matrix A has some negative eigenvalues, then these values are given by*

$$\lambda_k=-\frac{4}{h^2}\sinh^2\frac{\beta_k h}{2}, \tag{4.20}$$

where β_k are positive roots of the equation

$$\frac{24\sinh\beta l}{B\beta l}\left(\frac{\sinh\beta l}{A\beta^2}-\frac{l\cosh\beta l}{B\beta}\right)\gamma_1\gamma_2+$$

$$\left(\frac{2l\sinh^2\beta l}{B\beta}-6\cosh\beta l\left(\frac{\sinh\beta l}{A\beta^2}-\frac{l\cosh\beta l}{B\beta}\right)\right)(\gamma_1+\gamma_2)-2l^2\cosh\beta l\sin\beta l=0, \tag{4.21}$$

$$A=\frac{4}{\beta^2h^2}\sinh^2\frac{\beta h}{2}, \quad B=\frac{2}{\beta h}\tan\frac{\beta h}{2}. \tag{4.22}$$

The proof is analogous to the proof of Theorem 4.3.

5. Computating experiment

The computating experiment allows us to gain a deeper insight into the spectral properties of the matrix A. It gives an opportunity to clarify some of the commonalities or fenomena characteristic of nonlocal problems. One of the most important goals of the computating experiment is to determine the number of negative and positive eigenvalues of the matrix A and establish the dependence of that number on γ_1, γ_2. In the problems of thermoelasticity [5] it is usually $\gamma_1 = \gamma_2 < 0$. More exactly,

$$\gamma_1 = \gamma_2 = -\frac{\Theta_0 B^2}{2cA},$$

where Θ_0 is a uniform reference temperature, c is the specific heat at constant strain, A is the flexural regidity, and B is a measure of the cross-coupling between the thermal and mechanical effects. So the case $\gamma_1 = \gamma_2$ was analyzed most thoroughly. Here we present some of the rezults of the experiment.

We observe first of all that two branches of the hyperbola defined by(4.5) divide the whole γ_1, γ_2 plane into 3 infinite domains (Fig. 4.1). The domain S_1 is located below the branches of the hyperbola. This domain includes the origin ($\gamma_1 = 0, \gamma_2 = 0$), it contains a complete set of values γ_1, γ_2, characteristic of thermoelasticity, i.e., $\gamma_1 = \gamma_2 < 0$. The domain S_2 is located between the branches of the hyperbola, and the domain S_3 is above the branches. If the point (γ_1, γ_2) belongs to the hyperbola, then $\lambda = 0$ is an eigenvalue of the matrix A for γ_1, γ_2.

In the case where $\gamma_1 = \gamma_2 = 0$, the supplementary conditions of problem (4.1)–(4.3) are of the classical type, therefore all eigenvalues of the matrix A are real and distinct. Since the eigenvalues are continuous functions of the matrix elements all the real eigenvalues of A are positive over the domain S_1 (there are no trivial or negative eigenvalues in this case).

As γ_1 and γ_2, vary continuously, the point (γ_1, γ_2) in the domain S_1 reaches the hyperbola, the positive eigenvalue, varying continuously, too, turns to a trivial one. Computation of the positive roots of Eq. (4.14) for different values of γ_1, γ_2, h gives the dependence of the number of roots of the equation on the values of γ_1, γ_2, h. The following patterns were determined. In the case of $(\gamma_1, \gamma_2) \in S_2$, Eq. (4.21) has a single positive root, i.e., there exists only one negative eigenvalue of the matrix A (inequality (3.5) holds for any value of h, as $\gamma_1+\gamma_2 < 0$; if $\gamma_1 + \gamma_2 > 0$, h has to be sufficiently small).

In the case of $(\gamma_1, \gamma_2) \in S_3$, the matrix A has 2 negative eigenvalues provided h is sufficiently small, i.e., if

$$h < h_1 = \frac{\gamma_1 + \gamma_2 - \sqrt{\gamma_1^2 - \gamma_1\gamma_2 + \gamma_2}}{3\gamma_1\gamma_2}. \tag{5.1}$$

Consequently, the sufficient stability condition of the difference scheme (3.1) is $(\gamma_1, \gamma_2) \in S_1$.

Typical cases for the occurrence of negative eigenvalues are illustrated in Figs. 5.1 – 5.3. These figures show the graph of the function $f(\beta,\gamma_1,\gamma_2)$ on the left-hand side of Eq. (4.21) with $h = 1/10$. Notice that according to Theorem 1, $\lambda = 0$, when $\gamma = \gamma_1 = \gamma_2 \approx 0.4982$ and $\gamma_1 = \gamma_2 \approx 0.5018$. Therefore, as $\gamma < 0.4982$, Eq. (4.21) has no positive roots; for $0,4982 < \gamma < 0.5018$, there exists a single root; for $\gamma > 0.5018$, there exist two roots. This means that as $\gamma < 0.4982$, the matrix A has no negative eigenvalues; for $0.4982 < \gamma < 0.5018$, there exists a single negative eigenvalue λ_1 defined by formula (4.20); for $\gamma > 0.5018$, there exist two negative eigenvalues. In case $(\gamma_1,\gamma_2) \in S_3$, matrix A has two negative eigenvalues, if h satisfies inequality (5.1). If h does not satisfy this inequality (it is too large compared to γ_1,γ_2), the matrix A can posses either a single or no negative eigenvalues at all. If $(\gamma_1,\gamma_2) \in S_3$, sufficient stability condition $\rho(S) < 1$ of the difference scheme does not hold for sufficiently small values of h. This condition can be fulfilled if h is not too small. This fact was revealed earlier in [16] for problems subject to the nonlocal conditions (2.2), (2.3), in the case that $\alpha = \text{const}$, $\beta = \text{const}$.

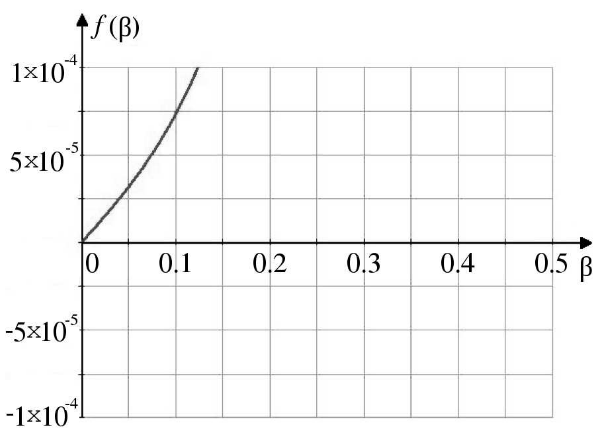

Fig. 5.1. The function $f(\beta,\gamma_1,\gamma_2)$, $\gamma_1 = \gamma_2 = 0.49$, $l = 1$, $N = 10$, $\gamma_1,\gamma_2) \in S_1$. There are no roots in the interval $\beta \in (0,\infty)$

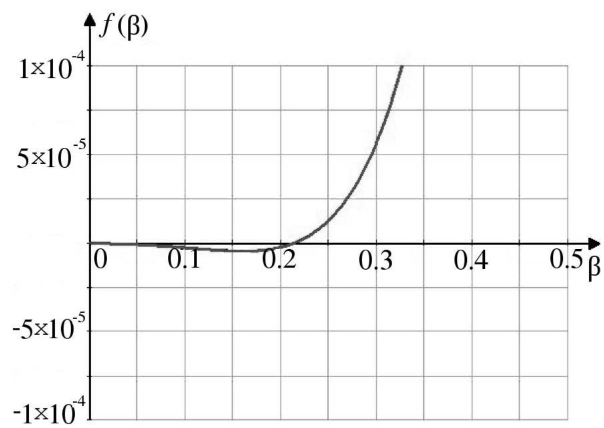

Fig. 5.2. The function $f(\beta,\gamma_1,\gamma_2)$, $\gamma_1 = \gamma_2 = 0.499$, $l = 1$, $N = 10$, $(\gamma_1,\gamma_2) \in S_2$. There is one root in the interval $\beta \in (0,\infty)$

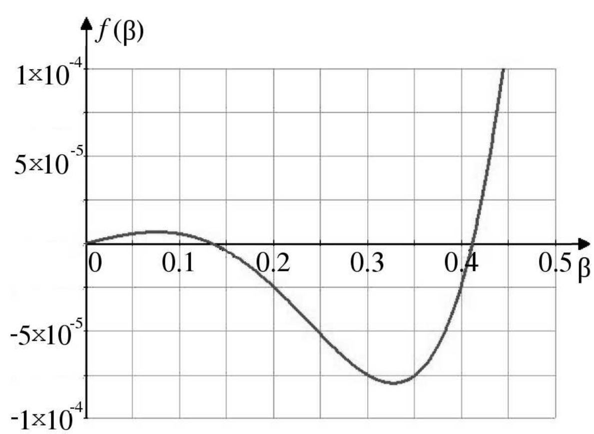

Fig. 5.3. The function $f(\beta,\gamma_1,\gamma_2)$, $\gamma_1 = \gamma_2 = 0.503$, $l = 1$, $N = 10$, $(\gamma_1,\gamma_2) \in S_3$. There are two roots in the interval $\beta \in (0,\infty)$

Figures 5.4 – 5.6 show the graph of the function $\Phi(\alpha,\gamma_1,\gamma_2)$ (the left-hand side of Eq. (4.14)). It gives the number of positive roots of Eq. (4.14), i.e., the number of positive eigenvalues of the matrix A. Hence, in all three cases ($\gamma = 0.49; 0.499; 0,503$) the total number of

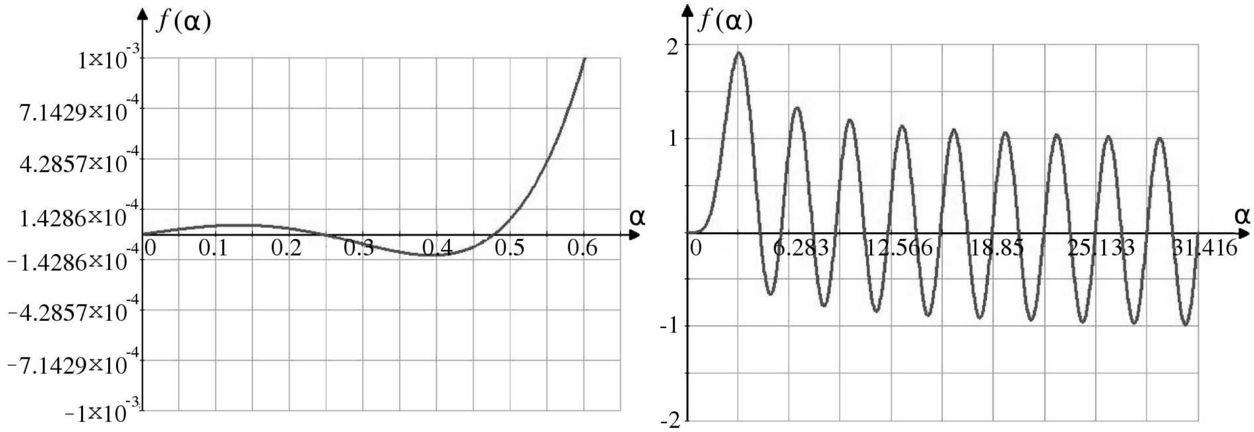

Fig. 5.4. The function $\Phi(\alpha, \gamma_1, \gamma_2)$, $\gamma_1 = \gamma_2 = 0.49$, $l = 1$, $N = 10$, $(\gamma_1, \gamma_2) \in S_1$ and $\alpha \in (0, 10\pi)$. There are 19 roots $(2N - 1 = 19)$ in the interval $\alpha \in (0, \infty)$

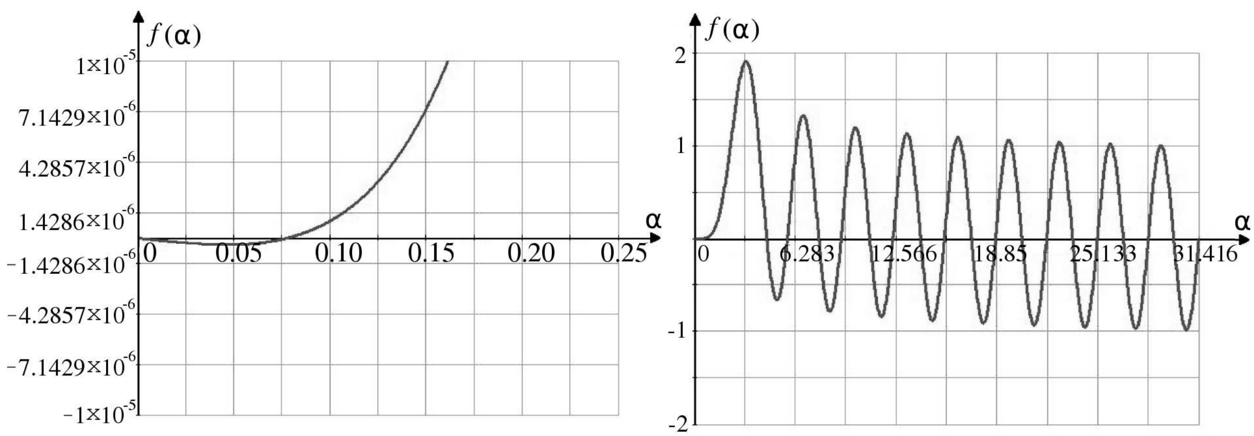

Fig. 5.5. The function $\Phi(\alpha, \gamma_1, \gamma_2)$, $\gamma_1 = \gamma_2 = 0.499$, $l = 1$, $N = 10$, $(\gamma_1, \gamma_2) \in S_2$. There are 18 roots in the interval $\alpha \in (0, 10\pi)$

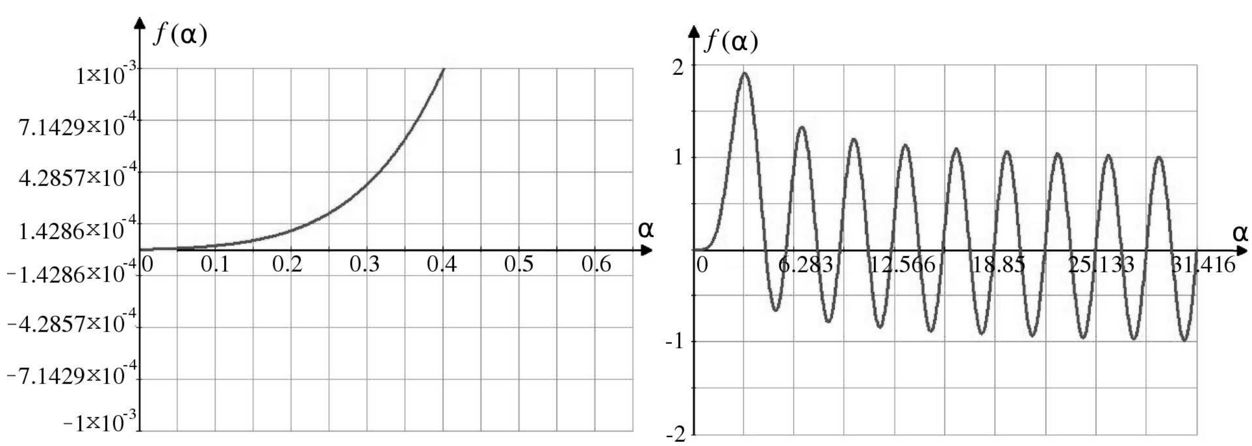

Fig. 5.6. The function $\Phi(\alpha, \gamma_1, \gamma_2)$, $\gamma_1 = \gamma_2 = 0.503$, $l = 1$, $N = 10$, $(\gamma_1, \gamma_2) \in S_3$. There are 17 roots in the interval $\alpha \in (0, 10\pi)$

negative and positive eigenvalues of A is $2N-1$, i.e., it coincides with the order of the matrix. There are no complex eigenvalues. If $(\gamma_1,\gamma_2)\in S_2$, the sufficient stability condition $\rho(S)<1$ of the difference scheme does not hold.

As was proved in Section 4 (Lemma 4.1), the eigenvalue problem (4.1)–(4.3) subject to nonlocal conditions is equivalent to the eigenvalue problem (4.4) of a matrix of A order $(2N-1)$ provided (3.5) holds. If it does not hold, i.e.,

$$1-\frac{2h(\gamma_1+\gamma_2)}{l}+\frac{3h^2\gamma_1\gamma_2}{l^2}=0 \tag{5.2}$$

the eigenvalue problem (4.1)–(4.3) cannot be reduced to problem (4.4). In the case that $\gamma_1=\gamma_2=\gamma$, (5.2) is true for two values of h

$$h_1=\frac{1}{3\gamma l},\quad h_2=\frac{1}{\gamma l}. \tag{5.3}$$

For example, if $l=1$, $N=10$, then equality (5.3) yields $\gamma^{(1)}=10/3$, $\gamma^{(2)}=10$.

As γ increases and attains the value $\gamma_1^{(1)}=10/3$, one of the negative eigenvalues of problem (4.1)–(4.3) moves from $-\infty$ to $+\infty$. There appears one positive eigenvalue not satisfying the condition $0<\lambda<4/h^2$. A similar situation arises as γ increases and reaches the point $\gamma_1^{(2)}=10$. In this case, the second negative eigenvalue dissappears. This effect is not present at all provided h is sufficiently small, i.e., $h<h_1=0.1$. This holds for the poles of the characteristic function (for more detail see [18]).

One more important corolary of the computating experiment is that the stability of the difference scheme does not depend on the parameter l.

To show the stability of the difference scheme, we have solved the illustrative problem (2.1)–(2.4). The functions $f(x,t),\mu_1(t),\mu_2(t)$ and $\varphi(x)$ were chosen so that $u(x,t)=e^{x+t}$ is the solution of the differential problem.

Tables 5.1 and 5.2 give the maximal error of the finite difference method

$$\varepsilon=\max_{-N\leqslant i\leqslant N}|u(x_i,t^j)-U_i^j|$$

for $t=2$.

Table 5.1. **Values of the error ε for different γ_1, γ_2 ($h=0.025$, $\tau=h^2/2$, $t=2$, $l=1$).**

γ_1	γ_2	ε
-100	-100	$0.425\cdot10^{-2}$
-10	-10	$0.402\cdot10^{-2}$
-10	0.25	$0.312\cdot10^{-2}$
0.25	-10	$0.443\cdot10^{-2}$
-1	-1	$0.275\cdot10^{-2}$
0	0	$0.326\cdot10^{-3}$
0.25	0.25	$0.355\cdot10^{-2}$
0.5	0.5	$0.487\cdot10^{-2}$
0.51	0.51	$0.623\cdot10^{-1}$
0.55	0.55	0.245

Table 5.2. **Values of the error ε for different N ($\gamma_1=\gamma_2=1000$, $h=l/N$, $\tau=h^2/2$, $t=2, l=1$).**

N	ε
10	$0.728\cdot10^{-1}$
20	$0.182\cdot10^{-1}$
40	$0.415\cdot10^{-2}$
80	$0.132\cdot10^{-2}$
160	$0.501\cdot10^{-2}$
320	$0.197\cdot10^{-1}$
640	$0.721\cdot10^{-1}$
1280	0.272
2560	unstable

The results of the computation illustrate well the fact that $(\gamma_1,\gamma_2)\in S_1$ is the sufficient stability condition of the difference scheme. Moreover, the results show, that for sufficiently large values of $\gamma = \gamma_1 = \gamma_2$, $(\gamma_1, \gamma_2) \in S_3$, one can compute the solution of the difference scheme with sufficient accuracy provided h is a relatively large number.

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