COMPUTATIONAL METHODS IN APPLIED MATHEMATICS, Vol. 8 (2008), No. 4, pp. 374–385

NUMERICAL SOLUTION OF A HYPERBOLIC TRANSMISSION PROBLEM

B. S. JOVANOVIĆ[1] AND L. G. VULKOV[2]

Abstract — In this paper we investigate an initial boundary value problem for a one-dimensional hyperbolic equation in two disconnected intervals. A finite difference scheme approximating this problem is proposed and analyzed. An estimate of the convergence rate has been obtained.

2000 Mathematics Subject Classification: 65M15.
Keywords: transmission problem, initial-boundary value problem, disconnected domains, Sobolev spaces, finite differences, convergence rate.

1. Introduction

Layers with material properties which significantly differ from those of the surrounding medium appear in a variety of applications [3,4,6]. The layer may have a structural, thermal, electromagnetic or optical role, etc. The processes in domains with layers can be modelled by boundary value problems whose solutions are defined in two (or more) domains. In some cases these domains are disconnected. For example, such a situation occurs when the solution in the intermediate region is known or can be determined from a simpler equation. The effect of the intermediate region can be taken into account by means of nonlocal jump conditions [6, 7].

In this paper, as a model example, an initial boundary value problem for a one-dimensional hyperbolic equation in two disconnected intervals is considered. A finite difference scheme for its solution is proposed and analyzed. An estimate of the convergence rate is obtained. Analogous parabolic problem has been considered in [12, 17].

2. Formulation of the boundary value problem

We consider the following initial-boundary-value problem (IBVP): Find functions $u_1(x,t)$ and $u_2(x,t)$ that satisfy the system of hyperbolic equations

$$\varrho_1(x)\frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x}\left(p_1(x)\frac{\partial u_1}{\partial x}\right) = f_1(x,t), \quad x \in \Omega_1 \equiv (a_1, b_1), \quad t > 0, \tag{2.1}$$

$$\varrho_2(x)\frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x}\left(p_2(x)\frac{\partial u_2}{\partial x}\right) = f_2(x,t), \quad x \in \Omega_2 \equiv (a_2, b_2), \quad t > 0, \tag{2.2}$$

[1] *Faculty of Mathematics, University of Belgrade*, Studentski trg 16, 11000 Belgrade, Serbia. E-mail: bosko@matf.bg.ac.yu
[2] *Department of Mathematics, University of Rousse*, Studentska str. 8, 7017 Rousse, Bulgaria. E-mail: vulkov@ami.ru.acad.bg

where $-\infty < a_1 < b_1 < a_2 < b_2 < +\infty$, the internal boundary conditions of Robin-Dirichlet type are

$$p_1(b_1)\frac{\partial u_1(b_1,t)}{\partial x} + \alpha_1 u_1(b_1,t) = \beta_1 u_2(a_2,t), \tag{2.3}$$

$$-p_2(a_2)\frac{\partial u_2(a_2,t)}{\partial x} + \alpha_2 u_2(a_2,t) = \beta_2 u_1(b_1,t), \tag{2.4}$$

the simplest external Dirichlet boundary conditions are

$$u_1(a_1,t) = 0, \quad u_2(b_2,t) = 0, \tag{2.5}$$

and the initial conditions are

$$u_1(x,0) = u_{10}(x), \quad u_2(x,0) = u_{20}(x), \tag{2.6}$$

$$\frac{\partial u_1}{\partial t}(x,0) = u_{11}(x), \quad \frac{\partial u_2}{\partial t}(x,0) = u_{21}(x). \tag{2.7}$$

Throughout the paper we assume that the data satisfy the usual regularity and ellipticity conditions

$$\varrho_i(x) \in L_\infty(\Omega_i), \quad 0 < \varrho_{0i} \leqslant \varrho_i(x) \quad \text{a.e. in} \quad \Omega_i, \quad i = 1,2, \tag{2.8}$$

$$p_i(x) \in L_\infty(\Omega_i), \quad 0 < p_{0i} \leqslant p_i(x) \quad \text{a.e. in} \quad \Omega_i, \quad i = 1,2. \tag{2.9}$$

In particular, we require that p_1 be continuous in a suitable left neighbourhood of b_1 and p_2 be continuous in a suitable right neighbourhood of a_2. We also assume that

$$\alpha_i > 0, \quad \beta_i > 0, \quad i = 1,2, \quad \text{and} \quad \beta_1\beta_2 \leqslant \alpha_1\alpha_2. \tag{2.10}$$

By C we denote a generic positive constant independent of the solution of the IBVP and the mesh-sizes, which can take different values in different formulas.

The aim of the present paper is to investigate the properties of the IBVP (2.1)–(2.7) and construct an efficient finite difference scheme (FDS) for its numerical solution.

The layout of the paper is as follows. Section 3 is devoted to the analysis of the existence and the uniqueness of the weak solution to the IBVP (2.1)–(2.7). In Section 4 we introduce an FDS approximating IBVP (2.1)–(2.7) and investigate its convergence.

3. Existence and uniqueness of a weak solution

Let conditions (2.10) hold. We introduce the product space

$$L = L_2(\Omega_1) \times L_2(\Omega_2) = \{v = (v_1, v_2) \mid v_i \in L_2(\Omega_i)\},$$

endowed with the inner product and the associated norm

$$(u,v)_L = \beta_2(u_1,v_1)_{L_2(\Omega_1)} + \beta_1(u_2,v_2)_{L_2(\Omega_2)}, \quad \|v\|_L = (v,v)_L^{1/2},$$

where

$$(u_i,v_i)_{L_2(\Omega_i)} = \int_{\Omega_i} u_i v_i \, dx, \quad i = 1,2.$$

We also define the spaces

$$H^k = \{v = (v_1,v_2) \mid v_i \in H^k(\Omega_i)\}, \quad k = 1,2,\ldots,$$

endowed with the inner products and norms

$$(u,v)_{H^k} = \beta_2(u_1,v_1)_{H^k(\Omega_1)} + \beta_1(u_2,v_2)_{H^k(\Omega_2)}, \quad \|v\|_{H^k} = (v,v)_{H^k}^{1/2},$$

where

$$(u_i,v_i)_{H^k(\Omega_i)} = \sum_{j=0}^{k} \left(\frac{d^j u_i}{dx^j}, \frac{d^j v_i}{dx^j}\right)_{L_2(\Omega_i)}, \quad i=1,2, \quad k=1,2,\dots$$

In particular, we set

$$H_0^1 = \{v=(v_1,v_2)\in H^1 \mid v_1(a_1)=0, \;\; v_2(b_2)=0\}.$$

Finally, we define the following bilinear form:

$$A(u,v) = \beta_2 \int\limits_{\Omega_1} \left(p_1 \frac{du_1}{dx}\frac{dv_1}{dx}\right) dx + \beta_1 \int\limits_{\Omega_2} \left(p_2 \frac{du_2}{dx}\frac{dv_2}{dx}\right) dx +$$

$$\beta_2\alpha_1 u_1(b_1)v_1(b_1) + \beta_1\alpha_2 u_2(a_2)v_2(a_2) - \beta_1\beta_2\big[u_1(b_1)v_2(a_2) + u_2(a_2)v_1(b_1)\big]. \tag{3.1}$$

Lemma 3.1. *Under conditions* (2.9) *and* (2.10) *the bilinear form* A*, defined by* (3.1)*, is symmetric and bounded on* $H^1\times H^1$*. Moreover, this form is also coercive on* H_0^1*, i.e., there exists a constant* $c_0>0$ *such that*

$$A(u,u) \geqslant c_0\|u\|_{H^1}^2, \quad \forall v\in H_0^1.$$

Proof. The symmetry of A is obvious, while its boundedness follows from (2.9) and the embeddings $H^1(\Omega_i)\subset C(\bar\Omega_i)$, $i=1,2$. Under condition (2.10) we have

$$\beta_2\alpha_1 u_1^2(b_1) + \beta_1\alpha_2 u_2^2(a_2) - 2\beta_1\beta_2 u_1(b_1)u_2(a_2) \geqslant 0,$$

which, together with (2.9) and Poincaré type inequalities

$$\int\limits_{\Omega_i} u_i^2(x)\,dx \leqslant \frac{b_i^2-a_i^2}{2}\int\limits_{\Omega_i}\left(\frac{du_i}{dx}(x)\right)^2 dx, \quad i=1,2,$$

ensures the coercivity of A. □

Let Ω be a domain in $\mathbb{R}^n$ and $u(t)$ a function mapping Ω into a Hilbert space H. In a standard manner (see [13]) we define the Sobolev space of the vector-valued functions $H^k(\Omega,H)$, endowed with the inner product

$$(u,v)_{H^k(\Omega,H)} = \int\limits_{\Omega}\sum_{|\alpha|\leqslant k}(D^\alpha u(t),\, D^\alpha v(t))_H\,dt\,, \quad k=0,1,2,\dots$$

For $k=0$ we set $L_2(\Omega,H)=H^0(\Omega,H)$. We also introduce the space of continuous vector-valued functions $C(\bar\Omega,H)$, endowed with the norm

$$\|u\|_{C(\bar\Omega,H)} = \max_{t\in\bar\Omega}\|u(t)\|_H\,.$$

Multiplying equation (2.1) by $v_1(x)$, such that $v_1(a_1) = 0$, and integrating by parts using condition (2.3), we obtain the identity

$$\left(\varrho_1 \frac{\partial^2 u_1}{\partial t^2}(\cdot, t),\, v_1\right)_{L_2(\Omega_1)} + \int\limits_{\Omega_1} \left(p_1 \frac{\partial u_1}{\partial x}\frac{dv_1}{dx}\right) dx +$$

$$\alpha_1 u_1(b_1, t) v_1(b_1) - \beta_1 u_2(a_2, t) v_1(b_1) = (f_1(\cdot, t), v_1)_{L_2(\Omega_1)}.$$

Analogously, multiplying (2.2) by $v_2(x)$, such that $v_2(b_2) = 0$, and integrating by parts we obtain

$$\left(\varrho_2 \frac{\partial^2 u_2}{\partial t^2}(\cdot, t),\, v_2\right)_{L_2(\Omega_2)} + \int\limits_{\Omega_2} \left(p_2 \frac{\partial u_2}{\partial x}\frac{dv_2}{dx}\right) dx +$$

$$\alpha_2 u_2(a_2, t) v_2(a_2) - \beta_2 u_1(b_1, t) v_2(a_2) = (f_2(\cdot, t),\, v_2)_{L_2(\Omega_2)}.$$

Now multiplying the first of these identities by β_2 and the second one by β_1 and summing up we get the weak form of (2.1)–(2.5)

$$\left(\varrho\, \frac{\partial^2 u}{\partial t^2}(\cdot, t),\, v\right)_L + A(u(\cdot, t),\, v) = (f(\cdot, t),\, v)_L\,, \quad \forall\, v \in H_0^1. \tag{3.2}$$

Applying Theorem 29.1 from [18] to (3.2) we immediately obtain the following assertion.

Theorem 3.1. *Let assumptions* (2.8)–(2.10) *hold and suppose that* $u_0 = (u_{10}, u_{20}) \in H_0^1$, $u_1 = (u_{11}, u_{21}) \in L$, $f = (f_1, f_2) \in L_2((0,T), L)$. *Then the IBVP* (2.1)–(2.7) *has a unique weak solution* $u \in L_2((0,T), H_0^1)$, *and this depends continuously on* f, u_0 *and* u_1. *Also* $\partial u/\partial t \in L_2((0,T), L)$ *and the a priori estimate*

$$\int\limits_0^T \left\| \frac{\partial u}{\partial t}(\cdot, t)\right\|_L^2 dt + \int\limits_0^T \|u(\cdot, t)\|_{H^1}^2 dt \leqslant C\, e^T \left(\|u_0\|_{H^1}^2 + \|u_1\|_L^2 + \int\limits_0^T \|f(\cdot, t)\|_L^2 dt \right)$$

holds.

A more precise a priori estimate which guarantee the global stability (see [10, 11]) of problem (2.1)–(2.7) can be obtained in the following way. Setting in (3.2) $v = 2\,\partial u/\partial t$ we obtain

$$\frac{d}{dt}\left(\left\| \sqrt{\varrho}\, \frac{\partial u}{\partial t}(\cdot, t)\right\|_L^2 + A\big(u(\cdot, t), u(\cdot, t)\big)\right) = 2\left(f(\cdot, t), \frac{\partial u}{\partial t}(\cdot, t)\right)_L \leqslant$$

$$2 \left\| \frac{f(\cdot, t)}{\sqrt{\varrho}}\right\|_L \left\| \sqrt{\varrho}\, \frac{\partial u}{\partial t}(\cdot, t)\right\|_L \leqslant 2 \left\| \frac{f(\cdot, t)}{\sqrt{\varrho}}\right\|_L \left(\left\| \sqrt{\varrho}\, \frac{\partial u}{\partial t}(\cdot, t)\right\|_L^2 + A\big(u(\cdot, t), u(\cdot, t)\big)\right)^{1/2}.$$

Integrating this inequality from 0 to t we obtain

$$\left(\left\| \sqrt{\varrho} \frac{\partial u}{\partial t}(\cdot, t)\right\|_L^2 + A(u(\cdot, t), u(\cdot, t))\right)^{1/2} - \left(\left\| \sqrt{\varrho} \frac{\partial u}{\partial t}(\cdot, 0)\right\|_L^2 + A(u(\cdot, 0), u(\cdot, 0))\right)^{1/2} \leqslant \int\limits_0^t \left\| \frac{f(\cdot, s)}{\sqrt{\varrho}}\right\|_L ds,$$

whereby, using Lemma 3.1, follows

$$\max_{t\in[0,T]} \left(\left\| \frac{\partial u}{\partial t}(\cdot, t)\right\|_L + \big\|u(\cdot, t)\big\|_{H^1}\right) \leqslant C\left(\|u_0\|_{H^1} + \|u_1\|_L + \int\limits_0^T \|f(\cdot, t)\|_L dt\right). \tag{3.3}$$

In such a manner, we proved the next assertion.

Theorem 3.2. *Let assumptions* (2.8)–(2.10) *hold and suppose that* $u_0 \in \overset{\circ}{H}{}^1$, $u_1 \in L$, $f \in L_1((0,T),L)$. *Then the IBVP* (2.1)–(2.7) *has a unique weak solution* $u \in C([0,T],\overset{\circ}{H}{}^1)$, $\partial u/\partial t \in C([0,T],L)$ *and the a priori estimate* (3.3) *holds.*

4. Finite difference approximation

4.1. Meshes and finite difference operators. Let $\bar{\omega}_{1,h_1}$ be a uniform mesh in $\bar{\Omega}_1$ with the stepsize $h_1=(b_1-a_1)/n_1$, $\omega_{1,h_1}=\bar{\omega}_{1,h_1}\cap\Omega_1$, $\omega^-_{1,h_1}=\omega_{1,h_1}\cup\{a_1\}$, $\omega^+_{1,h_1}=\omega_{1,h_1}\cup\{b_1\}$. Analogously we define a uniform mesh $\bar{\omega}_{2,h_2}$ in $\bar{\Omega}_2$ with the stepsize $h_2=(b_2-a_2)/n_2$ and its submeshes $\omega_{2,h_2}=\bar{\omega}_{2,h_2}\cap\Omega_2$, $\omega^-_{2,h_2}=\omega_{2,h_2}\cup\{a_2\}$, $\omega^+_{2,h_2}=\omega_{2,h_2}\cup\{b_2\}$. Finally, we introduce a uniform mesh $\bar{\omega}_\tau$ in $[0,T]$ with the step-size $\tau=T/n$ and set $\omega_\tau=\bar{\omega}_\tau\cap(0,T)$, $\omega^-_\tau=\omega_\tau\cup\{0\}$, $\omega^+_\tau=\omega_\tau\cup\{T\}$. We shall consider vector-functions of the form $v=(v_1,\,v_2)$ where v_i is a mesh function defined on $\bar{\omega}_{i,h_i}\times\bar{\omega}_\tau$, $i=1,2$. We define the difference quotients in the usual way (see [16])

$$v_{i,x}(x,t)=\frac{v_i(x+h_i,\,t)-v_i(x,\,t)}{h_i}=v_{i,\bar{x}}(x+h_i,\,t),$$

$$v_{i,t}(x,t)=\frac{v_i(x,\,t+\tau)-v_i(x,\,t)}{\tau}=v_{i,\bar{t}}(x,\,t+\tau).$$

We shall use the following notational conventions:

$$v_i=v_i(x,t)\,,\quad \overline{v}_i=\frac{1}{2}\left[v_i(x,t)+v_i(x,t+\tau)\right],$$

$$v_i^{(\sigma)}=\sigma\,v_i(x,t+\tau)+(1-2\sigma)\,v_i(x,t)+\sigma\,v_i(x,t-\tau)\,.$$

In this section, we shall assume that u_i belongs to $H^4(Q_i)$, where $Q_i=\Omega_i\times(0,T)$, while $p_i\in H^3(\Omega_i)$, $\varrho_i\in H^2(\Omega_i)$, $i=1,2$. We shall also assume that the quotient h_1/h_2 remains constant when $h_i,\,h_2\to 0$ and

$$\frac{\varrho_2(a_2)\,p_1(b_1)}{\varrho_1(b_1)\,p_2(a_2)}=\frac{h_1^2}{h_2^2}\,. \tag{4.1}$$

The last assumption can be ensured, without loss of generality, by a suitable change of the variable x.

4.2. Finite difference scheme. We approximate equations (2.1) and (2.2) in the following manner:

$$\varrho_1\,v_{1,\bar{t}t}-(\bar{p}_1 v_{1,\bar{x}}^{(1/4)})_x=f_1,\quad x\in\omega_{1,h_1},\quad t\in\omega_\tau, \tag{4.2}$$

$$\varrho_2\,v_{2,\bar{t}t}-(\bar{p}_2 v_{2,\bar{x}}^{(1/4)})_x=f_2,\quad x\in\omega_{2,h_2},\quad t\in\omega_\tau, \tag{4.3}$$

where $\bar{p}_i(x)=[p_i(x)+p_i(x-h_i)]/2$, $i=1,2$. To ensure the same order of approximation for $x=b_1$ and $x=a_2$ we set

$$\varrho_1(b_1)\,v_{1,\bar{t}t}(b_1,t)+\frac{h_1}{3}\,\frac{\varrho_1(b_1)}{p_1(b_1)}\left[\tilde{\alpha}_1\,v_{1,\bar{t}t}(b_1,t)-\tilde{\beta}_1\,v_{2,\bar{t}t}(a_2,t)\right]-\frac{h_1}{3}\,\varrho_1'(b_1)\,v_{1,\bar{t}t}(b_1,t)+$$

$$\frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\,\varrho_1(b_1)\,v_{1,\bar{t}t}(b_1,t)+\frac{2}{h_1}\left[\bar{p}_1(b_1)v_{1,\bar{x}}^{(1/4)}(b_1,t)+\tilde{\alpha}_1 v_1^{(1/4)}(b_1,t)-\tilde{\beta}_1 v_2^{(1/4)}(a_2,t)\right]=$$

$$f_1(b_1,t)-\frac{h_1}{3}\,f_{1,\bar{x}}(b_1,t)+\frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\,f_1(b_1,t),\quad t\in\omega_\tau, \tag{4.4}$$

$$\varrho_2(a_2)\, v_{2,\bar{t}t}(a_2,t) + \frac{h_2}{3}\,\frac{\varrho_2(a_2)}{p_2(a_2)}\Big[\tilde{\alpha}_2\, v_{2,\bar{t}t}(a_2,t) - \tilde{\beta}_2\, v_{1,\bar{t}t}(b_1,t)\Big] + \frac{h_2}{3}\,\varrho_2'(a_2)\, v_{2,\bar{t}t}(a_2,t) -$$
$$\frac{h_2}{6}\frac{p_2'(a_2)}{p_2(a_2)}\,\varrho_2(a_2)\, v_{2,\bar{t}t}(a_2,t) - \frac{2}{h_2}\Big[\bar{p}_2(a_2+h_2)v_{2,x}^{(1/4)}(a_2,t) - \tilde{\alpha}_2 v_2^{(1/4)}(a_2,t) + \tilde{\beta}_2 v_1^{(1/4)}(b_1,t)\Big] =$$
$$f_2(a_2,t) + \frac{h_2}{3}\, f_{2,x}(a_2,t) - \frac{h_2}{6}\,\frac{p_2'(a_2)}{p_2(a_2)}\, f_2(a_2,t), \quad t \in \omega_\tau, \tag{4.5}$$

where $\tilde{\alpha}_i = \alpha_i\, D_i$, $\tilde{\beta}_i = \beta_i\, D_i$, $i = 1,2$,

$$D_1 = \left[1 + \frac{h_1^2}{12}\left(\frac{p_1''(b_1)}{p_1(b_1)} + \frac{(p_1'(b_1))^2}{p_1^2(b_1)}\right)\right], \quad D_2 = \left[1 + \frac{h_2^2}{12}\left(\frac{p_2''(a_2)}{p_2(a_2)} + \frac{(p_2'(a_2))^2}{p_2^2(a_2)}\right)\right].$$

The Dirichlet boundary conditions (2.5) and the initial conditions (2.6) can be satisfied exactly

$$v_1(a_1,t) = 0, \quad v_2(b_2,t) = 0, \quad t \in \bar{\omega}_\tau, \tag{4.6}$$

$$v_i(x,0) = u_{i0}(x), \quad x \in \omega_{i,h_i}^{\pm}, \quad i = 1,2, \tag{4.7}$$

while we approximate the initial conditions (2.7) with

$$v_{i,t}(x,0) = u_{i1}(x) + \frac{\tau}{2\,\varrho_i(x)}\left[\frac{d}{dx}\left(p_i\,\frac{du_{i0}}{dx}\right) + f_i(x,0)\right], \quad x \in \omega_{i,h_i}^{\pm}, \quad i = 1,2. \tag{4.8}$$

At each time level $t = j\tau$ the FDS (4.2)–(4.8) reduces to a tridiagonal linear system with $n_1 + n_2$ unknowns. In such a way, the FDS (4.2)–(4.8) is computationally efficient.

4.3. Convergence of the finite difference scheme. Let $u = (u_1, u_2)$ be the solution of the IBVP (2.1)–(2.7) and $v = (v_1, v_2)$ the solution of the FDS (4.2)–(4.8). Then the error $z = u - v$ satisfies the following FDS:

$$\varrho_1\, z_{1,\bar{t}t} - (\bar{p}_1 z_{1,\bar{x}}^{(1/4)})_x = \varphi_1, \quad x \in \omega_{1,h_1}, \quad t \in \omega_\tau, \tag{4.9}$$

$$\varrho_1(b_1)\, z_{1,\bar{t}t}(b_1,t) + \frac{h_1}{3}\,\frac{\varrho_1(b_1)}{p_1(b_1)}\Big[\tilde{\alpha}_1\, z_{1,\bar{t}t}(b_1,t) - \tilde{\beta}_1\, z_{2,\bar{t}t}(a_2,t)\Big] - \frac{h_1}{3}\,\varrho_1'(b_1)\, z_{1,\bar{t}t}(b_1,t) + \frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)} \times$$
$$\varrho_1(b_1)\, z_{1,\bar{t}t}(b_1,t) + \frac{2}{h_1}\Big[\bar{p}_1(b_1) z_{1,\bar{x}}^{(1/4)}(b_1,t) + \tilde{\alpha}_1 z_1^{(1/4)}(b_1,t) - \tilde{\beta}_1 z_2^{(1/4)}(a_2,t)\Big] = \varphi_1(b_1,t), \quad t \in \omega_\tau, \tag{4.10}$$

$$\varrho_2\, z_{2,\bar{t}t} - (\bar{p}_2 z_{2,\bar{x}}^{(1/4)})_x = \varphi_2, \quad x \in \omega_{2,h_2}, \quad t \in \omega_\tau, \tag{4.11}$$

$$\varrho_2(a_2)\, z_{2,\bar{t}t}(a_2,t) + \frac{h_2}{3}\,\frac{\varrho_2(a_2)}{p_2(a_2)}\Big[\tilde{\alpha}_2\, z_{2,\bar{t}t}(a_2,t) - \tilde{\beta}_2\, z_{1,\bar{t}t}(b_1,t)\Big] + \frac{h_2}{3}\,\varrho_2'(a_2)\, z_{2,\bar{t}t}(a_2,t) - \frac{h_2}{6}\frac{p_2'(a_2)}{p_2(a_2)} \times$$
$$\varrho_2(a_2)\, z_{2,\bar{t}t}(a_2,t) - \frac{2}{h_2}\Big[\bar{p}_2(a_2+h_2) z_{2,x}^{(1/4)}(a_2,t) - \tilde{\alpha}_2 z_2^{(1/4)}(a_2,t) + \tilde{\beta}_2 z_1^{(1/4)}(b_1,t)\Big] = \varphi_2(a_2,t), \quad t \in \omega_\tau, \tag{4.12}$$

$$z_1(a_1,t) = 0, \quad z_2(b_2,t) = 0, \quad t \in \bar{\omega}_\tau, \tag{4.13}$$

$$z_i(x,0) = 0, \quad x \in \omega_{i,h_i}^{\pm}, \quad i = 1,2, \tag{4.14}$$

$$z_{i,t}(x,0) = \zeta_i(x), \quad x \in \omega_{i,h_i}^{\pm}, \quad i = 1,2, \tag{4.15}$$

where

$$\varphi_i = \psi_i + \chi_i\,, \quad x \in \omega_{i,h_i}, \quad t \in \omega_\tau, \quad i = 1,2,$$

$$\varphi_1(b_1,t) = \psi_1(b_1,t) + \frac{h_1}{6}\,\frac{p_1'(b_1)}{p_1(b_1)}\,\psi_1(b_1,t) + \frac{h_1}{3}\,\eta_1(b_1,t) + \tilde{\chi}_1(b_1,t)\,,$$

$$\varphi_2(a_2,t) = \psi_2(a_2,t) - \frac{h_2}{6}\,\frac{p_2'(a_2)}{p_2(a_2)}\,\psi_2(a_2,t) + \frac{h_2}{3}\,\eta_2(a_2,t) + \tilde{\chi}_2(a_2,t)\,,$$

$$\psi_i = \varrho_i\biggl(u_{i,\bar{t}t} - \frac{\partial^2 u_i}{\partial t^2}\biggr), \quad \chi_i = \frac{\partial}{\partial x}\biggl(p_i\,\frac{\partial u_i}{\partial x}\biggr) - (\bar{p}_i\,u_{i,\bar{x}}^{(1/4)})_x,$$

$$\tilde{\chi}_1(b_1,t) = \biggl\{\frac{\partial}{\partial x}\biggl(p_1\,\frac{\partial u_1}{\partial x}\biggr) - \frac{h_1}{3}\biggl[\frac{\partial}{\partial x}\biggl(p_1\,\frac{\partial u_1}{\partial x}\biggr)\biggr]_{\bar{x}} + \frac{h_1}{6}\,\frac{p_1'}{p_1}\,\frac{\partial}{\partial x}\biggl(p_1\,\frac{\partial u_1}{\partial x}\biggr)\biggr\}\biggr|_{(b_1,t)}$$

$$+\frac{2}{h_1}\Bigl(\bar{p}_1(b_1)u_{1,\bar{x}}^{(1/4)}(b_1,t) + \tilde{\alpha}_1 u_1^{(1/4)}(b_1,t) - \tilde{\beta}_1 u_2^{(1/4)}(a_2,t)\Bigr),$$

$$\tilde{\chi}_2(a_2,t) = \biggl\{\frac{\partial}{\partial x}\biggl(p_2\,\frac{\partial u_2}{\partial x}\biggr) + \frac{h_2}{3}\biggl[\frac{\partial}{\partial x}\biggl(p_2\,\frac{\partial u_2}{\partial x}\biggr)\biggr]_{\bar{x}} - \frac{h_2}{6}\,\frac{p_2'}{p_2}\,\frac{\partial}{\partial x}\biggl(p_2\,\frac{\partial u_2}{\partial x}\biggr)\biggr\}\biggr|_{(a_2,t)}$$

$$-\frac{2}{h_2}\Bigl(\bar{p}_2(a_2+h_2)u_{2,x}^{(1/4)}(a_2,t) - \tilde{\alpha}_2 u_2^{(1/4)}(a_2,t) + \tilde{\beta}_2 u_1^{(1/4)}(b_1,t)\Bigr),$$

$$\eta_1(b_1,t) = \biggl(\varrho_1\,\frac{\partial^2 u_1}{\partial t^2}\biggr)_{\bar{x}}\biggr|_{(b_1,t)} - \varrho_1'(b_1)\,u_{1,\bar{t}t}(b_1,t) + \frac{\varrho_1(b_1)}{p_1(b_1)}\Bigl[\tilde{\alpha}_1 u_{1,\bar{t}t}(b_1,t) - \tilde{\beta}_1 u_{2,\bar{t}t}(a_2,t)\Bigr],$$

$$\eta_2(a_2,t) = -\biggl(\varrho_2\,\frac{\partial^2 u_2}{\partial t^2}\biggr)_{x}\biggr|_{(a_2,t)} + \varrho_2'(a_2)\,u_{2,\bar{t}t}(a_2,t) + \frac{\varrho_2(a_2)}{p_2(a_2)}\Bigl[\tilde{\alpha}_2 u_{2,\bar{t}t}(a_2,t) - \tilde{\beta}_2 u_{1,\bar{t}t}(b_1,t)\Bigr],$$

$$\zeta_i(x) = u_{i,t}(x,0) - u_{i1}(x) - \frac{\tau}{2\varrho_i(x)}\biggl[\frac{d}{dx}\biggl(p_i\,\frac{du_{i0}}{dx}\biggr) + f_i(x,0)\biggr], \quad x \in \omega_{i,h_i}^{\pm}, \quad i = 1,2.$$

To obtain the discrete analogue of the a priori estimate (3.3), we introduce the discrete inner products and the associated norms

$$(v,w)_{L_h} = \tilde{\beta}_2 \sum_{x\in\omega_{1,h_1}^{+}} v_1 w_1 \hbar_1 + \tilde{\beta}_1 \sum_{x\in\omega_{2,h_2}^{-}} v_2 w_2 \hbar_2, \quad \|v\|_{L_h}^2 = (v,v)_{L_h},$$

$$(v,w)_{L_{h'}} = \tilde{\beta}_2\, h_1 \sum_{x\in\omega_{1,h_1}^{+}} v_1 w_1 + \tilde{\beta}_1\, h_2 \sum_{x\in\omega_{2,h_2}^{+}} v_2 w_2, \quad \|v\|_{L_{h'}}^2 = (v,v)_{L_{h'}},$$

where denoted $\hbar_i = h_i$, $x \in \omega_{i,h_i}$, $i = 1,2$, $\hbar_1(b_1) = h_1/2$, and $\hbar_2(a_2) = h_2/2$. We also define the following discrete norm:

$$\|v\|_{H_h^1}^2 = \|v_{\bar{x}}\|_{L_{h'}}^2 + \|v\|_{L_h}^2.$$

Analogously as in the continuous case, one obtains the next assertion.

Lemma 4.1. *Let $p_i \in C(\bar{\Omega}_i)$, for $i = 1,2$, and let assumptions* (2.9), (2.10) *and* (4.1) *hold. Then the solution z of FDS* (4.9)–(4.15) *satisfies the a priori estimate*

$$\max_{t\in\omega_\tau}\Bigl(\bigl\|z_t(\cdot,t)\bigr\|_{L_h} + \bigl\|\overline{z}(\cdot,t)\bigr\|_{H_h^1}\Bigr) \leqslant C\Bigl(\bigl\|z_t(\cdot,0)\bigr\|_{L_h} + \bigl\|\overline{z}(\cdot,0)\bigr\|_{H_h^1} + \tau \sum_{t\in\omega_\tau} \|\varphi(\cdot,t)\|_{L_h}\Bigr). \quad (4.16)$$

Therefore, in order to determine the convergence rate of the FDS (4.2)–(4.8), it is enough to estimate the right-hand side terms in inequality (4.16).

From the integral representation

$$\psi_1(x,t)=\frac{\varrho_1(x)}{\tau h_1}\int\limits_{x-h_1}^{x}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\int\limits_{t''}^{t}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial t^4}(x',t''')\,dt'''\,dt''\,dt'\,dx'-$$

$$\frac{\varrho_1(x)}{\tau h_1}\int\limits_{x-h_1}^{x}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\int\limits_{x'}^{x}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial x\partial t^3}(x'',t'')\,dx''\,dt''\,dt'\,dx'$$

we immediately obtain

$$|\psi_1(x,t)|\leqslant\frac{(\tau^2+h_1\tau)\sqrt{2}}{\sqrt{\tau h_1}}\,\|\varrho_1\|_{C(\bar\Omega_1)}\bigg(\Big\|\frac{\partial^4 u_1}{\partial t^4}\Big\|_{L_2(e_{10}(x,t))}+\Big\|\frac{\partial^4 u_1}{\partial x\partial t^3}\Big\|_{L_2(e_{10}(x,t))}\bigg),$$

where $e_{10}(x,t)=(x-h_1,x)\times(t-\tau,t+\tau)$. Summation over the mesh yields

$$\bigg(\tau\sum_{t\in\omega_\tau}\sum_{x\in\omega^+_{1,h_1}}\hbar_1|\psi_1|^2\bigg)^{1/2}\leqslant C(\tau^2+h_1^2)\|\varrho_1\|_{C(\bar\Omega_1)}\|u_1\|_{H^4(Q_1)}\leqslant C(\tau^2+h_1^2)\|\varrho_1\|_{H^1(\Omega_1)}\|u_1\|_{H^4(Q_1)}. \tag{4.17}$$

For $x\in\omega_{1,h_1}$, $t\in\omega_\tau$ the term χ_1 can be represented in the following way:

$$\chi_1(x,t)=-\frac{p_1(x)}{2\tau h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{x'}^{x}\int\limits_{x''}^{x}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|x'-x|}{h_1}\Big)\frac{\partial^4 u_1}{\partial x^4}(x''',t')\,dt'\,dx'''\,dx''\,dx'+$$

$$\frac{p_1(x)}{2\tau h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{t-\tau}^{t+\tau}\int\limits_{x'}^{x}\int\limits_{t'}^{t}\Big(1-\frac{|x'-x|}{h_1}\Big)\frac{\partial^4 u_1}{\partial t\partial x^3}(x'',t'')\,dt''\,dx''\,dt'\,dx'-$$

$$\frac{p_1'(x)}{2h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{x'}^{x}\int\limits_{x''}^{x}\frac{\partial^3 u_1}{\partial x^3}(x''',t)\,dx'''\,dx''\,dx'-\frac{h_1}{4}\,p_1''(x)\int\limits_{x-h_1}^{x+h_1}\Big(1-\frac{|x'-x|}{h_1}\Big)\frac{\partial^2 u_1}{\partial x^2}(x',t)\,dx'-$$

$$\frac{1}{2h_1^2}\bigg(\int\limits_{x}^{x+h_1}\int\limits_{x}^{x'}\int\limits_{x}^{x''}p_1'''(x''')\,dx'''\,dx''\,dx'\bigg)\bigg(\int\limits_{x}^{x+h_1}\frac{\partial u_1}{\partial x}(x',t)\,dx'\bigg)-$$

$$\frac{1}{2h_1^2}\bigg(\int\limits_{x-h_1}^{x}\int\limits_{x'}^{x}\int\limits_{x''}^{x}p_1'''(x''')\,dx'''\,dx''\,dx'\bigg)\bigg(\int\limits_{x-h_1}^{x}\frac{\partial u_1}{\partial x}(x',t)\,dx'\bigg)-$$

$$\frac{\tau p_1(x)}{h_1}\int\limits_{x-h_1}^{x+h_1}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|x'-x|}{h_1}\Big)\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial t^2\partial x^2}(x',t')\,dt'\,dx'-$$

$$\frac{\tau}{2h_1^2}\bigg(\int\limits_{x}^{x+h_1}p_1'(x')\,dx'\bigg)\bigg(\int\limits_{x}^{x+h_1}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^3 u_1}{\partial t^2\partial x}(x',t')\,dt'\,dx'\bigg)-$$

$$\frac{\tau}{2h_1^2}\Big(\int\limits_{x-h_1}^{x} p_1'(x')\,dx'\Big)\Big(\int\limits_{x-h_1}^{x}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^3 u_1}{\partial t^2\partial x}(x',t')\,dt'\,dx'\Big).$$

Consequently,

$$|\chi_1(x,t)| \leqslant \Big(\frac{\tau^2+h_1^2}{\sqrt{\tau h_1}}\,\|p_1\|_{C^1(\bar\Omega_1)}\|u_1\|_{H^4(e_{11}(x,t))}+\frac{h_1^2}{\sqrt{h_1}}\,\|p_1\|_{C^2(\bar\Omega_1)}\|u_1(\cdot,t)\|_{H^3(i_{11}(x))}+$$

$$\frac{h_1^2}{\sqrt{h_1}}\,\|p_1\|_{H^3(i_{11}(x))}\|u_1(\cdot,t)\|_{C^1(\bar\Omega_1)}\Big),$$

where $e_{11}(x,t)=(x-h_1,x+h_1)\times(t-\tau,t+\tau)$ and $i_{11}(x)=(x-h_1,x+h_1)$. Summation over the mesh with the application of the Sobolev embedding theorem yields

$$\Big(\tau\sum_{t\in\omega_\tau}h_1\sum_{x\in\omega_{1,h_1}}|\chi_1|^2\Big)^{1/2}\leqslant C(\tau^2+h_1^2)\,\|p_1\|_{H^3(\Omega_1)}\,\|u_1\|_{H^4(Q_1)}. \tag{4.18}$$

Similarly, the term $\tilde\chi_1$ can be represented as follows:

$$\tilde\chi_1(b_1,t)=\frac{p_1(b_1)}{\tau h_1^3}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x''''}^{x'''}\int\limits_{t-\tau}^{t+\tau}\frac{\partial^4 u_1}{\partial x^4}(x''''',t')\,dt'\,dx'''''dx''''dx'''\,dx''\,dx'+$$

$$\frac{2p_1(b_1)}{3\tau h_1^2}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\frac{\partial^4 u_1}{\partial x^3\partial t}(x''',t'')\,dt''dt'dx'''dx''dx'-$$

$$\frac{p_1(b_1)}{3\tau h_1^2}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{b_1-h_1}^{x''}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\frac{\partial^4 u_1}{\partial x^3\partial t}(x''',t'')\,dt''dt'dx'''dx''dx'+$$

$$\Big(\int\limits_{b_1-h_1}^{b_1}p_1'(x')\,dx'\Big)\Big(\int\limits_{b_1-h_1}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x',t)\,dx'\Big)+\frac{p_1'(b_1-h_1)}{3}\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x'',t)\,dx''dx'\Big)-$$

$$\frac{p_1'(b_1)}{h_1}\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{x''}^{b_1}\frac{\partial^3 u_1}{\partial x^3}(x''',t)\,dx'''\,dx''\,dx'\Big)+$$

$$\Big(\frac{h_1}{3}\int\limits_{b_1-h_1}^{b_1}p_1''(x')\,dx'+\frac{1}{3}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}p_1''(x'')\,dx''dx'\Big)\frac{\partial^2 u_1}{\partial x^2}(b_1,t)-$$

$$\frac{1}{h_1^2}\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}p_1''(x'')\,dx''dx'\Big)\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\frac{\partial^2 u_1}{\partial x^2}(x'',t)\,dx''\,dx'\Big)+$$

$$\frac{1}{h_1^2}\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'''}^{x''}p_1'''(x'''')dx''''dx'''dx''dx'\Big)\frac{\partial u_1}{\partial x}(b_1,t)-\Big(\frac{1}{6}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}p_1'''(x'')dx''dx'\Big)\frac{\partial u_1}{\partial x}(b_1,t)-$$

$$\frac{\tau}{4h_1^2}\Big(\int\limits_{b_1-h_1}^{b_1} p_1'(x')\,dx'\Big)\Big(\int\limits_{b_1-h_1}^{b_1}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^3 u_1}{\partial x\partial t^2}(x',t')\,dt'\,dx'\Big)-$$

$$\frac{\tau}{2h_1^2}\,p_1(b_1)\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial x^2\partial t^2}(x'',t')\,dt'\,dx''\,dx'+$$

$$\frac{\alpha_1\tau h_1}{24}\Big(\frac{p_1''(b_1)}{p_1(b_1)}+\frac{(p_1'(b_1))^2}{p_1^2(b_1)}\Big)\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^2 u_1}{\partial t^2}(b_1,t')\,dt'-$$

$$\frac{\beta_1\tau h_1}{24}\Big(\frac{p_1''(b_1)}{p_1(b_1)}+\frac{(p_1'(b_1))^2}{p_1^2(b_1)}\Big)\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^2 u_2}{\partial t^2}(a_2,t')\,dt'\,,$$

wherefrom, for sufficiently small h_1, follows

$$\Big(\tau\sum_{t\in\omega_\tau}\frac{h_1}{2}|\tilde\chi_1(b_1,t)|^2\Big)^{1/2}\leqslant C(\tau^2+h_1^2)\,\|p_1\|_{H^3(\Omega_1)}\big(\|u_1\|_{H^4(Q_1)}+\|u_2\|_{H^4(Q_2)}\big). \qquad (4.19)$$

From the integral representation

$$\eta_1(b_1,t)=\Big(-\frac{1}{h_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\varrho_1''(x'')\,dx''\,dx'\Big)\frac{\partial^2 u_1}{\partial t^2}(b_1,t)+$$

$$\frac{\varrho_1'(b_1)}{\tau h_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^3 u_1}{\partial t^3}(x',t'')\,dt''\,dt'\,dx'+$$

$$\frac{\varrho_1'(b_1)}{\tau h_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\int\limits_{x'}^{b_1}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial t^3\partial x}(x'',t'')\,dx''\,dt''\,dt'\,dx'-$$

$$\Big(\int\limits_{b_1-h_1}^{b_1}\varrho_1'(x')\,dx'\Big)\Big(\frac{1}{h_1}\int\limits_{b_1-h_1}^{b_1}\frac{\partial^3 u_1}{\partial t^2\partial x}(x',t)\,dx'\Big)+$$

$$\frac{\varrho_1(b_1)}{\tau h_1}\int\limits_{t-\tau}^{t+\tau}\int\limits_{t'}^{t}\int\limits_{b_1-h_1}^{b_1}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial t^3\partial x}(x',t'')\,dx'\,dt''\,dt'-$$

$$\frac{\varrho_1(b_1)}{\tau h_1}\int\limits_{b_1-h_1}^{b_1}\int\limits_{x'}^{b_1}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^4 u_1}{\partial t^2\partial x^2}(x'',t')\,dt'\,dx''\,dx'+$$

$$\alpha_1\,\frac{h_1^2}{12}\,\frac{\varrho_1(b_1)}{p_1(b_1)}\Big(\frac{p_1''(b_1)}{p_1(b_1)}+\frac{(p_1'(b_1))^2}{p_1^2(b_1)}\Big)\frac{1}{\tau}\int\limits_{t-\tau}^{t+\tau}\Big(1-\frac{|t'-t|}{\tau}\Big)\frac{\partial^2 u_1}{\partial t^2}(b_1,t')\,dt'-$$

$$\beta_1 \frac{h_1^2}{12}\frac{\varrho_1(b_1)}{p_1(b_1)}\left(\frac{p_1''(b_1)}{p_1(b_1)}+\frac{(p_1'(b_1))^2}{p_1^2(b_1)}\right)\frac{1}{\tau}\int\limits_{t-\tau}^{t+\tau}\left(1-\frac{|t'-t|}{\tau}\right)\frac{\partial^2 u_2}{\partial t^2}(a_2,t')\,dt'$$

for sufficiently small h_1 follows

$$\left(\tau\sum_{t\in\omega_\tau}\frac{h_1}{2}|h_1\,\eta_1(b_1,t)|^2\right)^{1/2}\leqslant C(\tau^2+h_1^2)\,\|\varrho_1\|_{H^2(\Omega_1)}\big(\|u_1\|_{H^4(Q_1)}+\|u_2\|_{H^4(Q_2)}\big). \tag{4.20}$$

Finally, from the integral representation

$$\zeta_1(x)=\frac{1}{\tau h_1}\int\limits_{x-h_1}^{x}\int\limits_0^{\tau}\int\limits_0^{t'}\int\limits_0^{t''}\frac{\partial^3 u_1}{\partial t^3}(x',t''')\,dt'''\,dt''\,dt'\,dx'+$$

$$\frac{1}{\tau h_1}\int\limits_{x-h_1}^{x}\int\limits_0^{\tau}\int\limits_0^{t'}\int\limits_0^{t''}\int\limits_{x'}^{x}\frac{\partial^4 u_1}{\partial t^3\partial x}(x'',t''')\,dx''\,dt'''\,dt''\,dt'\,dx',$$

using inequality [15]

$$\|g\|_{L_2(0,\varepsilon)}\leqslant C\sqrt{\varepsilon}\,\|g\|_{H^1(0,1)}\,,\quad 0<\varepsilon<1,$$

we obtain

$$\left(\sum_{x\in\omega_{1,h_1}^+}|\zeta_1|^2\,\hbar_1\right)^{1/2}\leqslant C(\tau^2+h_1^2)\,\|u_1\|_{H^4(Q_1)}, \tag{4.21}$$

whereas from

$$\zeta_{1,\bar x}(x)=\frac{1}{\tau h_1}\int\limits_{x-h_1}^{x}\int\limits_0^{\tau}\int\limits_0^{t'}\int\limits_0^{t''}\frac{\partial^4 u_1}{\partial t^3\partial x}(x',t''')\,dt'''\,dt''\,dt'\,dx'$$

follows

$$\left(h_1\sum_{x\in\omega_{1,h_1}^+}|\overline{z}_{1,\bar x}(x,0)|^2\right)^{1/2}=\left(h_1\sum_{x\in\omega_{1,h_1}^+}\left|\frac{\tau}{2}\,\zeta_{1,\bar x}(x)\right|^2\right)^{1/2}\leqslant C\tau^2\,\|u_1\|_{H^4(Q_1)}. \tag{4.22}$$

From (4.16)–(4.22) and analogous estimates for ψ_2, χ_2, $\tilde\chi_2$, η_2 and ζ_2 one obtains the next assertion.

Theorem 4.1. *Let $p_i\in H^3(\Omega_i)$, $\varrho_i\in H^2(\Omega_i)$, $i=1,2$, and let assumptions* (2.8)*–*(2.10) *hold. Let further the solution of IBVP* (2.1)*–*(2.7) *belong to the space H^4. Then the solution v of the FDS* (4.2)*–*(4.8) *converges to the solution u of the IBVP* (2.1)*–*(2.7) *and the following convergence rate estimate holds:*

$$\max_{t\in\omega_\tau}\left(\big\|z_t(\cdot,t)\big\|_{L_h}+\big\|\overline{z}(\cdot,t)\big\|_{H_h^1}\right)\leqslant C(h^2+\tau^2)\left(\max_i\|p_i\|_{H^3(\Omega_i)}+\max_i\|\varrho_i\|_{H^2(\Omega_i)}\right)\|u\|_{H^4},$$

where $z=u-v$ and $h=\max\{h_1,h_2\}$.

Remark 4.1. For $u \in H^s$, $3 < s < 4$, using the Bramble — Hilbert lemma (see [2,5]) and the methodology proposed in [8, 9], one can obtain the convergence rate $O(h^{s-2})$ assuming $\tau \asymp h$. The same result holds for $2 < s \leqslant 3$, but in this case a FDS with averaged data must be used, because the right-hand sides f_i may be discontinuous functions. Generalization to the finite difference schemes on nonuniform meshes (cf. [1, 14]) is also possible and will be the subject of the future work.

Acknowledgement. The research of the first author was supported by the Ministry of Science of the Republic of Serbia under project # 144005A and the Bulgarian Foundation for Science under Project VU-MI-106/2005. The research of the second author was supported by the Bulgarian Foundation for Science under Project VU-MI-106/2005.

References

1. D. Bojović and B. S. Jovanović, *Fractional order convergence rate estimates of finite difference method on non-uniform meshes,* Comput. Methods Appl. Math. 1, (2001), no. 3, pp. 213–221.
2. J. H. Bramble and S. R. Hilbert, *Bounds for a class of linear functionals with application to Hermite interpolation,* Numer. Math., **16** (1971), pp. 362–369.
3. L. Caffarelli, *A monotonicity formula for heat functions in disjoint domains,* in: B. V. Problems for PDEs and Appls., RMA Res. Notes Appl. Math. **29**, pp. 53–60, Masson, Paris, 1993.
4. A. K. Datta, *Biological and bioenvironmental heat and mass transfer*, Marcel Dekker, 2002.
5. T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces,* Math. Comput., **34** (1980), pp. 441–463.
6. D. Givoli, *Exact representation on artificial interfaces and applications in mechanics,* Appl. Mech. Rev., **52** (1999), pp. 333–349.
7. D. Givoli, *Finite element modeling of thin layers,* Comput. Model. Eng. Sci., **5** (2004), no. 6, pp. 497–514.
8. B. S. Jovanović, *Finite difference method for boundary value problems with weak solutions,* Posebna izdanja Mat. Instituta 16, Belgrade 1993.
9. B. S. Jovanović, *Finite difference schemes for partial differential equations with weak solutions and irregular coefficients,* Comput. Methods Appl. Math., **4** (2004), no. 1, pp. 48–65.
10. B. S. Jovanović, S. V. Lemeshevskiĭ, and P. P. Matus, *Global stability of second-order operator-differential equations and three-level operator-difference schemes,* Doklady NAN Belarusi, **46** (2002), no. 4, pp. 30–34 (in Russian).
11. B. S. Jovanović, S. V. Lemeshevskiĭ, and P. P. Matus, *On the stability of differential-operator equations and operator-difference schemes as* $t \to \infty$*,* Comput. Methods Appl. Math., **2** (2002), no. 2, pp. 153–170.
12. B. S. Jovanović and L. G. Vulkov, *Finite difference approximation of strong solutions of a parabolic interface problem on disconnected domains,* Publ. Inst. Math. **84** (2008).
13. J. L. Lions and E. Magenes, *Non homogeneous boundary value problems and applications,* Springer-Verlag, Berlin and New York, 1972.
14. P. Matus and E. Zyuzina, *Three-level difference schemes on non-uniform in time grids,* Comput. Methods Appl. Math., **1** (2001), no. 3, pp. 265–284.
15. L. A. Oganesyan, L. A. Rukhovets, *Variational-difference methods for the solution of elliptic equations,* Armenian Acad. Sci., Erevan, 1979 (in Russian).
16. A. A. Samarskiĭ, *Theory of Difference Schemes,* Nauka, Moscow, 1989 (in Russian).
17. L. G. Vulkov, *Well posedness and a monotone iterative method for a nonlinear interface problem on disjoint intervals,* Amer. Inst. of Physics, Proceedings Series, **946** (2007).
18. J. Wloka, *Partial differential equations,* Cambridge Univ. Press, Cambridge, 1987.