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TWO-SIDED APPROXIMATIONS FOR NONLINEAR OPERATOR EQUATIONS

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Abstract — We propose an analytical-numerical method for nonlinear operator equations which converges exponentially and provides two-sided approximations. A numerical example confirms the theoretical results.

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1. Introduction

The present paper describes an analytical-numerical method for nonlinear operator equations which converges exponentially and provides two-sided approximations. Our method is closely related to the Adomian decomposition method (ADM) and to the FD-method [10, 11] and uses essentially Adomian's polynomials [1–3, 14, 18]. Both these methods are closely related to the homotopy idea. Such methods are semi-analytic and provide inclusions of an arbitrary accuracy order for the exact solution of the problem under consideration. The inclusions can be used, e.g., to prove the existence of solutions of various problems and to construct a posteriori error estimators.

There are inclusion methods for solutions of nonlinear differential and operator equations based on ideas different from ADM or FD-methods. They are very actual and widely discussed in the modern mathematical literature (we mention only [4, 13] and the references therein).

The last decade has seen many publications devoted to the application of the Adomian decomposition method [3] for both linear and nonlinear operator and differential operator equations [9–12, 17].

Let us call to mind the idea of the ADM which can be also interpreted as the FD-method proposed in [9] for the Sturm — Liouville problems. If we have to solve the operator equation

$$u = -N(u) + F, \tag{1.1}$$

then we can imbed it into the family of equations

$$u(t) = -tN(u(t)) + F, \quad t \in [0, 1], \tag{1.2}$$

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and obtain

$$u(1) = u. \tag{1.3}$$

We look for the solution of (1.1) in the form

$$u(t) = \sum_{j=0}^{\infty} t^j u^{(j)}, \tag{1.4}$$

and represent

$$N\left(\sum_{j=0}^{\infty} t^j u^{(j)}\right) = \sum_{j=0}^{\infty} t^j A_j, \tag{1.5}$$

where

$$A_j = A_j(N;\ u^{(0)}, u^{(1)}, \ldots, u^{(j)}) = \frac{1}{j!} \frac{\partial^j N\left(\sum\limits_{k=0}^{\infty} t^k u^{(k)}\right)}{\partial t^j}\Bigg|_{t=0}. \tag{1.6}$$

Substituting (1.4) into (1.1) we have

$$\sum_{j=0}^{\infty} t^j u^{(j)} = -tN\left(\sum_{j=0}^{\infty} t^j u^{(j)}\right) + F. \tag{1.7}$$

Applying successively the operator $\frac{1}{(j+1)!}\frac{d^{j+1}}{dt^{j+1}}$ to this equality and then setting $t = 0$, we obtain the following recurrence formulas:

$$u^{(j+1)} = -A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right), \quad j = 0, 1, \ldots,$$

$$A_0\left(N; u^{(0)}\right) = N\left(u^{(0)}\right), \quad u^{(0)} = F. \tag{1.8}$$

Here $A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right)$ are the Adomian polynomials with the following explicit representation:

$$A_j\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right) = \sum_{\alpha_1+\ldots+\alpha_j=j} N^{(\alpha_1)}\left(u^{(0)}\right) \frac{\left(u^{(1)}\right)^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \times \ldots \times \frac{\left(u^{(j-1)}\right)^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!} \frac{\left(u^{(j)}\right)^{\alpha_j}}{(\alpha_j)!}, \tag{1.9}$$

where the sequence of natural indices α_i is not increasing, $N^{(i)}(u)$ is the i-th (Fréchet) derivative of the operator N.

The solution of (1.1) can be now represented (if the series is convergent) by

$$u = u(1) = \sum_{j=0}^{\infty} u^{(j)} \tag{1.10}$$

and the truncated sum

$$\overset{m}{u} = \sum_{j=0}^{m} u^{(j)} \tag{1.11}$$

represents an approximation to the exact solution. The following theorem from [12] gives some sufficient conditions for the convergence of (1.4) for all $t \in [0, 1]$.

Theorem 1.1. *Let H be a Banach space and $F \in H$. If the operator $N(u): \ H \to H$ is analytic in a ball $\|u - u_0\| < R$ with the centrum u_0 and if for all $n \geqslant 0$ there hold $\|N^{(n)}(u_0)\| \leqslant n! M \alpha^n$ with some $M > 0$, $\alpha > 0$, then the conditions*

1) $4M\alpha \leqslant 1$, *for* $R = \infty$,

2) $5M\alpha \leqslant 1$, *for* $R < \infty$,

provide the convergence of (1.4) *for all* $t \in [0,1]$ *and, therefore, the convergence of* (1.10).

In the recent paper [17], the following modification of the ADM was proposed. One looks for the summands of (1.10) in accordance with the recurrence formulas

$$u^{(j+1)} = -\overline{A_j}\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right), \quad j = 0, 1, \ldots,$$

$$u^{(0)} = F, \tag{1.12}$$

where $\overline{A_j}\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right)$ are the modified Adomian polynomials given by

$$\overline{A_j}\left(N; u^{(0)}, u^{(1)}, \ldots, u^{(j)}\right) = N\left(u^{(0)} + \ldots + u^{(j)}\right) - N\left(u^{(0)} + \ldots + u^{(j-1)}\right). \tag{1.13}$$

In [17], for the problem

$$\frac{d^k y(t)}{dt^k} + \beta(t) f(y(t)) = \kappa(t), \quad t \in (0, \ T), \quad \frac{d^p y(t)}{dt^p} = c_p, \quad p = \overline{0, k-1}, \tag{1.14}$$

with given c_p, $p = \overline{0, k-1}$, with $M = \max\limits_{t \in [0, \ T]} \|\beta(t)|$ and with the right-hand side $f(y)$ satisfying the Lipschitz condition with a constant L it was shown that the modified Adomian method converges as a geometrical progression with the quotient α and with the error estimate

$$\left| y(t) - \sum_{j=0}^{m} y_j(t) \right| \leqslant \frac{\alpha^m}{1-\alpha} \|y_1\|_\infty \tag{1.15}$$

provided that

$$\alpha = \frac{LMT^k}{k!} < 1. \tag{1.16}$$

Numerical experiments have shown that the modified ADM converges faster than the ordinary one. But the author of [17] does not notice that the modified ADM in fact coincides with the usual fixed point iteration. Actually, relations (1.12), (1.13) imply

$$u^{m+1} = -N(u^m) + F, \quad m = 0, 1, \ldots,$$

$$u^0 = F. \tag{1.17}$$

The advantages of the modified ADM have been known for a long time (see, e.g., [16]), now the conclusions of [17] about them have been confirmed.

The goal of the present paper is to find sufficient conditions under which the fixed point iteration (1.17) provides two-sided approximations on a cone K from some Banach space X. The numerical examples show the corresponding inclusions and the exponential convergence rate.

2. Two-sided iteration method

Let $\overline{S}_r(a) = \{x \in X : \ \|x - a\| \leqslant r\}$ be a closed ball in a Banach space X. Then the following assertion about the fixed point iteration for equation (1.1) holds true (see, e.g., [6]).

Theorem 2.1. *Let the operator N satisfy the conditions:*
1°) *for $u, v \in \overline{S}_r(a)$*

$$\|N(u) - N(v)\| \leqslant q\|u - v\|, \quad q \in (0;1) \ \ \textit{holds;} \tag{2.1}$$

2°) *for $F \in \overline{S}_r(a)$*

$$\|N(F)\| \leqslant (1 - q)r \ \ \textit{holds.} \tag{2.2}$$

Then the equation

$$u = -N(u) + F, \quad N(0) = 0, \tag{2.3}$$

possesses a unique solution $u_\star \in \overline{S}_r(a)$, which can be obtained by the fixed point iteration

$$u_{n+1} = -N(u_n) + F, \quad n = 0, 1, \ldots, \tag{2.4}$$

with the error estimate

$$\|u_\star - u_n\| \leqslant \frac{q^n r}{1 - q}. \tag{2.5}$$

Let us clarify the conditions on the operator N under which iterations (2.4) provide two-sided approximations to $u_\star$.

Let $K \subset X$ be a cone with a partial order $\preceq$, i.e., we write $v \preceq u$ when $u - v \in K$. Further we make the following assumptions:
3°) $F \in K$;
4°) the operator N is positive in the sense that

$$N(K) \subset K; \tag{2.6}$$

5°) there exists a Frechét derivative $N'(v)$ with the property

$$\|N'(v)\| \leqslant q, \quad 0 \preceq N'(v)u \quad \forall u, v \in \overline{S}_r(F) \cap K; \tag{2.7}$$

6°)

$$0 \preceq u_1 = -N(u_0) + F = -N(F) + F. \tag{2.8}$$

Then the following assertion holds true.

Theorem 2.2. *Let conditions* 2°)–6°) *hold. Then the fixed point iteration* (2.4) *converges to the unique solution $u_\star$ of equation* (2.3) *and provides a two-sided approximation, i.e.,*

$$u_\star \preceq \ldots \preceq u_{2k} \preceq \ldots \preceq u_2 \preceq u_0,$$

$$u_1 \preceq u_3 \preceq \ldots \preceq u_{2k+1} \preceq \ldots \preceq u_\star. \tag{2.9}$$

Proof. First of all, using 2°), 4°)–6°) we prove that all terms of sequence (2.4) belong to the closed set $\overline{S}_r(F) \cap K$. Actually, we have $u_0 = F \in \overline{S}_r(F) \cap K$ and due to 6°)

$$0 \preceq u_1 = -N(u_0) + F = -N(F) + F,$$

$$0 \preceq -u_1 + F = N(F) + N(0) = N'(\theta F)F \in \overline{S}_r(F) \cap K.$$

Further we use the method of mathematical induction. We have already shown that

$$u_n \in \overline{S}_r(F) \cap K, \tag{2.10}$$

$$0 \preceq -u_n + F, \tag{2.11}$$

then

$$u_{n+1} = -N(u_n) + F = -\left[N(u_n) - N(F)\right] - N(F) + F =$$
$$-N'\left(\theta u_n + (1-\theta)F\right)(u_n - F) - N(F) + F,$$

which due to (2.7),(2.8) as well as to assumption (2.11) leads to

$$0 \preceq u_{n+1}.$$

Besides, condition (2.7) and assumption (2.10) imply

$$\|u_{n+1} - F\| = \|N(u_n) - N(0)\| = \|N'(\theta u_n)u_n\| \leqslant qr < r,$$

which together with the above considerations prove that

$$u_{n+1} \in \overline{S}_r(F) \cap K.$$

Further we have

$$-u_{n+1} + F = N(u_n) - N(0) = N'(\theta u_n)u_n$$

and condition (2.8) together with assumption (2.11) lead to

$$0 \preceq -u_{n+1} + F.$$

Therefore, in accordance with the method of mathematical induction we have (2.10), (2.11) for all $n \in \mathbb{N}$.

Using 4°)–6°) as well as Theorem 2.1, we obtain

$$u_2 - u_0 = -N(u_1), \quad u_2 - u_0 \preceq 0,$$

$$u_3 - u_1 = -N(u_2) + N(u_0) = -N'(\theta u_2 + (1-\theta)u_0)(u_2 - u_0), \quad 0 \preceq u_3 - u_1,$$

and further by induction

$$u_{2k} - u_{2k-2} \preceq 0, \quad 0 \preceq u_{2k+1} - u_{2k-1}, \quad k = 0, 1, \ldots \tag{2.12}$$

This completes the proof of the theorem. □

Example 2.1. Let us consider the Dirichlet boundary value problem

$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0;\ 1),$$
$$u(0) = u(1) = 0, \tag{2.13}$$

with $f(x) = \pi^2 \sin \pi x + M(\sin \pi x)^3$ and a given constant $M \geqslant 0$. The exact solution of (2.13) is

$$u(x) = \sin \pi x. \tag{2.14}$$

Problem (2.13) is equivalent to the following nonlinear Fredholm integral equation:

$$u(x) = \int_0^1 G(x,\xi)[-Mu^3(\xi) + f(\xi)]d\xi, \quad G(x,\xi) = \begin{cases} x(1-\xi), \; x \leqslant \xi \\ \xi(1-x), \; \xi < x. \end{cases} \tag{2.15}$$

This equation is of the form (2.3) with $N(u) = M\int_0^1 G(x,\xi)u^3(\xi)\,d\xi$, $F = \int_0^1 G(x,\xi)f(\xi)\,d\xi$. For $M = 1$, $r = 1.2$, $q = 0.54$ all assumptions of Theorem 2.2 are fulfilled where

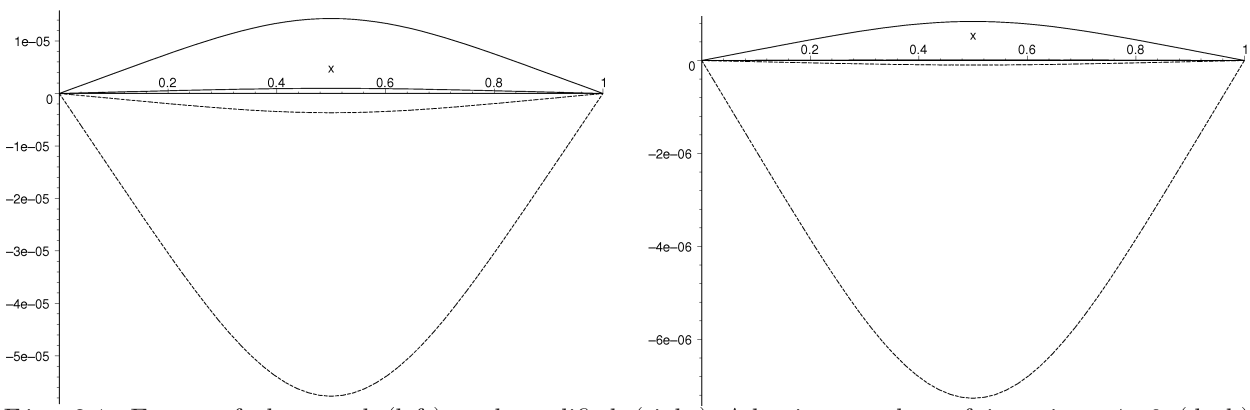
$$X = C[0;\ 1], \quad K = \{v(x) \in C[0;\ 1]:\ v(x) \geqslant 0,\ x \in [0;\ 1]\},$$

$$\|F\|_\infty = 1.087\ldots < r, \quad \|N(F) - F\|_\infty = 0.101\ldots < (1-q)r,$$

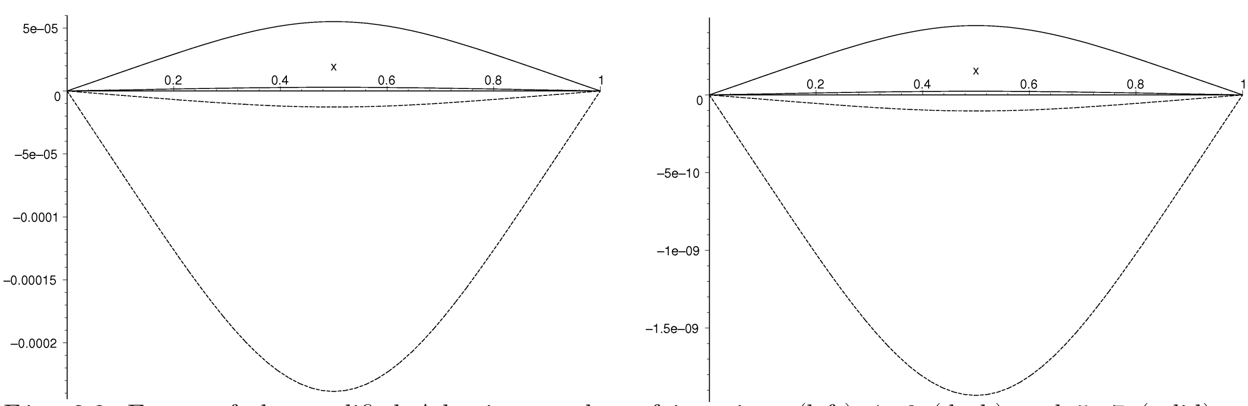
$$0 \leqslant N(v)u = 3\int_0^1 G(x,\xi)v^2(\xi)u(\xi)d\xi \leqslant \frac{3r^3}{8} = \frac{3*1.44}{8}r = qr,$$

$$u_1 = -N(u_0) + F = -\int_0^1 G(x,\xi)f(\xi)d\xi + f(x) \geqslant 0.$$

The numerical results (obtained with Maple) are presented in Table and in Figs. 2.1 and 2.2. The table exhibits the advantages of the fixed point iteration over the usual Adomian's method. The figures demonstrate that the even and the odd Adomian's iterations include the exact solution.



F i g. 2.1. Errors of the usual (left) and modified (right) Adomian number of iterations 4, 6 (dash) and 5, 7 (solid) for $M = 0.5$



F i g. 2.2. Errors of the modified Adomian number of iterations (left) 4, 6 (dash) and 5, 7 (solid) as well as (right) 12, 14 (dash) and 13,15 (continuous) for $M = 1$

Error of the usual (ε_A) and modified (ε_{mA}) Adomian method for $M = 0.5$

n	ε_A	ε_{mA}
0	.4477593957331e-1	.394026825275758e-1
1	.6148944791993e-2	.473705632178769e-2
2	.114506764843e-2	.544572291857201e-3
3	.246398359542e-3	.629246333480784e-4
4	.5768833894e-4	.726644599750120e-5
5	.14273551991e-4	.839176371831571e-6
6	.367192582e-5	.969127578152334e-7
7	.972342751e-6	.111920351333820e-7
8	.26329372e-6	.129251941616996e-8
9	.72571985e-7	.149267628681118e-9
10	.2029413e-7	.172381275559081e-10

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