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DOMAIN DECOMPOSITION METHODS WITH OVERLAPPING SUBDOMAINS FOR THE TIME-DEPENDENT PROBLEMS OF MATHEMATICAL PHYSICS

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Abstract — At the present time, the domain decomposition methods are considered as the most promising ones for parallel computer systems. Nowadays success is attained mainly in solving approximately the classical boundary value problems for second-order elliptic equations. As for the time-dependent problems of mathematical physics, there are, in common use, approaches based on ordinary implicit schemes and implemented via iterative methods of the domain decomposition. An alternative technique is based on the non-iterative schemes (region-additive schemes). On the basis of the general theory of additive schemes a wide class of difference schemes (alternative directions, locally one-dimensional, factorized schemes, summarized approximation schemes, vector additive schemes, *etc.*) as applied to the domain decomposition technique for time-dependent problems with synchronous and asynchronous implementations has been investigated.

For nonstationary problems with self-adjoint operators, we have considered three different types of decomposition operators corresponding to the Dirichlet and Neumann conditions on the subdomain boundaries. General stability conditions have been obtained for the region-additive schemes. We focused on the accuracy of domain decomposition schemes. In particular, the dependence of the convergence rate on the width of subdomain overlapping has been investigated as the primary property.

In the present paper, new classes of domain decomposition schemes for nonstationary problems, based on the subdomain overlaping and minimal data exchange in solving problems in subdomains, have been constructed.

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1. Introduction

Effective computational algorithms for approximate solution of the boundary value problems of mathematical physics in complicated three-dimensional calculated domains on modern parallel-architecture computers are based on the domain decomposition methods. In the theory of domain decomposition methods (see, for instance, [10, 11, 21, 22]), the most impressive results are obtained for boundary value problems for elliptic equations of the second order. Iterative domain decomposition schemes with and without overlapping of subdomains and with various types of exchange conditions at the boundaries of subdomains (interfaces)

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have been considered. Particular emphasis was placed on the asynchronous (parallel) implementation of calculations.

In constructing domain decomposition methods for time-dependent problems two main approaches [13] were used. The first approach is based on using the classical implicit schemes and involves domain decomposition methods in order to solve an elliptic grid problem at a new time level. Taking into account the specificity of the problem on a new time layer, optimal iterative domain decomposition methods are constructed when the number of iterations does not depend on discretization steps in time and space [1, 2].

In the second approach, the features of the transient problems are taken into account in larger measure in using noniterative algorithms for solving a parabolic problem. In particular, in solving boundary value problems with subdomain overlaping one can be limited to one iteration of the Shwarts algorithm [5, 6]. A more general approach is connected with the construction of additive schemes (schemes of splitting in subdomains). In this case, we mean region-additive schemes [23, 24].

The construction and investigation of convergence of the region-additive schemes is based on the general theory of splitting schemes [9, 12, 32]. Under the general conditions of problem operator splitting into a sum of noncommutative not self-adjoint operators, additive difference schemes are constructed in the simplest way for two-component splitting. A more complicated situation takes place in the case of multicomponent splitting (into three or more operators). For these problems, the most interesting results are obtained by using the concept of summarized approximation. A new class of operator-difference schemes of splitting — vector additive schemes of full approximation at general multicomponent splitting have been developed ((see monograph [18] and the bibliography therein). Unconditionally stable additive difference schemes of full approximation have been also constructed on the basis of the small multiplicative perturbation of each operator of splitting with an arbitrary number of mutually noncommutative operator terms. There are additive difference schemes based on the splitting with respect to separate directions (locally-one-dimensional schemes), schemes employing the splitting with respect to the physical processes and regionally-additive schemes of domain decomposition oriented to the construction of parallel algorithms. Special attention should be given to the additively-averaged schemes [4], which can be treated as asynchronous (parallel) variants of component-wise splitting schemes.

The specificity of the domain decomposition schemes becomes apparent when constructing splitting operators. One can define two main directions. The first one is connected with the use of decomposition together with subdomain overlaping and with particular exchange boundary conditions.

The investigation of such regional-additive schemes is carried out on the basis of the maximum principle [12]. In this case, the closeness of the approximate solution to the exact one is considered in the uniform norm. In [20], two classes of such regional-additive finite difference schemes with overlapping of subdomains are considered. The first class of them is similar to the usual schemes of component-wise decomposition, and it is treated as one-iteration synchronous (multiplicative) variant of the classical alternating Schwarz method. Such schemes are close to the above-mentioned schemes and the one-iteration overlapping Schwarz algorithm for the numerical solution of the elliptic equation. The second class of regional-additive finite difference schemes corresponds to the application of additive-averaged finite difference schemes. It can be considered as a one-iterative asynchronous (additive) variant of the Schwarz method.

The second direction is connected with the partition of unity for the calculated domain when in the domain decomposition method with subdomain overlaping a separate subdomain is connected with the function and its value is within zero and one.

In the extreme case, the width of the overlaping domain is equal to the step of discretization. Here such regional additive schemes can be interpreted as a decomposition scheme without overlaping subdomains with the corresponding exchange boundary conditions. The decomposition schemes with the operators of the domain decomposition are investigated in the grid Hilbert spaces. Among the works in this direction let us quote [3,7,8,14,23–26,28,30]. We emphasize the works [16,17,19,27] on the domain decomposition methods for nonstationary convection-diffusion and also the paper [31] in which the problems for evolution second-order equations have been considered. The results of our investigations of the domain decomposition methods for nonstationary problems of mathematical physics are presented in [13,18].

The regional additive schemes constructed on the basis of the partition of unity have certain defects. The most important among them is connected with the use of the uniquely defined operators with the generating coefficients in the overlaping domains. This leads to the fact that, for example, in the problems with the constant (piece-wise constant in the subdomain of partition) coefficients of the equations one needs to use the algorithm with strongly varying coefficients. The main goal of the paper is construction of unconditionally stable regional additive schemes with domain overlaping which are more comfortable for the practical use than traditional schemes constructed on the basis of the unit partition.

2. A model problem

Let us consider as a model a two-dimensional problem in the rectangle

$$\Omega = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), \quad 0 < x_i < l_i, \quad i = 1, 2\}.$$

The solution of the parabolic equations is sought in the domain Ω:

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{2} \frac{\partial}{\partial x_\alpha}\left(k_\alpha(\mathbf{x})\frac{\partial u}{\partial x_\alpha}\right) = 0, \quad \mathbf{x} = (x_1, x_2) \in \Omega, \quad t > 0. \tag{2.1}$$

Equation (2.1) is supplemented by the homogeneous boundary condition (the Dirichlet problem)

$$u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad t > 0, \tag{2.2}$$

and the initial condition

$$u(\mathbf{x}, 0) = u_0(x), \quad \mathbf{x} \in \Omega. \tag{2.3}$$

We will think that the equation coefficient (2.1) is sufficiently smoothed and

$$0 < m \leqslant k(x) \leqslant M, \quad \mathbf{x} \in \Omega.$$

Let us introduce into the domain Ω a uniform grid x_α with a fixed spacing $h_\alpha, \alpha = 1, 2$. Let ω be a set of internal nodes. We can easily pass to more general problems similarly as is done in the present paper. Let us define on the set of functions $y \in H$ such as $y(\mathbf{x}) \equiv 0$, $\mathbf{x} \notin \omega$ the difference operator A

$$Ay = -\sum_{\alpha=1}^{2} (a_\alpha(x) y_{\overline{x}})_x, \quad \mathbf{x} \in \omega. \tag{2.4}$$

Here the standard index-free notation of the difference scheme theory [12] is used, for example,

$$w_{\bar{x}} = \frac{w(x) - w(x-h)}{h}, \quad w_x = \frac{w(x+h) - w(x)}{h},$$

$$a_1(\mathbf{x}) = 0.5(k_1(x) + k_1(x_1 - h_1, x_2)), \quad a_2(\mathbf{x}) = 0.5(k_2(x) + k_2(x_1, x_2 - h_2)).$$

Let us introduce a scalar product and a norm in the Hilbert space H as follows:

$$(y, v) = \sum_{\mathbf{x} \in \omega} y v h_1 h_2, \quad \|y\| = (y, y)^{1/2}.$$

Note that in H the operator A is self-adjoint and positive definite [12], i.e., $A = A^* > 0$.

From equations (2.1)–(2.3) for the given $y(\mathbf{x}, 0), \mathbf{x} \in \omega$ we obtain the following equation:

$$\frac{dy}{dt} + Ay = 0, \quad \mathbf{x} \in \omega. \tag{2.5}$$

Equation (2.5) in accordance with (2.3) is supplemented with an initial condition. In the simplest case,

$$y(0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{2.6}$$

For problems (2.5), (2.6) difference schemes of domain decomposition schemes are constructed. The numerical implementation of these schemes is based on the solution of problems in separate subdomains of the calculation domain Ω at every time-level.

3. Region-additive schemes

Let the domain Ω consist of m separate subdomains

$$\Omega = \Omega_1 \cup \Omega_2 \cup \ldots \cup \Omega_p.$$

The separate subdomains may have overlapping. The work of one of the elementary processor is indended for solving the problem in an individual subdomain or on a set of separate subdomains. We assume that exactly the splitting of the whole domain into separate subdomains can lead to an effective usage of a parallel computer. The domain decomposition methods make it possible to adapt the numerical algorithms of an approximate solution of nonstationary problems on the arbitrary parallel computer architecture and also for the arbitrary memory configurations and commutation of different computer units.

Let ω_α be the subsets of nodes ω, lying in subdomains Ω_α, $\alpha = 1, 2, \ldots, p$. Let us construct decomposition difference schemes on the basis of the unit splitting for the domain Ω. Let us define the following functions:

$$\chi_\alpha(\mathbf{x}) = \begin{cases} > 0, & \mathbf{x} \in \Omega_\alpha, \\ 0, & \mathbf{x} \notin \Omega_\alpha, \end{cases} \quad \alpha = 1, 2, \ldots, p, \tag{3.1}$$

where

$$\sum_{\alpha=1}^{p} \chi_\alpha(\mathbf{x}) = 1, \quad \mathbf{x} \in \Omega. \tag{3.2}$$

Let us consider the class of decomposition schemes, where for the operator A the following additive presentation takes place:

$$A = \sum_{\alpha=1}^{p} A_\alpha, \tag{3.3}$$

here the operators A_α, $\alpha = 1, 2, \ldots, p$ are associated with isolated subdomains and also with splitting (3.1), (3.2) and with the solution of individual subproblems in subdomains $\Omega_\alpha, \alpha = 1, 2, \ldots, p$.

In dividing the initial domain into subdomains, it is necessary to construct such a decomposition (3.3) which inherits all basic properties of the operators of the initial problem in each subdomain. It seems reasonable enough to use the following decomposition operators:

$$A_\alpha = A_\alpha^* \geqslant 0, \quad \alpha = 1, 2, \ldots, p, \tag{3.4}$$

in presentation (3.3).

Taking into account (2.5), let us assume

$$A_\alpha y = -\sum_{\beta=1}^{2} (a_\beta^\alpha(x) y_{\overline{x}_\beta})_{x_\beta}, \quad \mathbf{x} \in \omega, \quad \alpha = 1, 2, \ldots, p. \tag{3.5}$$

The grid operators A_α, $\alpha = 1, 2, \ldots, p$ are approximated by the difference degenerating elliptic operators

$$\mathcal{A}_\alpha u \equiv -\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta} \left(\chi_\alpha(\mathbf{x}) k_\beta(\mathbf{x}) \frac{\partial u}{\partial x_\beta} \right), \quad \alpha = 1, 2, \ldots, p,$$

where the coefficients a_β^α are defined as a_β. With expressions (3.1),(3.2) we obtain (3.3), (3.4). The possibility of using other decomposition operators is shortly discussed below.

For an approximate solution of the Couchy problem for Eq. (2.5) under conditions (3.3), (3.4) particular splitting schemes are used. For example, different variants of alternative directions methods, factorized region-additive schemes, synchronous and asynchronous variants of additive schemes with component-wise splitting are analogues to the classical locally one-dimensional schemes, vector schemes, and regularized additive schemes [13, 18].

Let us give a typical result on the stability and accuracy of regional additive schemes. Under more general domain decomposition the grid operator A in Eq. (2.5) has the form (3.3) with a number of operators $p > 2$. For such problems the difference schemes of summarized approximation [12, 18] have some advantages. The investigation of the difference schemes of summarized approximation shows that these schemes have a low accuracy of the spatial approximation.

Let $\tau > 0$ be the time grid step and $y^n = y(t^n)$, $t^n = n\tau$. For instance, for the following fully implicit scheme of multicomponent splitting

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + A_\alpha y^{n+\alpha/p} = 0, \quad \alpha = 1, 2, \ldots, p, \tag{3.6}$$

the error estimate has the form

$$\left\| z^{n+1} \right\| \leqslant M\left(\left(1 + \sum_{\alpha=1}^{p} \|A\chi_\alpha\| \right) \tau + |h|^2 \right). \tag{3.7}$$

As for the parallel implementation, additive-averaged schemes of domain decomposition should be mentioned separately. For example, the implicit scheme has the following form (cf. (3.6)):

$$\frac{\tilde{y}^{n+\alpha/p} - y^n}{\tau} + A_\alpha \tilde{y}^{n+\alpha/p} = 0, \quad \alpha = 1, 2, \dots, p, \tag{3.8}$$

$$y^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} \tilde{y}^{n+\alpha/p} \tag{3.9}$$

and for the error we have estimate (3.7). The principal point here is the possibility of calculating $\tilde{y}^{n+\alpha/p}$, $\alpha = 1, 2, \dots, p$ independently (asynchronously).

The accuracy of decomposition schemes depends on the width of subdomain overlapping Ω_1 and Ω_2 (see the term at $\|A\chi_\alpha\|$ in estimate (3.6). At a minimal overlapping of subdomains (the width of subdomain overlapping equals $O(|h|)$) we have from estimate (13) that the convergence rate is $O(|h|^{-3/2}\tau + |h|^2)$. One succeeds to obtain better estimates of convergence for regional additive schemes (see [13, 18]) for two-components splitting ($p = 2$ (3.4)), but these estimates are poorly adjusted to problems with general domain decomposition.

4. Regularized schemes

The solution of problem (2.1)–(2.3) by the considered variants of the domain decomposition method is connected with the inversion of grid elliptic operators $E + \tau A_\alpha$. We put a problem — to construct unconditionally stable regional additive schemes with subdomain overlaping which, as distinct from the schemes of the type (3.6) and (3.8),(3.9) are based on the inversion of grid operators with a simpler structure. The problem is solved by using schemes constructed on the basis of the principle of regularization of the operator difference scheme [12]. Nowadays the regularization principle is considered as the main methodological principle for improoving difference schemes. The construction of unconditionally stable difference schemes on the basis of the regularization principle is realized as follows.

1. For the original problem one simplest difference scheme (generating difference scheme) having no necessary properties, i.e., a conditionally stable or even absolutely unstable scheme is constructed.

2. The difference scheme is written in unified (canonic) form for which the conditions stability are known.

3. The properties of the difference scheme (its stability) are improved due to the perturbation of the difference scheme operators.

The regularization principle thereby is based on using the already known results on the stability conditions. The general theory of stability of difference schemes by A.A. Samarskii [12] gives the same criteria. From this point of view, we can consider the regularization principle as an element of constructive use of the general results of the theory of stability of difference schemes. This is possible due to the writing of the difference schemes in a general enough canonical form and the formulation of criteria convenient for verification.

With reference to the model problem (2.5), (2.6) let us first choose some difference scheme from which we start. As such a generating scheme, it is natural to consider the simplest explicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + A y^n = 0, \quad \mathbf{x} \in \omega, \quad n = 0, 1, \dots, \tag{4.1}$$

$$y^0 = u_0(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{4.2}$$

Let us write the difference scheme (4.1), (4.2) in canonical form of the two-layer operator and the difference schemes

$$B\frac{y^{n+1}-y^n}{\tau}+Ay^n=0, \quad n=0,1,\dots \tag{4.3}$$

under a given y^0 with the operators

$$A=A, \quad B=E,$$

where E is a unit (identical) operator.

The stability conditions of the operator and the difference scheme (4.3) were formulated by A. A. Samarskii [12]. For the difference scheme (4.3) with the operator $A=A^*>0$ the condition

$$B \geqslant \frac{\tau}{2}A \tag{4.4}$$

is necessary and sufficient for stability in H_A, i.e., for the estimate

$$\|y^{n+1}\|_A \leqslant \|y^n\|_A, \quad n=0,1,\dots, \quad \text{to be fulfilled.} \tag{4.5}$$

If the operator $B=B^*>0$, then condition (4.4) is necessary and sufficient for stability in H_B as well.

Due to the inequality $A \leqslant \|A\|E$, from the necessary and sufficient stability conditions (4.4) we obtain the following limitations on the time step for the explicit scheme (4.1), (4.2)

$$\tau \leqslant \frac{2}{\|A\|}.$$

In our case $\|A\|=O(|h|^{-2})$, where $|h|^2=h_1^2+h_2^2$ and the maximum allowable step $\tau_0 = O(|h|^2)$.

In accordance with (4.4), an increase in the stability can be attained in two ways. In the first case — due to an increase in energy (By,y) of the operator B (left-hand side of inequality (4.4)) or due to a decrease in the energy of the operator A (right-hand side of inequality (4.4)). Note first the possibilities connected with the addition of the operator terms to the operators B and A. In this case, we will speak of additive regularization.

It is more natural to start with an additive perturbation of the operator B, i.e., with the change $B \longmapsto B+\alpha R$, where R is the regularization operator, α is the regularization parameter. Taking into account that for our generating scheme $B=E$, let

$$A=A, \quad B=E+\alpha R, \tag{4.6}$$

To save the first order of approximation in scheme (4.3), (4.6), it is necessary to choose $\alpha=O(\tau)$.

As most typical ways, let us consider two ways for choosing the regularization operator:

$$R=A, \tag{4.7}$$

$$R=A^2. \tag{4.8}$$

One can directly verify whether the regularized difference scheme (4.3), (4.6) is stable in H_A under $\alpha \geqslant \tau/2$ in case (4.7) and $\alpha \geqslant \tau^2/16$ under (4.8).

The regularized scheme (4.3), (4.6), (4.7) corresponds to the standard scheme with weights

$$\frac{y^{n+1}-y^n}{\tau}+A(\sigma y^{n+1}+(1-\sigma)y^n)=0, \quad \mathbf{x}\in\omega, \quad n=0,1,\dots,$$

if $\alpha=\sigma\tau$ is chosen.

The standard approach to the construction of stable schemes is based on the use of additive regularization. The second possibility is connected with multiplicative perturbation of the grid operators of the generating scheme. Let us give some simplest examples of using such an approach. Some of the examples can be considered as a new interpretation of the above regularized schemes.

Let us replace, e.g., $B\longmapsto B(1+\alpha R)$ or $B\longmapsto(1+\alpha R)B$ under multiplicative regularization of the operator B. Under such perturbation we still deal with the class of schemes with self-adjoint operators if $R=R^*$. Here we have the regularized scheme (4.3), (4.6) investigated above.

An example of a more complicated regularization is given by the transformation

$$B\longmapsto(E+\alpha R^*)B(E+\alpha R).$$

In the case of $R=A$, the stability condition is $\alpha\geqslant\tau/8$. Another interesting example of such regularization corresponds to [12] when $A=R^*+R$, $\alpha\geqslant\tau/2$.

Likewise, one can perform multiplicative regularization at the expense of perturbation of the operator B. Taking into account inequality (4.4), one can realize the transformation $A\longmapsto A(1+\alpha R)^{-1}$ or $A\longmapsto(1+\alpha R)^{-1}A$. For the simplest two-layer schemes such a regularization can be considered as a new formulation of the operator B regularization. To remain in the class of schemes with self-adjoint operators, it is sufficient to choose $R=R(A)$. Many possibilities are given by the regularization

$$A\longmapsto(E+\alpha R^*)^{-1}A(E+\alpha R)^{-1}. \tag{4.9}$$

In this case, the regularizaed operator R cannot be directly related to the operator A.

Let us construct (see [18]) additive schemes on the basis of the regularization principle of difference schemes. In constructing unconditionally stable additive schemes for problem (2.5), (2.6), (3.3), (3.4) as generating schemes, we will consider the simplest explicit scheme

$$\frac{y^{n+1}-y^n}{\tau}+\sum_{\alpha=1}^{p}A_\alpha y^n=0, \quad \mathbf{x}\in\omega, \quad n=0,1,\dots \tag{4.10}$$

At multiplicative regularization we will perturb each operator term in (4.10)

$$\frac{y^{n+1}-y^n}{\tau}+\sum_{\alpha=1}^{p}(E+\sigma_\alpha\tau A_\alpha)^{-1}A_\alpha y^n=0, \quad \mathbf{x}\in\omega, \quad n=0,1,\dots \tag{4.11}$$

Under $\sigma_\alpha\geqslant p/2$, $\alpha=1,2,\dots,p$ and any $\tau>0$ (4.2), (4.11) is a valid a priori estimate

$$\|y^{n+1}\|\leqslant\|u_0\|. \tag{4.12}$$

The considered regularized scheme (4.11) can be realized as the additive average total-approximation scheme (3.8), (3.9). Note that the construction of such an additive scheme of multi-components splitting is carried out now without total approximation.

5. General regulators

In constructing stable difference schemes on the basis of the regularization principle, one of the most important problems is the problem of choosing the regularizing operator R. We have discussed above some possible structures connected with a particular perturbation of the difference scheme operators. Here to set regularizing operators, in fact only two possibilities have been used (4.7), (4.8). Under more general conditions one can put $R = R(A)$. The real limitation is connected with the fulfilment of the commutativity condition of the regulator B and operator A ($RA = AR$). Let us consider issues of constructing the regularized operator and difference schemes when the operators R and A do not commutate with each other, i.e.,

$$RA \neq AR. \tag{5.1}$$

It is natural to choose the regulator to fulfil the requirement of simpler computational realization. For example, in [12] in constructing of the schemes for problems with variable coefficients the regulating operator corresponds to the problem with constant coefficients. In this case, the grid problem on a new time layer is essentially simplified.

For the model problem (2.1)–(2.3) it is natural to connect the choice of the regulating operator with the Laplace operator

$$A_0 y = -\sum_{i=1}^{2} y_{\bar{x}_i x_i},$$

which is determined on the set of grid functions $y(\mathbf{x})$ such that $y(\mathbf{x}) = 0$, $\mathbf{x} \neq \omega$.

Taking into account the limitations of the coefficient $k(x)$ formulated above, we will get

$$mA_0 \leqslant A \leqslant MA_0, \quad m > 0.$$

Now let us choose the regulating operator $R = MA_0$. For the regularized scheme (4.3), (4.6) at such R the stability condition takes the form $\alpha \geqslant \tau/2$.

In accordance with the considered problem (4.1), (4.2), (4.6), for the sake of simplicity, we will restrict ourselves to the important class of regulators

$$R = R^* \geqslant A. \tag{5.2}$$

The regularized operator and the difference scheme (4.3), (4.6), (5.2) are unconditionally stable at $\alpha \geqslant \tau/2$. In constructing regularized additive schemes, stability is established for schemes of the type of (4.3), (4.6) for each individual operator term $A_\alpha, \alpha = 1, 2, \ldots, p$ (3.3), (3.4). For obtaining a general estimate of stability from the stability for separate subproblems a stability should take place in a unique norm. In accordance with the considered problem (2.5), (2.6) such a norm is H.

At a sufficiently small τ the explicit scheme (4.1), (4.2), as well as the solution of problem (2.5), (2.6), is stable in any norm of the space of grid functions H_D generated by the operator $D = D^* > 0$:

$$(y, v)_D = (Dy, v), \quad \|y\|_D = (Dy, y)^{1/2}.$$

at $DA = AD$. For the regularized scheme (4.3), (4.6), (5.2) a stability takes place in H_D at

$$D = A, \quad D = E + \alpha R - \frac{\tau}{2} A.$$

and some of their combinations. In particular, stability takes place at $D = E + \alpha R$. It is essentially for us that for both regulators (5.1) one does not succeed to prove stability in H. By virtue of that we cannot use a scheme of the type of (4.3), (4.6) with general regulators (5.1).

To construct a regularized operator and difference schemes with regulators (5.1), (5.2) stable in H, we will use a regularization of the type of (4.9). For the considered generating scheme (4.1), (4.2) let us set

$$\frac{y^{n+1} - y^n}{\tau} + \tilde{A} y^n = 0, \quad n = 0, 1, \dots, \tag{5.3}$$

where

$$\tilde{A} = (E + \alpha R)^{-1} A (E + \alpha R)^{-1}. \tag{5.4}$$

To get necessary and sufficient conditions of stability, we will use inequality (4.4). Under our assumptions (5.2) it is equivalent to the inequality

$$(E + \alpha R)(E + \alpha R) - \frac{\tau}{2} R \geqslant 0,$$

which is fulfilled at $\alpha \geqslant \tau/8$.

Regularization of the explicit additive scheme (4.10) by scheme (5.3), (5.4) yields

$$\frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} \tilde{A}_\alpha y^n = 0, \quad \mathbf{x} \in \omega, \quad n = 0, 1, \dots, \tag{5.5}$$

$$\tilde{A}_\alpha = (E + \sigma_\alpha \tau R_\alpha)^{-1} A_\alpha (E + \sigma_\alpha \tau R_\alpha)^{-1}, \quad \alpha = 1, 2, \dots, p, \tag{5.6}$$

with the regulator assignment

$$R_\alpha = R_\alpha^* \geqslant A_\alpha \quad \alpha = 1, 2, \dots, p. \tag{5.7}$$

Let us rewrite the stability conditions (4.4) of scheme (5.5)–(5.7) as

$$\sum_{\alpha=1}^{p} \frac{1}{p} E \geqslant \sum_{\alpha=1}^{p} \frac{\tau}{2} \tilde{A}_\alpha.$$

They will be satisfied, e.g., at

$$\frac{1}{p} E \geqslant \frac{\tau}{2} \tilde{A}_\alpha. \tag{5.8}$$

For the fulfillment of inequality (5.8), we will choose $\sigma_\alpha \geqslant p/8$, $\alpha = 1, 2, \dots, p$. With such restrictions on $\sigma_\alpha, \alpha = 1, 2, \dots, p$ the regularized additive scheme (4.2), (5.5)–(5.7) is stable in H, and for the solution the a priori estimate of stability in the sense of the initial data (4.12) is valid. Considering the problem for the error [18] the convergence of an approximate solution to the exact one with the rate $O(\tau)$ has been established.

One can make similar conclusions about the additive scheme of multi-component splitting (see (3.6)). In this case,

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + \tilde{A}_\alpha y^{n+(\alpha-1)/p} = 0, \quad \alpha = 1, 2, \dots, p, \tag{5.9}$$

using (5.6). The stability conditions of scheme (5.6), (5.9) are $\sigma_\alpha \geqslant 1/8$, $\alpha = 1, 2, \dots, p$.

6. Region-additive schemes with subdomain overlaping

With the use of the general regulators of the type of (5.1), (5.2) we will construct new region-additive schemes. They are based on the solution of the subproblems in the subdomain with subdomains overlaping. In our case, the decomposition operators $A_\alpha, \alpha = 1, 2, \dots, p$ are constructed on the basis of the partition of unity for the calculated domain Ω (3.1), (3.2) and determined according to (3.3)–(3.5).

For constructing the regulators R_α, $\alpha = 1, 2, \dots, p$, the domain decomposition is used

$$\Omega = \tilde{\Omega}_1 \cup \tilde{\Omega}_2 \cup \ldots \cup \tilde{\Omega}_p$$

at

$$\Omega_\alpha \subset \tilde{\Omega}_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{6.1}$$

Thus, the regulators are constructed along large subdomains $\tilde{\Omega}_\alpha$, $\alpha = 1, 2, \dots, p$.

For the explicit construction of the regulators let us determine the functions

$$\tilde{\chi}_\alpha(\mathbf{x}) = \begin{cases} > 0, & \mathbf{x} \in \tilde{\Omega}_\alpha, \\ 0, & \mathbf{x} \notin \tilde{\Omega}_\alpha, \end{cases} \quad \alpha = 1, 2, \dots, p, \tag{6.2}$$

at

$$\tilde{\chi}_\alpha(\mathbf{x}) \geqslant \chi_\alpha(\mathbf{x}), \quad \mathbf{x} \in \Omega_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{6.3}$$

In an important particular case,

$$\tilde{\chi}_\alpha(\mathbf{x}) = 1, \quad \mathbf{x} \in \Omega_\alpha, \quad \alpha = 1, 2, \dots, p.$$

The operators $R_\alpha, \alpha = 1, 2, \dots, p$ are constructed by analogy with the operators of decomposition $A_\alpha, \alpha = 1, 2, \dots, p$. Let us define the grid operators

$$R_\alpha y = -\sum_{\beta=1}^{2} (r_\beta^\alpha(x) y_{\overline{x}_\beta})_{x_\beta}, \quad \mathbf{x} \in \omega, \quad \alpha = 1, 2, \dots, p, \tag{6.4}$$

which correspond to the differential operators

$$\mathcal{R}_\alpha u \equiv -\sum_{\beta=1}^{2} \frac{\partial}{\partial x_\beta} \left(\tilde{\chi}_\alpha(\mathbf{x}) k_\beta(\mathbf{x}) \frac{\partial u}{\partial x_\beta} \right), \quad \alpha = 1, 2, \dots, p.$$

Under restrictions (6.2), (6.3) for operators (6.4) we have

$$R_\alpha = R_\alpha^* \geqslant A_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{6.5}$$

Inequalities (6.5) provide the belonging of the regulators to class (5.1), (5.2).

The region-additive schemes (5.5), (5.6) and (5.9) with operators (3.3)–(3.5) and (6.1)–(6.4) are unconditionally stable in H. Their main advantage over the region-additive schemes (3.6) (3.8), (3.9) is the larger freedom of choosing a problem on a new time layer. Instead of the operator inversion $E + \sigma_\alpha A_\alpha$, $\alpha = 1, 2, \dots, p$, the more general grid operators $E + \sigma_\alpha R_\alpha$, $\alpha = 1, 2, \dots, p$, are inversed.

7. Generalizations

Among the most important generalizations of the constructed region-additive schemes let us note the choice of other operators of decomposition. Likewise [13], with respect to the partition of unity (3.1), (3.2) we define the operators of domain decomposition A_α, $\alpha = 1, 2, \dots, p$, in the following way:

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1, 2, \dots, p. \tag{7.1}$$

The following presentation for the decomposition operators can be used:

$$A_\alpha = A\chi_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{7.2}$$

Clearly, for such a splitting the operator A is not self-adjoint, i.e., $A_\alpha \neq A_\alpha^*$, $\alpha = 1, 2, \dots, p$. For (7.1), (7.2) stability of the region-additive schemes takes place in H_A, $H_{A^{-1}}$ respectively.

The regulators R_α, $\alpha = 1, 2, \dots, p$, are constructed by analogy with the operators A_α, $\alpha = 1, 2, \dots, p$:

$$R_\alpha = \tilde{\chi}_\alpha A, \quad \alpha = 1, 2, \dots, p, \tag{7.3}$$

$$R_\alpha = A\tilde{\chi}_\alpha, \quad \alpha = 1, 2, \dots, p. \tag{7.4}$$

The main properties of the operators of domain and regulator decomposition are directly established

$$R_\alpha = R_\alpha^* \geqslant A_\alpha = A_\alpha^* \geqslant 0, \quad \alpha = 1, 2, \dots, p, \tag{7.5}$$

in H_A for the choice of (7.1), (7.3) and in $H_{A^{-1}}$ ufor (7.2), (7.4). With (7.5) region-additive schemes are constructed according to the scheme considered above.

Generalizations of the considered region-additive schemes over not self-adjoint problems (convection-diffusion problems) are carried out by analogy with [15]. The schemes of the second order on time are constructed [18] on the basis of the regularization of the explicit three-layer Adams scheme. Analogously to [31] problems for second-order evolution equations are considered.

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