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WORKS OF A. A. SAMARSKII ON COMPUTATIONAL MATHEMATICS

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Abstract — This is a review of the main results in computational mathematics that were obtained by the eminent Russian mathematician Alexander Andreevich Samarskii (February 19, 1919 – February 11, 2008). His outstanding research output addresses all the main questions that arise in the construction and justification of algorithms for the numerical solution of problems from mathematical physics.
The remarkable works of A. A. Samarskii include statements of the main principles required in the construction of difference schemes, rigorous mathematical proofs of the stability and convergence of these schemes, and also investigations of their algorithmic implementation. A. A. Samarskii and his collaborators constructed and applied in practical calculations a large number of algorithms for solving various problems from mathematical physics, including thermal physics, gas dynamics, magnetic gas dynamics, plasma physics, ecology and other important models from the natural sciences.

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Introduction

The effective solution of applied problems today presupposes the wide availability of computers and, therefore, the development of computer-oriented numerical methods. The works of A. A. Samarskii that deal with the construction and mathematical justification of algorithms for the numerical solution of problems from mathematical physics began at the end of 1940s and laid the foundation for the modern theory of difference schemes. His theory of difference methods for the solution of mathematical physics problems has the following main strands:

• construction of discrete analogues that inherit the basic properties of the differential problem;

• investigation of stability (well-posedness) of the difference problem;

• effective computational implementation on modern computers.

In his papers, A. A. Samarskii obtained the principal results that underpin the current development of the theory of difference schemes.

For the construction of difference schemes he formulated a general concept of conservative difference schemes, i.e., those schemes that satisfy the corresponding discrete conservation law. Homogeneous difference schemes were derived for problems with discontinuous coefficients and with weak solutions. For the construction of difference schemes on an arbitrary

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grids he proposed the method of support operators that deal with conforming approximations of differential operators from vector analysis (gradient, divergence and curl). His papers suggest and use a general methodology for the construction of difference schemes of any desired quality, namely, the principle of regularization of difference schemes based on small perturbations of the operators (coefficients) of these schemes.

The general theory of stability (well-posedness) of operator-difference schemes is widely used when investigating difference schemes for the non-stationary problems of mathematical physics. In the works by A. A. Samarskii precise necessary and sufficient conditions for stability in finite-dimensional Hilbert spaces are obtained for a wide class of two-level and three-level difference schemes. In particular it should be emphasized that a general theory of stability is developed in which stability criteria are stated in the form of easy-to-test inequalities for the operator. Among important generalizations let us note the use of a general theory of stability for ill-posed time-dependent problems and for the investigation of projection-difference schemes (finite-element schemes). New *a priori* estimates of stability were also obtained in norms based on time integrals, from which convergence of difference schemes for problems with weak solutions is studied. Special attention is paid to *a priori* estimates of strong (coefficient) stability under various perturbation assumptions on the differential and difference operators.

When solving approximately initial-boundary value problems for multi-dimensional partial differential equations, great attention is paid to the construction of additive schemes. This reduction to a sequence of simpler problems (splitting with respect to spatial variables) permits the construction of economical difference schemes. In some cases it is useful to split subproblems of different natures (splitting with respect to physical processes). Recently regionally-additive schemes (domain decomposition schemes) have been examined, which are suitable when constructing computational algorithms for parallel computing. With multi-component splitting into three or more operators, unconditionally stable additive schemes are constructed using the idea of total approximation introduced by A. A. Samarskii — that is, one uses a separate initial-value problem for each operator summand. Furthermore, in some cases additive multi-component schemes are constructed without using total approximation.

To compute approximate solutions it is necessary to solve large systems of linear or nonlinear algebraic equations. Iterative methods for these equations were frequently considered in the works of A. A. Samarskii: criteria for the choice of iteration parameters in Chebyshev iterative methods were formulated, the optimal choice of iterative parameters in the approximate solution of non-selfadjoint problems was given for general operators, and new alternating-direction iterative methods were proposed. Special attention should be paid to the alternately-triangular iterative method, which is one of the fastest methods and is applied to general elliptic grid equations.

The power of these general results is seen in their widespread use today to solve major scientific and technical problems such as the calculation of nuclear and thermonuclear products and the theory of controlled fusion. The difference methods developed are applied in the numerical study of the processes of heat and mass transfer and in solving problems of mechanics.

The results obtained form the basis of the monographs and textbooks written by A. A. Samarskii and his disciples. His books are used to teach general and special courses on numerical methods and provide the foundation when training applied mathematicians at Russian universities. A special place is occupied by the books [1, 2], which are the standard reference sources for experts in numerical methods.

1. General principles for constructing difference schemes

To solve an applied problem approximately via a numerical method, one starts with the construction of its discrete analogue. The discrete problem must preserve the main properties of the differential problem. The issues involved when constructing discrete problems are dealt with in a variety of ways. In difference methods, the reduction to a finite-dimensional problem is achieved by replacing a continuous function by a discrete analogue. In projection-grid methods (finite-element methods and spectral methods) one considers finite-dimensional subspaces of functions of continuous arguments that are specified by some basis.

Various approaches are used to approximate the differential problem. The simplest of these is based on replacing the differential operator by a difference operator, but this approach is not immediately applicable to problems having discontinuous coefficients or non-smooth solutions. For such problems, the general principles for the construction of difference schemes of the desired quality formulated by A. A. Samarskii are widely used.

1. The principle of conservation was proposed when constructing difference schemes. The difference scheme is conservative if the corresponding conservation law is fulfilled on the discrete level. For partial differential equations and for systems of PDEs the principle of full conservation was formulated. It is associated with the fulfillment of all main conservation laws on the discrete level.

2. The requirement of uniformity of computational algorithm for a class of problems has led to the concept of a uniform difference scheme. A uniform difference scheme is a difference scheme whose form depends neither on the concrete problem nor on the mesh used; its coefficients are defined as functionals of the coefficients of the differential equation.

3. Applied mathematical models are often based on the equations of mathematical physics written in terms of invariant first-order operators (div, grad, rot and their combinations). The method of support operators was proposed for the construction of discrete analogues of these operators on arbitrary computational grids.

4. The principle of regularization of difference schemes provides opportunities for constructing difference schemes of a desired accuracy exploiting small perturbations of the coefficients (operators) of the difference scheme. In this way stable difference schemes are constructed for a wide class of problems of mathematical physics and iterative methods for solving the grid equations are obtained.

1.1. Conservative difference schemes. The differential equations of continuum mechanics are conservation laws over small volumes (conservation laws in integral form) where the volume shrinks to zero. The construction of discrete analogues is, in fact, based on returning from the differential to the integral model. In this transformation it is natural to require that conservation laws are fulfilled. Difference schemes that express conservation laws on the grid level are called *conservative difference schemes* [3, 4].

Let us give an example (A. A. Samarskii, 1954) of a nonconservative scheme for the simplest diffusion equation. This scheme diverges in the case of a discontinuous diffusion coefficient. Consider the problem

$$-\frac{d}{dx}\left(k(x)\frac{du}{dx}\right) = 0, \quad 0 < x < 1, \tag{1.1}$$

$$u(0) = 1, \quad u(0) = 0. \tag{1.2}$$

To construct the difference scheme we use the following equation obtained from (1.1) by formal differentiation:

$$-k(x)\frac{d^2u}{dx^2} - \frac{dk}{dx}\frac{du}{dx} = 0, \quad 0 < x < 1. \tag{1.3}$$

Let us associate the problem (1.2), (1.3) with the following difference scheme (which is second-order accurate for smooth coefficients and smooth solution):

$$-ky_{\bar{x}x} - k_{\overset{\circ}{x}}y_{\overset{\circ}{x}} = 0, \quad x \in \omega, \tag{1.4}$$

$$y_0 = 1, \quad y_N = 0, \tag{1.5}$$

where ω is the set of interior mesh-points of a uniform mesh. Here we use the standard notation of the theory of difference schemes: for left and right difference derivatives in the variable x we set

$$w_x = \frac{w(x+h) - w(x)}{h}, \quad w_{\bar{x}} = \frac{w(x) - w(x-h)}{h},$$

and for the central derivative one uses

$$w_{\overset{\circ}{x}} = \frac{1}{2}(w_x + w_{\bar{x}}) = \frac{w(x+h) - w(x-h)}{2h}.$$

Consider the simplest case of piecewise-constant coefficients when

$$k(x) = \begin{cases} k_1, & 0 < x < \xi, \\ k_2, & \xi < x < 1. \end{cases}$$

At the point of discontinuity $x = \xi$ one has the conditions of ideal contact (viz., solution and flux continuity):

$$[u] \equiv u(\xi + 0) - u(\xi - 0) = 0, \quad \left[k\frac{du}{dx}\right] = 0. \tag{1.6}$$

Under these circumstances, the problem (1.1),(1.2) has the piecewise-linear solution

$$u(x) = \begin{cases} 1 - \alpha_0 x, & 0 < x < \xi, \\ \beta_0(1 - x), & \xi < x < 1. \end{cases} \tag{1.7}$$

Here the constants α_0 and β_0 can be found from the coupling conditions (1.6):

$$\alpha_0 = \frac{1}{\chi + (1 - \chi)\xi}, \quad \beta_0 = \chi\alpha_0, \quad \chi = \frac{k_1}{k_2}.$$

Using linear interpolation extend the grid function to the entire interval $0 \leqslant x \leqslant 1$. Then as $h \to 0$ we get the limit function

$$\bar{u}(x) = \begin{cases} 1 - \bar{\alpha}_0 x, & 0 < x < \xi, \\ \bar{\beta}_0(1 - x), & \xi < x < 1, \end{cases} \tag{1.8}$$

where

$$\bar{\alpha}_0 = \frac{1}{\mu + (1 - \mu)\xi}, \quad \bar{\beta}_0 = \mu\bar{\alpha}_0, \quad \mu = \lambda\frac{3 + \chi}{5 - \chi}.$$

Comparing (1.8) with (1.7) we see that the limit function $\bar{u}(x)$ coincides with the exact solution of the problem $u(x)$ only if $\bar{\alpha}_0 = \alpha_0$ and $\bar{\beta}_0 = \beta_0$. But this is possible only if $\chi = 1$, i.e., if $k_1 = k_2$. Hence for a discontinuous diffusion coefficient, i.e., $k_1 \neq k_2$, the difference scheme (1.4),(1.5) diverges.

It is shown that the property of conservation is necessary for the convergence of a difference scheme in the class of problems with discontinuous coefficients. To construct conservative difference schemes it is natural to proceed from conservation laws on separate cells of a difference mesh. The universal method of balance (integro-interpolation method) is the primary method for constructing the discrete problems. It was proposed by A. A. Samarskii in [5], and since the mid 1950s as been actively used in computational practice for the numerical solution of various applied problems. Currently, in the English literature the term "finite volume method" is used instead of integro-interpolation method when constructing discrete problems. The general constructive nature of the integro-interpolation method is apparent when devising difference schemes on irregular grids and for problems with discontinuous coefficients.

1.2. Homogeneous difference schemes. In *homogeneous difference schemes* [4,6] the coefficients are calculated by common formulas that do not change as one moves from one mesh-point to another and that do not depend on the coefficients of the differential equation. Such schemes are equally valid when one has smooth coefficients in the differential equation and when one has non-smooth (discontinuous) coefficients.

Homogeneous difference schemes are constructed for a wide class of ordinary differential equations and equations of mathematical physics. The techniques developed are used in constructing discrete analogues of the mathematical models of gas dynamics, magnetohydrodynamic, heat transfer, and problems with phase transitions.

Let us consider a typical example of a homogeneous difference scheme for the simplest second-order differential equation. Applying the integro-interpolation method, one constructs the following homogeneous difference scheme for problem (1.1), (1.2):

$$(au_{\bar{x}})_x = 0,$$

where

$$a_i = \left(\frac{1}{h} \int\limits_{x_{i-1}}^{x_i} \frac{dx}{k(x)} \right)^{-1}.$$

More general multidimensional partial differential equations can be handled in the same way.

Homogeneous difference schemes converge for problems with discontinuous coefficients. To prove this, it is necessary to modify the concept of accuracy and derive the error estimates using special negative norms. A new understanding of the role of approximation error in the evaluation of accuracy has been indispensable in the study of convergence of difference schemes on nonuniform meshes. It is shown that the global accuracy of difference schemes on nonuniform meshes is the same as for difference schemes on uniform meshes, even though the local accuracy of such schemes is worse.

Difference schemes for second-order differential equations usually have second-order accuracy. In [4] an *exact difference scheme* is constructed for ordinary differential equations with discontinuous coefficients, and the solution of this scheme at the mesh-points coincides with the solution of the original differential equation. One also obtains schemes of any desired order of accuracy, including schemes for boundary-value problems with weak solutions [7].

1.3. Method of support operators. To improve the accuracy of the approximate solution it has been traditional to use irregular computational meshes that are suited to the behaviour of the true solution. For this reason, in the theory of difference schemes great attention has been given to constructing difference schemes on arbitrary structured and unstructured meshes and studying the accuracy of the difference scheme solution. In the works by A. A. Samarskii for constructing difference schemes on unstructured meshes, the *method of support operators* has been proposed and rigorously justified on classes of problems from mathematical physics [8, 9].

Many applied problems are formulated in terms of the invariant vector operators div, grad, rot and combinations of them. As a typical example, consider the Dirichlet problem for an elliptic second-order equation:

$$\operatorname{div}(k(\mathbf{x})\operatorname{grad} u) = -f(\mathbf{x}), \quad \mathbf{x} \in \Omega,$$

$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \partial\Omega.$$

For this problem, the idea of the method of support operators is that one of the operators (support operator) div or grad is approximated directly and the other in such a way as to satisfy the difference analogue of the integral identity

$$\int\limits_{\Omega} u \operatorname{div} \mathbf{p}\, d\mathbf{x} + \int\limits_{\Omega} (\mathbf{p}, \operatorname{grad} u)\, d\mathbf{x} = \int\limits_{\partial\Omega} u\mathbf{p}\, d\mathbf{x}.$$

Thus we construct grid operators div_h, rad_h and rot_h. Then the difference problem corresponds to the differential problem; for example

$$\operatorname{div}_h(a(\mathbf{x})\operatorname{grad}_h y) = -\varphi(\mathbf{x}), \quad \mathbf{x} \in \omega.$$

At the level of finite differences, this form of consistent approximation permits us to retain important features such as conservation and the adjoint relationship between the operators div_h and grad_h.

1.4. Regularization principle for difference schemes. Nowadays the *regularization principle for difference schemes* [10] is regarded as the main principle for improving difference schemes, as illustrated by a large number of practical examples. For general two-level and three-level schemes the recipes are formulated so as to improve the quality (stability, accuracy, cost effectiveness) of difference schemes. This principle is used to study stability and convergence for a large class of difference schemes for the boundary-value problems of mathematical physics, and iterative algorithms for solving the discrete problems are constructed.

The regularization principle for difference schemes is widely used for the construction of stable difference schemes for the numerical solution of well-posed partial differential equation problems. The same approach is also used when constructing difference schemes for ill-posed time-dependent problems from mathematical physics. By means of small perturbations of operators of the problem one can control the growth of the norm of the solution at one moves from each time level to the next.

The construction of an unconditionally stable difference scheme based on the regularization principle is carried out as follows:

1) construct a simple difference scheme (a generating difference scheme) for the original problem; this scheme does not have the desired properties, e.g., it may be conditionally stable or even unstable;

2) write the difference scheme in a uniform (canonical) form for which the conditions of stability are known;

3) improve the quality of the difference scheme (its stability) by perturbations of its operators.

Thus the regularization principle for difference schemes is based on previously-known results for conditions ensuring stability. These criteria are provided by the general theory of stability of difference schemes. From this perspective, we can consider the regularization principle as a constructive use of general results in the stability theory of difference schemes. This is achieved by rewriting the difference schemes in a common canonical form and formulating the criteria of stability in a way that is easy to verify.

2. Stability of difference schemes

By a finite-difference or finite-element discretization in space of a problem from mathematical physics we obtain a Cauchy problem for a system of ordinary differential equations considered in a Hilbert space over the mesh (a differential-operator problem). Then discretization in time yields an operator-difference scheme. Here we describe some of the major strands in the development of the theory of operator-difference schemes for time-dependent problems from mathematical physics.

In the works of A. A. Samarskii a *general theory of stability (well-posedness) of operator difference schemes* was constructed and widely used. Among the most significant results from these studies are necessary and sufficient stability conditions for a wide class of two-level and three-level difference schemes in Hilbert spaces.

The general theory of stability of operator-difference schemes of A.A.Samarskii is constructive: it provides stability criteria that are formulated as easily-verifiable inequalities for operators. Similar stability criteria for projection-difference schemes (finite element schemes) are obtained in the form of inequalities for bilinear forms. The results of the general theory are used in the study of difference schemes for well-posed and ill-posed time-dependent problems and for problems with weak solutions.

For time-dependent initial-boundary value problems the fundamental theory deals not only with stability of the solution with respect to the initial data and right-hand side but also with the continuous dependence of the solution on perturbations of the operators of the problem, such as the coefficients of the differential equation (*strong stability*). In the works of A. A. Samarskii stability estimates are obtained for perturbations of the operator of the Cauchy problem, the right-hand side and the initial condition, for time-dependent first-order equations considered in Hilbert spaces. Estimates of strong stability are presented for two-level operator-difference schemes and these estimates are consistent with estimates for differential-operator equations.

2.1. Basic concepts. Let H be a Hilbert space with D and A linear operators acting on H. Let the inner product and norm in H be $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. Let H_D (where $D = D^* > 0$) denote a space H equipped with an inner product and norm

$$(y, v)_D = (Dy, v), \quad \|y\|_D = (Dy, y)^{1/2}.$$

Let $\tau > 0$ be the mesh width in time and set $y^n = y(t^n)$ where $t^n = n\tau$. Consider the uniform (for simplicity) two-level operator-difference scheme written in *canonical form*:

$$B\frac{y^{n+1} - y^n}{\tau} + Ay^n = 0, \quad n = 0, 1, \dots \tag{2.1}$$

where y^0 is given. Suppose that in (2.1) the operators A and B are constant (independent of n), and the operator A is self-adjoint and positive ($A = A^* > 0$).

A difference scheme (2.1) is called *ρ-stable* (uniformly stable) with respect to the initial data in H_D if there exist constants $\rho > 0$ and M, which are independent of τ and n, such that for any n and for all $y^n \in H$ the solution y^{n+1} of the difference equation (2.1) satisfies the following estimate:

$$\|y^{n+1}\|_D \leqslant \rho \|y^n\|_D, \quad n = 0, 1, \dots,$$

with moreover $\rho^n \leqslant M$. The constant ρ is often chosen to be one of the following:

$$\begin{aligned} \rho &= 1, \\ \rho &= 1 + c\tau, \quad c > 0, \\ \rho &= \exp(c\tau), \end{aligned}$$

where the constant c is independent of τ and n. In the case $\rho = 1$ the difference schemes (2.1) is *stable*.

2.2. Stability criteria. Let us formulate the principle results of A. A. Samarskii [11,12] concerning stability conditions for difference schemes:

Theorem 2.1. *For the difference scheme* (2.1) *with operator $A = A^* > 0$ the condition*

$$B \geqslant \frac{\tau}{2} A \tag{2.2}$$

is necessary and sufficient for stability in H_A, i.e., for fulfillment of the estimate

$$\|y^{n+1}\|_A \leqslant \rho \|y^n\|_A, \quad n = 0, 1, \dots \tag{2.3}$$

If $B = B^* > 0$, then the condition (2.2) *is necessary and sufficient for stability in H_B.*

When considering many time-dependent problems such as convection-diffusion-reaction problems one should use the following condition of ρ-stability.

Theorem 2.2. *For the difference scheme* (2.1) *with operators $A = A^*$ and $B = B^* > 0$, the conditions*

$$\frac{1-\rho}{\tau} B \leqslant A \leqslant \frac{1+\rho}{\tau} B \tag{2.4}$$

are necessary and sufficient for stability in H_B.

The convergence of difference schemes is established in various norms that must be consistent with the degree of smoothness of the solution of the differential problem. For this reason, it is necessary to have a set of estimates for the difference solution. When considering difference schemes for time-dependent boundary-value problems with weak solutions, special attention must be paid to estimates of the difference solution in norms that involve time integrals.

Let us discuss the possibility of obtaining stability conditions in norms that involve time integrals [13]. *A priori* estimates are obtained for two-level difference schemes written in canonical form. The fundamental point is to obtain estimates for the difference solution at midpoints of mesh intervals; the values here are defined by linear interpolation between the values at mesh points.

Theorem 2.3. *For the difference scheme* (2.1) *with operators* $A = A^* > 0$ *and* $B = B^* > 0$, *under conditions* (2.2) *one has the following* a priori *estimate:*

$$\sum_{k=0}^{n} \tau \left\| \frac{1}{2}(y^{k+1} + y^k) \right\|_A^2 \leqslant \frac{1}{2} \left(\left(B - \frac{\tau}{2} A \right) y^0, y^0 \right). \tag{2.5}$$

The estimate (2.5) is obtained for the difference solution at midpoints of temporal mesh intervals where the solution is defined by $y^{k+1/2} = (y^{k+1} + y^k)/2$. Note that stability in a time-integral norm is established under conditions (2.2), which are necessary and sufficient for stability in uniform in time norms; see (2.3). Using a cruder estimate than (2.2), one can derive an *a priori* estimate of stability with respect to the right-hand side in integral in time norms for the difference solution at the mesh-points of time mesh.

2.3. Three-level difference schemes. When computing approximate solutions to time-dependent problems from mathematical physics, three-level difference schemes are often used along with two-level difference schemes. Three-level schemes are typical when considering second-order time-dependent equations such as wave equations. We now present some fundamental stability conditions for three-level operator-difference schemes [14].

Consider the following *canonical form of a three-level difference scheme*:

$$\begin{aligned} B\frac{y^{n+1} - y^{n-1}}{2\tau} &+ R(y^{n+1} - 2y^n + y^{n-1}) + \\ &+ Ay^n = 0, \quad n = 1, 2, \ldots \end{aligned} \tag{2.6}$$

where one is given

$$y^0 = u_0, \quad y^1 = u_1. \tag{2.7}$$

We formulate stability conditions with respect to initial data for constant (independent of n) self-adjoint operators A, B and R.

Theorem 2.4. *Suppose that in the operator-difference scheme* (2.6), (2.7) *the operators* R *and* A *are self-adjoint. Then under the conditions*

$$B \geqslant 0, \quad A > 0, \quad R > \frac{1}{4} A \tag{2.8}$$

the following estimate holds true:

$$\begin{aligned} &\frac{1}{4}\|y^{n+1} + y^n\|_A^2 + \|y^{n+1} - y^n\|_R^2 - \frac{1}{4}\|y^{n+1} - y^n\|_A^2 \leqslant \\ &\frac{1}{4}\|y^n + y^{n-1}\|_A^2 + \|y^n - y^{n-1}\|_R^2 - \frac{1}{4}\|y^n - y^{n-1}\|_A^2, \end{aligned} \tag{2.9}$$

i.e., the operator-difference scheme (2.6), (2.7) *is stable with respect to its initial data.*

The norms in the estimate of stability (2.9) depend on the values of the solution of problem (2.6), (2.7) at both the nth and $(n+1)$st time levels. In some important cases, by restricting the class of difference schemes we can use simpler norms.

An investigation of a multilevel difference scheme can conveniently be based on reducing it to an equivalent two-level scheme. The most profound results have been obtained for two-level difference schemes; in particular, necessary and sufficient conditions of stability have been found. Let us note some developments in this direction for the three-level operator-difference schemes (2.6), (2.7).

Denote by H^2 the direct sum of spaces H : $H^2 = H \oplus H$. For vectors $U = \{u^1, u^2\}$ and $V = \{v^1, v^2\}$ the operations of addition and multiplication in H^2 are defined coordinate-wise and inner product is

$$(U, V) = (u^1, v^1) + (u^2, v^2).$$

On the space H^2 define the operators (operator matrices)

$$\mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where $G_{\alpha\beta}$ is an operator on H. To each self-adjoint positive operator $\mathbf{G}$ assign the Hilbert space $H^2_{\mathbf{G}}$ with the following inner product and norm:

$$(U, V)_{\mathbf{G}} = (\mathbf{G}U, V), \quad \|U\|_{\mathbf{G}} = (\mathbf{G}U, U)^{1/2}.$$

The three-level difference scheme (2.6) can be written in the form of a two-level vector scheme as

$$\mathbf{B}\frac{Y^{n+1} - Y^n}{\tau} + \mathbf{A}Y^n = 0, \quad n = 1, 2, \dots$$

with vectors

$$Y^n = \left\{ \frac{1}{2}(y^n + y^{n-1}),\ y^n - y^{n-1} \right\}.$$

Allowing an increase or decrease of the norm of the solution of the difference problem, let's focus on ϱ-stable schemes, whose stability condition with respect to initial data has the form

$$\|Y^{n+1}\|_{\mathbf{G}} \leqslant \varrho \|Y^n\|_{\mathbf{G}}, \quad \mathbf{G} = \mathbf{G}^* > 0,$$

with $\varrho > 0$.

Here is a typical result in this direction.

Theorem 2.5. *Suppose that in the difference scheme* (2.6), (2.7) *the operators* B, R *and* A *are self-adjoint. Then under the conditions*

$$\frac{\varrho^2 + 1}{2} B + \tau(\varrho^2 - 1) R \geqslant 0,$$

$$\frac{\varrho^2 - 1}{2\tau} B + (\varrho - 1)^2 R + \varrho A > 0,$$

$$\frac{\varrho^2 - 1}{2\tau} B + (\varrho + 1)^2 R - \varrho A > 0,$$

for $\varrho > 0$ *the operator-difference scheme is* ϱ*-stable with respect to initial data.*

Sufficient stability conditions are also obtained for a broad class of three-level operator-difference schemes with non-selfadjoint operators [15].

2.4. Special classes of operator-difference schemes.

When solving time-dependent problems from mathematical physics, special attention is given to schemes with weighting factors when, for example, in (2.1) one has

$$B = E + \tau\sigma A, \quad \sigma = \mathrm{const} \geqslant 0.$$

Adaptive computational algorithms on meshes that are locally refined in space and/or in time, regionally-additive difference schemes (domain decomposition schemes without iteration) that are suited to parallel computers, and difference schemes for equations of mixed type are constructed using schemes with variable weighting factors [16, 17].

Here is a typical result concerning the stability of these operator-difference schemes with respect to the initial data.

Theorem 2.6. *In* (2.1) *let A be a constant and positive operator, and*

$$B = E + \tau GA, \quad GA \neq A^* G^*. \tag{2.10}$$

Then the condition

$$G \geqslant \frac{1}{2} E \tag{2.11}$$

is sufficient for stability of the scheme in H_{A^*A}.

The diagonal operator G in (2.10) corresponds to schemes with variable weighting factors.

Under the restrictions (2.11), stability of the difference scheme (2.1) is established with

$$B = E + \tau AG, \quad GA \neq A^* G^*.$$

This choice of operator B is associated, for example, with a new type of weighting used by schemes with variable weighting factors.

For the self-adjoint operator A, schemes of the type

$$B = E + \tau T^* GT, \quad A = T^* T,$$

have been considered. An extensive study of three-level difference schemes with variable weighting factors is also carried out.

2.5. Strong stability. When studying the well-posedness of initial-boundary value problems for the time-dependent equations of mathematical physics, one pays special attention to the stability of the solution with respect to the initial data and right-hand side. In a more general setting, it is also necessary to require continuous dependence of the solution on perturbations of the operators of the problem (for example, the coefficients of the differential equation). This is called *strong stability* [18, 19]. *A priori* estimates that express the continuous variation of the solution of problem with respect to perturbations of the right-hand side and the operator are obtained under various conditions for stationary problems (operator equations of the first kind). Here we present some new strong stability estimates for two-level operator-difference schemes. These stability estimates correspond to estimates for the differential-operator equation.

Assume that A is a constant self-adjoint positive definite linear operator in H, i.e.,

$$A(t) = A = A^* \geqslant \delta E, \quad \delta = \text{const} > 0.$$

Consider the Cauchy problem for the first-order differential-operator equation

$$\frac{du}{dt} + Au = f(t), \quad 0 < t < T, \tag{2.12}$$

$$u(0) = u_0, \tag{2.13}$$

where f and u_0 are given, and $u(t)$ is the desired function with values in H. Let $\tilde{u}$ denote the solution of the problem with perturbed right-hand side, initial data and operator:

$$\frac{d\tilde{u}}{dt} + \tilde{A}\tilde{u} = \tilde{f}(t), \quad 0 < t < T, \tag{2.14}$$

$$\tilde{u}(0) = \tilde{u}_0. \tag{2.15}$$

Our aim is to estimate the value of the solution perturbation

$$z(t) = \tilde{u}(t) - u(t)$$

in terms of the perturbations of f, u_0 and A.

One makes the same assumptions for the perturbed operator as for the unperturbed operator:

$$\tilde{A}(t) = \tilde{A} = \tilde{A}^* \geqslant \delta E, \quad \delta = \text{const} > 0.$$

A measure of the perturbation is given by the positive constant α in the inequality

$$\|(\tilde{A} - A)v\| \leqslant \alpha \|\tilde{A}v\|. \tag{2.16}$$

Weaker assumptions are associated with the estimate of operator energy under an additional assumption concerning non-negativity of operator $\tilde{A} - A$:

$$0 \leqslant ((\tilde{A} - A)v, v) \leqslant \alpha(Av, v). \tag{2.17}$$

For the solution perturbation, from (2.12), (2.13) and (2.14), (2.15) we get

$$\frac{dz}{dt} + Az = \tilde{f}(t) - f(t) - (\tilde{A} - A)\tilde{u}, \quad 0 < t < T,$$

$$z(0) = \tilde{u}_0 - u_0.$$

Theorem 2.7. *Under the condition* (2.16) *on the perturbation of the operator* A, *the following a priori estimate holds true for the solution perturbation:*

$$\|z(t)\|_A^2 \leqslant \|\tilde{u}_0 - u_0\|_A^2 + \int_0^t \|\tilde{f}(\theta) - f(\theta)\|^2 d\theta + \alpha^2 \Big(\|\tilde{u}_0\|_{\tilde{A}}^2 + \int_0^t \|\tilde{f}(\theta)\|^2 d\theta \Big). \tag{2.18}$$

Under the condition (2.17) *one gets the stability estimate*

$$\|z(t)\|^2 \leqslant \|\tilde{u}_0 - u_0\|^2 + \int_0^t \|\tilde{f}(\theta) - f(\theta)\|_{A^{-1}}^2 d\theta + \alpha^2 \Big(\|\tilde{u}_0\|^2 + \int_0^t \|\tilde{f}(\theta)\|_{\tilde{A}^{-1}}^2 d\theta \Big). \tag{2.19}$$

Now let us present estimates of strong stability only for two-level operator-difference schemes. To the Cauchy problem for the differential-operator equation (2.12), (2.13) assign the following difference scheme with weights:

$$\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1 - \sigma) y^n) = f^n, \quad n = 0, 1, \ldots, \tag{2.20}$$

$$y^0 = u_0. \tag{2.21}$$

The difference scheme for the perturbed problem (2.14), (2.15) is

$$\frac{\tilde{y}^{n+1} - \tilde{y}^n}{\tau} + \tilde{A}(\sigma \tilde{y}^{n+1} + (1 - \sigma) \tilde{y}^n) = \tilde{f}^n, \quad n = 0, 1, \ldots, \tag{2.22}$$

$$\tilde{y}^0 = \tilde{u}_0. \tag{2.23}$$

Similarly to Theorem 2.7, one has strong stability of this difference scheme:

Theorem 2.8. *Consider the solution perturbation* $z^n = \tilde{y}^n - y^n$ *of difference schemes* (2.20), (2.21) *and* (2.22), (2.23). *If* $\sigma \geqslant 0.5$ *and* (2.16) *is satisfied, then the following a priori estimate holds true:*

$$\|z^{n+1}\|_A^2 \leqslant \|\tilde{y}^0 - y^0\|_A^2 + \sum_{k=0}^{n} \tau \|\tilde{f}^k - f^k\|^2 + \alpha^2 \left(\|\tilde{y}^0\|_{\tilde{A}}^2 + \sum_{k=0}^{n} \tau \|\tilde{f}^k\|^2 \right), \tag{2.24}$$

and under conditions (2.17) *one obtains*

$$\|z^{n+1}\|^2 \leqslant \|\tilde{y}^0 - y^0\|^2 + \sum_{k=0}^{n} \tau \|\tilde{f}^k - f^k\|_{A^{-1}}^2 + \alpha^2 \left(\|\tilde{y}^0\|^2 + \sum_{k=0}^{n} \tau \|\tilde{f}^k\|_{\tilde{A}^{-1}}^2 \right). \tag{2.25}$$

The estimates (2.24), (2.25) for the perturbation of the difference solution are complete analogues of the estimates (2.18), (2.19) for the Cauchy problem for the differential-difference equation.

2.6. Projection-difference schemes. In finding approximate solutions of the time-dependent problems of mathematical physics, one often uses an approach based on finite element approximation in space and finite difference approximation in time. For these *projection-difference schemes* it is essential to deal with the question of stability of the approximate solution with respect to the initial data and right-hand side. A general theory of stability of projection-difference schemes is developed in [20,21]. This theory supplements the general theory of stability of operator-difference schemes.

General conditions for the stability and ρ-stability of two-level and three-level projection-difference schemes are stated. In particular, sufficient conditions for stability are obtained for schemes with weights and estimates of stability with respect to right-hand side are given. As an example, here we consider stability conditions with respect to to initial data for three-level projection-difference schemes.

Find an approximate solution of an initial-boundary value problem in the domain Ω with boundary $\partial\Omega$. Let $(\cdot,\cdot)$ be an inner product and $\|\cdot\|$ a norm in $L_2(\Omega)$, i.e.,

$$(u, v) = \int_{\Omega} u(x)v(x)dx, \quad \|u\| = (u, u)^{1/2}.$$

To each symmetric bilinear positive definite form $d(u, v)$ such that

$$d(u, v) = d(v, u), \quad d(u, u) \geqslant \delta\|u\|^2, \quad \delta > 0,$$

assign the Hilbert space H_d with inner product

$$(u, v)_d = d(u, v)$$

and norm

$$\|u\|_d = (d(u, u))^{1/2}.$$

Denote by $\mathcal{V}^h$ the finite-dimensional space of finite elements. The approximate solution at the time level $t = t^n$ is denoted by y^n ($y^n \in \mathcal{V}^h$).

In accordance with the general theory of stability of difference schemes, we write three-level projection-difference schemes in the following *canonical form*:

$$b_n \left(\frac{y^{n+1} - y^{n-1}}{2\tau}, v \right) + r_n(y^{n+1} - 2y^n + y^{n-1}, v) + a_n(y^n, v) = (f^n, v), \tag{2.26}$$

$$\forall v \in \mathcal{V}^h, \quad n = 1, 2, \ldots,$$

where $b_n(\cdot,\cdot)$, $r_n(\cdot,\cdot)$, $a_n(\cdot,\cdot)$ are some real bilinear forms. An approximate solution y^n, $n = 2, 3, \ldots$, is found from (2.26) provided that y^0 and y^1 are given.

Without loss of generality it can be assumed that the bilinear forms in (2.26) are constant, i.e., independent of n. Consider the projection-difference scheme

$$b\left(\frac{y^{n+1} - y^{n-1}}{2\tau}, v\right) + r(y^{n+1} - 2y^n + y^{n-1}, v) + a(y^n, v) = 0, \tag{2.27}$$

$$\forall v \in \mathcal{V}^h, \quad n = 1, 2, \ldots,$$

i.e., we shall study stability with respect to the initial data.

For three-level difference schemes, stability is established in suitable norms. Let us present an *a priori* estimate for scheme (2.27) that expresses stability with respect to the initial data.

For the symmetric bilinear forms $r(\cdot,\cdot)$ ($r(u,v) = r(v,u)$) and $a(\cdot,\cdot)$ and a non-negative form $b(\cdot,\cdot)$ with $b(v,v) \geqslant 0$, one has the inequality

$$\mathcal{E}^{n+1} \leqslant \mathcal{E}^n, \tag{2.28}$$

where

$$\mathcal{E}^{n+1} = \frac{1}{4}a(y^{n+1} + y^n, y^{n+1} + y^n) + r(y^{n+1} - y^n, y^{n+1} - y^n) - \frac{1}{4}a(y^{n+1} - y^n, y^{n+1} - y^n). \tag{2.29}$$

Under some restrictions the quantity $\mathcal{E}_n$ defined by (2.29) determines a norm and thus inequality (2.28) supplies stability with respect to initial data for the projection-difference scheme. Moreover, the following statement is valid.

Theorem 2.9. *Suppose that in the projection-difference scheme* (2.27) *the bilinear forms* $r(\cdot,\cdot)$ *and* $a(\cdot,\cdot)$ *are symmetric. Then under the conditions*

$$b(v,v) \geqslant 0, \quad a(v,v) > 0, \quad r(v,v) - \frac{1}{4}a(v,v) > 0, \quad \forall v \in \mathcal{V}^h, \tag{2.30}$$

the following a priori estimate holds:

$$\begin{gathered} \frac{1}{4}\|y^{n+1} + y^n\|_a^2 + \|y^{n+1} - y^n\|_r^2 - \frac{1}{4}\|y^{n+1} - y^n\|_a^2 \leqslant \\ \frac{1}{4}\|y^n + y^{n-1}\|_a^2 + \|y^n - y^{n-1}\|_r^2 - \frac{1}{4}\|y^n - y^{n-1}\|_a^2, \end{gathered} \tag{2.31}$$

i.e., the projection-difference scheme (2.27) *is stable with respect to its initial data.*

The norms in the estimate of stability (2.31) depend on the values of the solution of problem (2.6), (2.7) at both the nth and $(n+1)$st time levels. In some important cases, by restricting the class of projection-difference schemes we can use simpler norms. Similarly, the conditions for ϱ-stability are stated for projection-difference schemes. Here the basis is Theorem 5 where stability conditions for operator-difference schemes are presented.

2.7. Ill-posed time-dependent problems. Many applied problems formulated as *inverse problems of mathematical physics* belong to the class of problems that are ill-posed in the classical sense. For their approximate solution *regularization methods* (A.N.Tikhonov et al.) are widely used. For inverse problems for time-dependent equations the generalized inverse method (R.Lattes and J.-L.Lions) is also used. Here we present some typical results on the construction of difference schemes for unstable problems; these are based on the regularization principle for difference schemes [22, 23].

Consider the model inverse problem for the first-order time-dependent equation. In a rectangle Ω we shall find the solution of the parabolic equation

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k(\mathbf{x})\frac{\partial u}{\partial x_i}\right) = 0, \quad \mathbf{x} \in \Omega, \quad 0 < t < T, \tag{2.32}$$

which differs from the standard parabolic equation only in the sign of the derivatives in space. This corresponds to replacing t by $-t$ (equation with inverse time). Boundary and initial conditions are taken in the form

$$u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad 0 < t < T, \tag{2.33}$$

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.34}$$

The inverse problem (2.32), (2.33), (2.34) is ill-posed as it is unstable with respect to relatively small perturbations of the initial data. For the class of bounded solutions, stability follows from the estimate

$$\|u(\mathbf{x}, t)\| \leqslant \|u(\mathbf{x}, 0)\|^{1-t/T} \|u(\mathbf{x}, T)\|^{t/T}.$$

To the differential problem corresponds a differential-difference problem that is discretized in space. For simplicity suppose that in the domain Ω we have a mesh that is uniform in each coordinate direction with mesh-sizes h_i, $i = 1, 2$. Let ω be the set of interior mesh-points.

On the set of mesh functions $y(\mathbf{x})$ such that $y(\mathbf{x}) = 0$, $\mathbf{x} \neq \omega$, define the mesh operator Λ by the relation

$$\Lambda y = -\sum_{i=1}^{2} (a_i y_{\bar{x}_i})_{x_i}, \tag{2.35}$$

putting, for example,

$$a_1(\mathbf{x}) = k(x_1 - 0.5h_1, x_2), \quad a_2(\mathbf{x}) = k(x_1, x_2 - 0.5h_2).$$

On the mesh Hilbert space H the inner product and norm are defined by

$$(y, w) = \sum_{\mathbf{x} \in \omega} y(\mathbf{x}) w(\mathbf{x}) h_1 h_2, \quad \|y\| = (y, y)^{1/2}.$$

In H we have $\Lambda = \Lambda^* > 0$.

To the differential problem (2.32), (2.33), (2.34) corresponds the Cauchy problem for the differential-operator equation

$$\frac{dy}{dt} - \Lambda y = 0, \quad \mathbf{x} \in \omega, \quad 0 < t < T, \tag{2.36}$$

where

$$y(0) = u_0, \quad \mathbf{x} \in \omega, \tag{2.37}$$

is given.

Let us construct unconditionally stable difference schemes for (2.36), (2.37) using the regularization principle for difference schemes. We start from the simplest explicit difference scheme

$$\frac{y^{n+1} - y^n}{\tau} - \Lambda y^n = 0, \quad \mathbf{x} \in \omega, \quad n = 0, 1, \dots, N-1, \tag{2.38}$$

supplemented by the initial condition

$$y^0 = u_0, \quad \mathbf{x} \in \omega, \tag{2.39}$$

where $N\tau = T$.

In accordance with the regularization principle, write the scheme (2.38) in the canonical form

$$B\frac{y^{n+1} - y^n}{\tau} + Ay^n = 0, \quad n = 0, 1, \dots \tag{2.40}$$

with

$$A = -\Lambda, \quad B = E, \tag{2.41}$$

i.e., $A = A^* < 0$.

The explicit scheme (2.38) is ϱ-stable in H with

$$\varrho = 1 + M\tau, \tag{2.42}$$

where, taking into account the upper bound of the problem operator $\Lambda \leqslant ME$, one has $M = O(h_1^{-2} + h_2^{-2})$. This result follows from the ϱ-stability condition

$$\frac{1-\varrho}{\tau} B \leqslant A \leqslant \frac{1+\varrho}{\tau} B \tag{2.43}$$

for the scheme (2.40), (2.41). From our assumptions $B > 0$ and $A < 0$, for $\varrho > 1$ the right-hand side of two-sided operator inequality (2.43) is obviously valid for all $\tau > 0$. The left-hand side of (2.43) becomes $(\varrho - 1)/\tau E \geqslant \Lambda$ and is valid under the choice of ϱ given by (2.42).

When approximate solving ill-posed problems the choice of regularization parameter must correspond to the amount of error in the input data. Here we merely construct stable computational algorithms for ill-posed time-dependent problems and study the influence of the regularization parameter only on the stability of the corresponding difference scheme. For the given regularization parameter α is stated the minimum value of ϱ according to (2.42).

Starting from the explicit scheme (2.38) for problem (2.36), (2.37), let us write the regularized scheme in the canonical form (2.40) with

$$A = -\Lambda, \quad B = E + \alpha R. \tag{2.44}$$

Theorem 2.10. *The regularized scheme* (2.40), (2.44) *is ϱ-stable in H_B with*

$$\varrho = 1 + \frac{\tau}{\alpha} \tag{2.45}$$

under the choice of regularizing operator

$$R = \Lambda, \tag{2.46}$$

and

$$\varrho = 1 + \alpha^{-1/2}\frac{\tau}{2} \tag{2.47}$$

if

$$R = \Lambda^2. \tag{2.48}$$

To prove this result it is sufficient to check only the left-hand side of the two-sided inequality (2.43), which for (2.44) takes the form

$$\frac{\varrho - 1}{\tau}(E + \alpha R) \geqslant \Lambda. \tag{2.49}$$

With $R = \Lambda$ and ϱ in the form of (2.45), inequality (2.49) is satisfied.

For $R = \Lambda^2$, inequality (2.49) is handled as follows:

$$E + \alpha\Lambda^2 - \frac{\tau}{\varrho - 1} = \left(\alpha^{1/2}\Lambda - \frac{\tau}{2(\varrho - 1)}\alpha^{-1/2}E\right)^2 + \left(1 - \frac{\tau^2}{4(\varrho - 1)^2}\alpha^{-1}\right)E \geqslant 0.$$

This inequality is true when ϱ is chosen in the form (2.47).

The regularized difference schemes constructed can be combined with any (standard or non-standard) variants of the generalized inverse method. In the same way other regularized difference schemes are obtained. This enables us to examine second-order time-dependent problems, problems with non-selfadjoint operators, additive schemes for multi-dimensional inverse problems, etc.

3. Additive difference schemes

For the efficient numerical solution of multi-dimensional time-dependent problems of mathematical physics, there are widely used *additive schemes* (splitting schemes — J. Douglas, D.W. Peaceman, H.H. Rachford, N.N. Yanenko, G.I. Marchuk et al.). Additive operator-difference schemes are based on a reduction to a sequence of simpler problems. In this way economical operator-difference schemes are constructed; these schemes are associated with a splitting of the spatial variables. Individual operator summands can have a separate meaning in the applied mathematical model and then we say we have splitting with respect to physical processes. In regional-additive schemes, the subproblems are based on the allocation of part of the computational domain (domain decomposition schemes).

Under the general conditions of splitting the problem operator into a sum of non-commutative non-selfadjoint operators, additive difference schemes can easily be constructed for two-component splitting. In this case, for the first-order time-dependent equation problem, under mild conditions the classical alternating-direction schemes, factored schemes and predictor-corrector schemes are unconditionally stable. A more complicated case is multi-component splitting (in three or more operators). In this case the most interesting results are obtained using the concept of total approximation introduced by A. A. Samarskii.

When passing from one layer to another, the original problem is divided into a sequence of subproblems where each of these subproblems is in general not a consistent approximation of the original problem. In this way unconditionally stable schemes of component-wise splitting (locally one-dimensional schemes with splitting with respect to the spatial variables) are constructed. For modern parallel computers the additively-averaged schemes of component-wise splitting deserve special attention.

Recently, a new class of operator-difference splitting scheme — vector-additive schemes — has been developed. Here the original scalar problem for one unknown function is reduced to the problem for a vector, where each component of this vector can be considered as a solution of the problem. In this way for time-dependent equations of the first and second order one obtains schemes of complete approximation under general multi-component splitting.

New additive difference schemes for first-order differential-operator equations have been constructed for the general case of additive splitting with an arbitrary number of pairwise non-commutative operator summands. The construction of unconditional stable schemes is based on regularization of the simplest explicit two-level scheme by small multiplicative perturbations of each splitting operator. For such schemes we have *a priori* estimates, with respect to the initial data and right-hand side, for the difference solution without having to consider intermediate problems. Hence we can obtain corresponding convergence estimates for specific splitting schemes. As a meaningful example one can consider schemes that use splitting with respect to spatial variables for multidimensional time-dependent convection-diffusion problem.

Unconditionally stable difference schemes are constructed for second-order time-dependent equations, for example, for the boundary-value problem for the second-order multidimensional hyperbolic equation. Complete approximation schemes are obtained using the regularization principle for difference schemes.

The class of additive difference schemes deserves special attention. This class is associated with a decomposition (splitting) of the computational domain into separate subdomains. Regionally-additive schemes obtained in this way are suited to computers with modern parallel architecture. The construction of domain decomposition schemes and the study of their accuracy when solving time-dependent problems from mathematical physics have been considered.

3.1. Statement of the problem. Consider the Cauchy problem for the first-order time-dependent equation given in the mesh Hilbert space. To solve this problem approximately we use a standard two-level scheme with weights. One has corresponding stability estimates with respect to the initial data and right-hand side; these estimates serve as a guide when constructing additive operator-difference schemes.

The function $y(t) \in H$ satisfies the equation

$$\frac{dy}{dt} + \Lambda y = f(t), \quad 0 < t \leqslant T, \tag{3.1}$$

and the initial condition

$$y(0) = u_0. \tag{3.2}$$

Suppose that the operator Λ is positive, non-selfadjoint and time-dependent, i.e., $\Lambda(t) \neq \Lambda^*(t) > 0$. Under these conditions the solution of problem (3.1), (3.2) satisfies the estimate

$$\|y(0)\| \leqslant \|u_0\| + \int\limits_0^t \|f(s)\| ds, \tag{3.3}$$

expressing stability with respect to the initial data and right-hand side.

For (3.1), (3.2) let us write the two-level difference scheme with a constant weight σ:

$$\frac{y^{n+1} - y^n}{\tau} + \Lambda(t^n)(\sigma y^{n+1} + (1-\sigma)y^n) = f^n, \quad n = 0, 1, \dots, \tag{3.4}$$

supplemented by the initial condition

$$y^0 = u_0. \tag{3.5}$$

To study the stability of this scheme we shall use the general theory of stability (well-posedness) of operator-difference schemes in finite-dimensional Hilbert spaces. This theory is based on obtaining necessary and sufficient stability conditions in the form of operator inequalities. First, write (3.4) in the canonical form

$$B\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n, \quad n = 0, 1, \dots \tag{3.6}$$

In this case, mesh operators $B = B(t^n)$ and $A = A(t^n)$ are of the form

$$B = E + \sigma\tau\Lambda(t^n), \quad A = \Lambda(t^n). \tag{3.7}$$

By virtue of (3.7) we have

$$B = E + \sigma\tau A, \quad A \neq A^*. \tag{3.8}$$

For the difference scheme with weights (3.6), (3.8) to be stable in H with $A > 0$ it is necessary and sufficient to have the operator inequality

$$A + \left(\sigma - \frac{1}{2}\right)\tau A^* A \geqslant 0. \tag{3.9}$$

Theorem 3.1. *If $\sigma \geqslant 0.5$ and $\tau > 0$, then for the solution of problem* (3.4), (3.5) *one has the a priori estimate*

$$\|y^{n+1}\| \leqslant \|u_0\| + \sum_{k=0}^{n} \tau \|f^k\|. \tag{3.10}$$

The implementation of the implicit scheme (3.4),(3.5) is related to inverting the operator $B = E + \sigma\tau\Lambda$. When the operator Λ is divided into a sum of separate (simpler) operators we can construct implicit schemes where passing to the next time-level does not depend on solving the problem with the operator Λ, but instead on solving a sequence of problems with separate operator summands. Among such additive schemes we note the classical unconditionally stable schemes with component-wise splitting (locally one-dimensional schemes) that are suited to general multicomponent splitting.

Suppose that for the operator Λ one has the additive representation

$$\Lambda = \sum_{\alpha=1}^{p} \Lambda_\alpha, \quad \Lambda_\alpha \geqslant 0, \quad \alpha = 1, 2, \dots, p. \tag{3.11}$$

Additive difference schemes are constructed using the representation (3.11), where passing from the time level t^n to the level $t^{n+1} = t^n + \tau$ depends on the solution of the problem for each separate operator $\Lambda_\alpha, \alpha = 1, 2, \dots, p$ from the additive splitting (3.11), i.e, the problem is divided into p subproblems.

These examples are classical schemes of component-wise splitting (locally one-dimensional schemes). Let us note also additively-averaged schemes with component-wise splitting. These schemes are constructed not only for time-dependent first-order equations but also for second-order equations. For a wide class of time-dependent problems vector-additive difference schemes are often used. New opportunities are provided for the construction of unconditionally stable factored schemes.

3.2. Schemes of component-wise splitting. Additive difference schemes with splitting into three and more pairwise noncommutative operators are constructed in [25,26] based on the new concept of *total approximation* (schemes of component-wise splitting, locally one-dimensional schemes). For the problem (3.1),(3.2),(3.11) we have

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + \Lambda_\alpha(\sigma_\alpha y^{n+\alpha/p} + (1-\sigma_\alpha) y^{n+(\alpha-1)/p}) = f_\alpha^n, \tag{3.12}$$

$$\alpha = 1,2,\dots,p, \quad n = 0,1,\dots,$$

where

$$f^n = \sum_{\alpha=1}^{p} f_\alpha^n.$$

If $\sigma_\alpha \geqslant 0.5$, then the scheme with component-wise splitting (3.12) is unconditionally stable. Let us present an *a priori* stability estimate with respect to the initial data and right-hand side. For right-hand sides f_α^n, $\alpha = 1,2,\dots,p$, use the special presentation

$$f_\alpha^n = \overset{\circ}{f}{}_\alpha^n + \overset{*}{f}{}_\alpha^n, \quad \alpha = 1,2,\dots,p, \quad \sum_{\alpha=1}^{p} \overset{\circ}{f}{}_\alpha^n = 0. \tag{3.13}$$

This form of the right-hand side is of fundamental importance when considering the error of the additive scheme. For the component-wise splitting scheme one has the following result.

Theorem 3.2. *If* $0.5 \leqslant \sigma_\alpha \leqslant 2$, $\alpha = 1,2,\dots,p$ *and* $\tau > 0$, *then the solution of the problem* (3.5), (3.12), (3.13) *satisfies the a priori estimate*

$$\|y^{n+1}\| \leqslant \|u_0\| + \sum_{k=0}^{n} \tau \sum_{\alpha=1}^{p} \left(\| \overset{*}{f}{}_\alpha^k \| + \tau \|\Lambda_\alpha \sum_{\beta=\alpha}^{p} \overset{\circ}{f}{}_\beta^k \| \right). \tag{3.14}$$

Focusing on modern parallel computers, we can use additively-averaged schemes of component-wise splitting [27]. In this case, the passage to each new time level is implemented in the following way:

$$\frac{y_\alpha^{n+1} - y^n}{p\tau} + \Lambda_\alpha(\sigma_\alpha y_\alpha^{n+1} + (1-\sigma_\alpha) y^n) = f_\alpha^n, \tag{3.15}$$

$$\alpha = 1,2,\dots,p, \quad n = 0,1,\dots.,$$

$$y^{n+1} = \frac{1}{p} \sum_{\alpha=1}^{p} y_\alpha^{n+1}.$$

Stability conditions for these schemes are the same as for standard component-wise splitting schemes. Similarly to Theorem 12 one has the following statement.

Theorem 3.3. *If* $\sigma_\alpha \geqslant 0.5$, $\alpha = 1,2,\dots,p$ *and* $\tau > 0$, *then the solution of problem* (3.5), (3.13), (3.15) *satisfies the a priori estimate*

$$\|y^{n+1}\| \leqslant \|u_0\| + \sum_{k=0}^{n} \tau \sum_{\alpha=1}^{p} \left(\| \overset{*}{f}{}_\alpha^k \| + p\tau\sigma_\alpha \|\Lambda_\alpha \overset{\circ}{f}{}_\alpha^k \| \right). \tag{3.16}$$

The potential advantage of additively-averaged schemes (3.15) is the opportunity to compute in parallel the auxiliary mesh functions $y_\alpha^{n+1}, \alpha = 1, 2, \dots, p$.

The above stability estimates (3.14) and (3.16) provide the basis for a study of the accuracy of the splitting schemes under consideration. The error of the solution satisfies a problem of the form (3.12), (3.15). The fundamental point is that when studying component-wise splitting schemes and additively-averaged schemes the stability estimates depend essentially on the splitting (3.13), i.e., how the individual components appear. In fact, this means that the accuracy of such additive schemes depends on the structure of the intermediate problems, their approximation, etc.

On the other hand, the intermediate problems (auxiliary mesh quantities $y^{n+\alpha/p}, y_\alpha^n, \alpha = 1, 2, \dots, p$) do not have any independent meaning. Ideally, we would like to do without them, i.e., to construct schemes that do not involve the concept of total approximation. Below we present some opportunities in this very promising direction [28].

3.3. Regularized additive schemes. The construction of difference schemes of a desired quality can be based on a methodological principle — the regularization principle for difference schemes. Here we use this principle when constructing unconditionally stable additive schemes.

First, consider the application of the regularization principle to the construction of stable schemes for the problem (3.1),(3.2). As a generator it is natural to take the explicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + \Lambda y^n = f^n, \quad n = 0, 1, \dots \tag{3.17}$$

with given y^0. It can be written in the canonical form (3.6) with

$$B = E, \quad A = \Lambda.$$

A necessary and sufficient stability condition in H_A for the scheme (3.6) with $A(t) = A,\ A = A^* > 0$ is the inequality

$$B \geqslant \frac{\tau}{2} A. \tag{3.18}$$

To apply this criterion let us reduce the scheme (3.17) to a more convenient form:

$$\Lambda^{-1} \frac{y^{n+1} - y^n}{\tau} + y^n = \Lambda^{-1} f^n, \quad n = 0, 1, \dots \tag{3.19}$$

This scheme is written in the canonical form with

$$B = \Lambda^{-1}, \quad A = E$$

and then the criterion (3.18) can be applied.

According to (3.18) to improve the stability condition we can focus on perturbing (increase the energy) the operator B or perturbing (decrease the energy) of the operator A.

Define the operators of the regularized scheme by

$$B = \Lambda^{-1}(E + \sigma\tau\Lambda), \quad A = E.$$

This is consistent with the fact that we pass from the scheme (3.19) to standard scheme with weights (3.4) as discussed above.

Let us rewrite the scheme with weights (3.4) in the somewhat different form

$$\frac{y^{n+1} - y^n}{\tau} + (E + \sigma\tau\Lambda)^{-1}\Lambda y^n = (E + \sigma\tau\Lambda)^{-1} f^n, \quad n = 0, 1, \ldots \tag{3.20}$$

This scheme can be interpreted as a regularized scheme obtained by multiplicative perturbation of A.

Based on the scheme (3.19), we put

$$B = \Lambda^{-1}, \quad A = (E + \sigma\tau\Lambda)^{-1}.$$

This leads to the scheme

$$\frac{y^{n+1} - y^n}{\tau} + (E + \sigma\tau\Lambda)^{-1}\Lambda y^n = f^n, \quad n = 0, 1, \ldots, , \tag{3.21}$$

that differs from the scheme with weights (3.20) only in its right-hand side. For the scheme (3.21) the *a priori* estimate (3.10) holds true.

The principal feature of the scheme (3.21) is that it is constructed based on a scheme with multiplicative regularization of the problem operator. In this new methodological basis one can also construct additive schemes.

As a generator in the construction of unconditionally stable additive schemes we consider the simplest explicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} \Lambda_\alpha y^n = f^n, \quad n = 0, 1, \ldots \tag{3.22}$$

Similarly to (3.21), we construct additive schemes based on a perturbation of each individual operator summand from the additive presentation (3.11):

$$\frac{y^{n+1} - y^n}{\tau} + \sum_{\alpha=1}^{p} (E + \sigma\tau\Lambda_\alpha)^{-1}\Lambda_\alpha y^n = f^n, \quad n = 0, 1, \ldots \tag{3.23}$$

If $\sigma_\alpha \geqslant p/2,\ \alpha = 1, 2, \ldots, p$ and $\tau > 0$, then a priori estimate (3.10) holds true for the solution of problem (3.5), (3.23).

The regularized scheme (3.23) is closely related to the additively-averaged scheme of total approximation. To demonstrate this, we introduce fictitious mesh values $y_\alpha^{n+1}, \alpha = 1, 2, \ldots, p$, which have no independent meaning. Let us implement the scheme (3.23) in the following form:

$$\frac{y_\alpha^{n+1} - y^n}{p\tau} + (E + \sigma\tau\Lambda_\alpha)^{-1}\Lambda_\alpha y^n = f_\alpha^n,$$

$$\alpha = 1, 2, \ldots, p, \quad n = 0, 1, \ldots,$$

$$y^{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} y_\alpha^{n+1}.$$

Thus we again come to an additively-averaged scheme, which is now constructed without using the concept of total approximation. Unlike the scheme (3.15) described above, it is connected with different approximations of right-hand sides.

A comparison of the stability estimates for the schemes (3.15) and (3.23) is significantly in favour of the regularized scheme. That is, for the scheme (3.23) the estimate (3.10) does not depend on a splitting of the right-hand side, i.e., the choice of function f_α^n, $\alpha = 1, 2, \ldots, p$. This feature often creates difficulties in the study of additive difference schemes of total approximation.

3.4. Additive schemes for the second-order equations. Certain difficulties arise when trying to construct splitting schemes for second-order time-dependent equations [29]. Here we consider the Cauchy problem for the second-order time-dependent equation with a self-adjoint operator acting in the mesh real Hilbert space. The study is performed from the perspective of stability theory for operator-difference schemes with the general methodological principle of constructing difference schemes of a desired quality — the regularization principle. A typical example is the multi-dimensional wave equation. For this equation one can construct schemes with splitting of the spatial variables (multicomponent schemes of alternating directions) and regionally-additive schemes (domain decomposition schemes).

We shall find a solution $u(t) \in H$ of the Cauchy problem for the second-order time-dependent equation, where

$$\frac{d^2u}{dt^2} + \Lambda u = f(t), \quad 0 < t \leqslant T, \tag{3.24}$$

$$u(0) = u_0, \tag{3.25}$$

$$\frac{du}{dt}(0) = u_1. \tag{3.26}$$

We confine ourselves to the simplest case of a positive self-adjoint and time-independent operator Λ, i.e., $\Lambda(t) = \Lambda = \Lambda^* > 0$.

For the problem (3.24)–(3.26) the following a priori estimate is valid:

$$\|u(t)\|_* \leqslant \|u_0\|_\Lambda + \|u_1\| + \int_0^t \|f(s)\| ds, \tag{3.27}$$

where

$$\|u(t)\|_*^2 \equiv \|u\|_\Lambda^2 + \left\|\frac{du}{dt}\right\|^2.$$

Let us present a stability estimate with respect to the initial data and right-hand side for the simple scheme with weights for the problem (3.24)–(3.26). For its approximate solution we use the second-order accurate scheme

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + \Lambda(\sigma y^{n+1} + (1 - 2\sigma)y^n + \sigma y^{n-1}) = f_n, \quad n = 1, 2, \ldots \tag{3.28}$$

with given y^0, y^1.

For the difference scheme (3.28) one has the a priori estimate

$$\|y_{n+1}\|_* \leqslant \|y_n\|_* + \tau \|f_n\|, \tag{3.29}$$

where now

$$\|y_{n+1}\|_*^2 \equiv \left\|\frac{y_{n+1} - y_n}{\tau}\right\|_{E + (\sigma - \frac{1}{4})\tau^2 \Lambda}^2 + \left\|\frac{y_{n+1} + y_n}{2}\right\|_\Lambda^2.$$

Estimate (3.29) agrees with estimate (3.27) for the solution of the differential problem and for $\sigma \geqslant 0.25$ guarantees unconditionally stability with respect to the initial data and right-hand side of the difference scheme with weights (3.28).

To construct regularized additive schemes of complete approximation based on the regularization principle for difference schemes, it is natural to take as generator for the problem (3.24)–(3.26) the following explicit scheme:

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + \Lambda y^n = f^n, \quad n = 1, 2, \ldots \tag{3.30}$$

Multiplicative regularization leads to the scheme

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + (E + \sigma\tau^2\Lambda)^{-1}\Lambda y^n = f^n, \quad n = 1, 2, \ldots, \tag{3.31}$$

which is similar to the simple scheme with weights for the second-order time-dependent equation. Checking the stability conditions leads to the conclusion that for $\sigma \geqslant 0.25$ the scheme (3.31) is stable.

A multicomponent analogue of the scheme (3.31) is the following additive scheme:

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + \sum_{\alpha=1}^{p}(E + \sigma\tau^2\Lambda_\alpha)^{-1}\Lambda_\alpha y^n = f^n, \quad n = 1, 2, \ldots \tag{3.32}$$

Theorem 3.4. *For $\sigma_\alpha \geqslant p/4$, $\alpha = 1, 2, \ldots, p$, the additive difference scheme* (3.32) *for the problem* (3.24)–(3.26) *is unconditionally stable.*

The scheme (3.32) can be implemented as

$$(E + \sigma\tau^2\Lambda_\alpha)\frac{y_\alpha^{n+1} - 2y^n + y^{n-1}}{p\tau^2} + \Lambda_\alpha y^n = \frac{1}{p}(E + \sigma\tau^2\Lambda_\alpha)f^n,$$

$$\alpha = 1, 2, \ldots, p, \quad n = 1, 2, \ldots,$$

$$y^{n+1} = \frac{1}{p}\sum_{\alpha=1}^{p} y_\alpha^{n+1}.$$

Thus we come to a special additive-averaged scheme.

4. Iterative methods for the solution of mesh equations

In the work of A. A. Samarskii the theory or iterative methods is developed from a single point of view. The original object of research is the operator equation of the first kind

$$A\,u = f, \tag{4.1}$$

where A is a linear operator acting on the linear finite-dimensional Hilbert space H, so $A : H \to H$, and $f \in H$.

The main results of the theory of stability of operator-difference schemes are naturally applied when developing the general theory of iterative methods for the solution of systems of linear equations [30]. Some characteristic features of this theory are:

• the interpretation of iterative schemes as operator-difference schemes with the operator defined in a Hilbert space;

• a refusal to examine the structure of the operators of the scheme. The theory uses a minimum of general information about the functional nature of the operators;

• a constructive theory that ultimately indicates the general principles of constructing optimal iterative methods depending on the a priori information available.

The principal results of A. A. Samarskii concerning iterative methods deal with the iterative solution of problems with non-selfadjoint operators. A. A. Samarskii proposed an alternately-triangular iterative method which can be considered as an additive operator-difference scheme for a specific two-component splitting for solving time-dependent problems. Let us also note the general approach to constructing iterative methods with aggregation of equations and unknown values; this approach is the foundation for establishing, in particular, the convergence of block iterative methods based on the Schwarz alternating method.

4.1. Iterative methods for problems with a non-selfadjoint operator. In the general theory of iterative methods for the solution of equation (4.1) iterative schemes of two types were studied: two-level and three-level. Any two-level iterative scheme connecting two iterative approximations can be written in the canonical form

$$B_{k+1}\,\frac{y_{k+1}-y_k}{\tau_{k+1}} + A\,y_k = f, \quad k \geqslant 0, \quad y_0 \in H. \tag{4.2}$$

Schemes of the following form are often considered among three-level schemes:

$$B_{k+1}\,y_{k+1} = \alpha_{k+1}(B_{k+1} - \tau_{k+1}A)y_k + (1-\alpha_{k+1})B_{k+1}\,y_{k-1} + \alpha_{k+1}\tau_{k+1}f, \quad k \geqslant 1,$$

$$B_1\,y_1 = (B_1 - \tau_1 A)y_0 + \tau_1 f, \quad y_0 \in H, \tag{4.3}$$

where y_k is the kth iterative approximation, the $\{B_k\}$ are a sequence of linear invertible operators acting on H, and $\{\tau_k\}$ and $\{\alpha_k\}$ are sequences of iteration parameters.

In constructing an iterative method, the operator A is fixed and parameters $\{\tau_k\}$, $\{\alpha_k\}$ and operators $\{B_k\}$ should be chosen to satisfy the condition of using a minimum of arithmetic operations to solve the problem (4.1) with a given accuracy $\epsilon > 0$. As the termination criterion one often uses the condition of smallness of the error $z_n = y_n - u$ in the energy space H_D:

$$\|z_n\|_D \leqslant \epsilon\,\|z_0\|_D, \quad \|y\|_D = (Dy,\, y)^{1/2},$$

where $D = D^* > 0$ is an operator acting in H.

For the core group of methods, the operator B_k is independent of number of iterations k, i.e., $B_k \equiv B$. This enables us to formulate conditions for the convergence of the corresponding iterative method, give optimal choices for the iteration parameters and derive an *a priori* estimate for the number of required iterations in terms of constants stating the energy equivalence of the operators A and B. There are two cases in the theory of iterative methods: first, when the operator $DB^{-1}A$ is selfadjoint in the space H, and second, when this does not hold true.

Iterative methods can be divided into two classes. The first includes methods that use a priori numerical information about the operators of the iterative scheme to choose its parameters. The second class consists of methods of variational type when the iteration parameters are chosen from the condition of minimization of a functional related to the original problem. In iteration methods the operators B_k are either fixed constructively and explicitly or constructed as a result of an auxiliary computational process when their explicit form does not matter.

In the case of a self-adjoint operator the following result is known [30].

Theorem 4.1. *Let the operator $DB^{-1}A$ be self-adjoint in H and let there be given constants γ_1 and γ_2 in the inequalities*

$$\gamma_1 D \leqslant DB^{-1}A \leqslant \gamma_2 D, \quad \gamma_1 > 0. \tag{4.4}$$

Method with a Chebyshev set of parameters: the scheme (4.2) *with*

$$\tau_k = \frac{\tau_0}{1+\rho_0\mu_k}, \quad k = 1, 2, \dots, n,$$

or the scheme (4.3) *with*

$$\tau_k \equiv \tau_0, \quad \alpha_{k+1} = \frac{4}{4-\rho_0^2\alpha_k}, \quad k = 1, 2, \dots, n, \quad \alpha_1 = 2,$$

converge in H_D, and the error z_n satisfies the estimate

$$\|z_n\|_D \leqslant q_n \, \|z_0\|_D.$$

For the number of iterations one has $n \geqslant n_0(\epsilon)$, where $n_0(\epsilon) = \dfrac{\ln 0.5\epsilon}{\ln \rho_1}$. Here

$$q_n = \frac{2\rho_1^n}{1+\rho_1^{2n}}, \quad \tau_0 = \frac{2}{\gamma_1+\gamma_2}, \quad \rho_0 = \frac{1-\xi}{1+\xi}, \quad \rho_1 = \frac{1-\xi^{1/2}}{1+\xi^{1/2}}, \quad \xi = \frac{\gamma_1}{\gamma_2},$$

$$\mu_k \in \mathfrak{M}_n = \{\cos\frac{2i-1}{2n}\pi, \quad i = 1, 2, \dots, n\} \quad k = 1, 2, \dots, n.$$

For a two-level scheme the set of zeros of the Chebyshev polynomials $\mathfrak{M}_n$ has to be sorted in a special way to ensure the computational stability of the algorithm.

If the operators A and B are self-adjoint in H and positive definite and if D is chosen as one of the operators A, B or $AB^{-1}A$, then the inequalities (4.4) are equivalent to the simpler condition

$$\gamma_1 (By, y) \leqslant (Ay, y) \leqslant \gamma_2 (By, y), \quad \gamma_1 > 0. \tag{4.5}$$

For non-selfadjoint operators the following results were obtained [31, 32].

Theorem 4.2. *Suppose that the constants γ_1 and γ_2 in the following inequalities are given:*

$$\gamma_1 D \leqslant DB^{-1}A, \quad \gamma_1 > 0,$$

$$(DB^{-1}Ay, B^{-1}Ay) \leqslant \gamma_2 (DB^{-1}Ay, y).$$

Then the method of iteration (4.2) *with $\tau = 1/\gamma_2$ converges in H_D, and for the error z_n one has*

$$\|z_n\|_D \leqslant \rho^n \, \|z_0\|_D, \quad \rho = (1-\xi)^{1/2}, \quad \xi = \frac{\gamma_1}{\gamma_2}.$$

The number of iterations satisfies the estimate $n \geqslant n_0(\epsilon)$, where $n_0(\epsilon) = \dfrac{\ln \epsilon}{\ln \rho}$.

We can construct a method with a better rate of convergence when more information is available a priori regarding the operators of the iterative scheme.

Theorem 4.3. *Suppose that the constants γ_1, γ_2 and γ_3 in the following inequalities are known:*

$$\gamma_1 D \leqslant DB^{-1}A \leqslant \gamma_2 D, \quad \gamma_1 > 0,$$

$$\left\| \frac{DB^{-1}A - (DB^{-1}A)^*}{2} y \right\|^2_{D^{-1}} \leqslant \gamma_3 (Dy, y).$$

Then the iterative method (4.2) *with $\tau = \tau_0(1-\varkappa\rho)$ converges in H_D and its error z_n satisfies*

$$\|z_n\|_D \leqslant \rho^n \|z_0\|_D, \quad \rho = \frac{1-\xi}{1+\xi}, \quad \xi = \frac{1-\varkappa}{1+\varkappa} \cdot \frac{\gamma_1}{\gamma_2},$$

where

$$\tau_0 = \frac{2}{\gamma_1 + \gamma_2}, \quad \varkappa = \frac{\gamma_3}{(\gamma_1\gamma_2 + \gamma_3^2)^{1/2}}.$$

For number of iterations satisfies the bound $n \geqslant n_0(\epsilon)$, where $n_0(\epsilon) = \dfrac{\ln\epsilon}{\ln\rho_0}$.

An application of the iterative methods described above requires a knowledge of the constants γ_1, γ_2 and γ_3. When these constants are not known accurately or are a priori unknown, and the operator B is self-adjoint and positive definite in H, it is appropriate to use the method of minimal corrections (4.2) with

$$\tau_k = \frac{(Aw_k, w_k)}{(B^{-1}Aw_k, Aw_k)}, \quad w_k = B^{-1}r_k, \quad r_k = Ay_k - f.$$

The error z_n of the method of minimal corrections satisfies the estimates of the above theorems where one sets $D = A^*B^{-1}A$. Thus the convergence of the method of minimal corrections in a class of arbitrary initial iterates is no worse than the method of iteration.

4.2. Alternately-triangular iterative method. The choice of operator B influences both the number of arithmetic operations used to perform one iteration, and the number of iterations needed to provide the required accuracy. When constructing the operator B in the case of a self-adjoint operator A, one often uses the following principle for the construction of stable difference schemes for time-dependent equations. Let $A = A^*$. Then $R = R^* > 0$ is a regularizer that is energy-equivalent to operators A and B if

$$c_1 R \leqslant A \leqslant c_2 R, \quad c_2 \geqslant c_1 > 0,$$

$$\mathring{\gamma}_1 B \leqslant R \leqslant \mathring{\gamma}_2 B, \quad \mathring{\gamma}_2 \geqslant \mathring{\gamma}_1 > 0.$$

One then has the inequalities (4.5) with $\gamma_1 = c_1 \mathring{\gamma}_1$, $\gamma_2 = c_2 \mathring{\gamma}_2$.

One possible way of constructing the factored operator B is implemented in the alternating-direction method. A significant contribution to the theory of iterative methods is A. A. Samarskii's universal *alternately-triangular method*, which is applicable to the solution of equation (4.1) with a self-adjoint positive definite operator A. In the iterative scheme (4.2) one uses the factored operator

$$B_k \equiv B = (D_0 + \omega R_1) D_0^{-1} (D_0 + \omega R_2),$$

which is constructed by splitting the regularizer R as a sum of mutually adjoint operators R_1 and R_2 : $R_1 = R_2^*$, $R_1 + R_2 = R$. Here $D_0 = D_0^* > 0$ is an arbitrary operator acting in H.

The operator D_0 is chosen to enhance the cost-effectiveness of the method. In particular, one can take R_1 to be the triangular part of the matrix R, and for D_0 any diagonal matrix with positive elements (in this case, D_0 is chosen so that the number of iterations is minimized).

For the alternately-triangular iterative method the following theorem is valid.

Theorem 4.4. *Suppose that $R_1 = R_2^*$, $R_1 + R_2 = R$ and the constants δ and Δ are known in the inequalities*

$$\delta\, D_0 \leqslant R, \quad R_1 D_0^{-1} R_2 \leqslant \frac{\Delta}{4} R, \quad \delta > 0.$$

With the optimal value $\omega = \omega_0 = \dfrac{2}{(\delta\Delta)^{1/2}}$ assign the values

$$\overset{\circ}{\gamma}_1 = \frac{\delta}{2(1+\eta^{1/2})}, \quad \overset{\circ}{\gamma}_2 = \frac{\delta}{4\eta^{1/2}}, \quad \overset{\circ}{\xi} = \frac{\overset{\circ}{\gamma}_1}{\overset{\circ}{\gamma}_2} = \frac{2\eta^{1/2}}{1+\eta^{1/2}}, \quad \eta = \frac{\delta}{\Delta}$$

for the constants from the inequalities

$$c_1 \overset{\circ}{\gamma}_1 B \leqslant A \leqslant c_2 \overset{\circ}{\gamma}_2 B.$$

The number of iterations of the alternately-triangular method with the Chebyshev set of parameters $\{\tau_k\}$ satisfies the estimate

$$n \geqslant n_0(\epsilon), \quad n_0(\epsilon) = \frac{c_2^{1/2}}{c_1^{1/2}} \cdot \frac{\ln(2/\epsilon)}{2\, \overset{\circ}{\xi}{}^{1/2}} \approx \frac{c_2^{1/2}}{c_1^{1/2}} \cdot \frac{\ln(2/\epsilon)}{2\sqrt{2}\eta^{1/4}}.$$

The alternately-triangular method proves to be very effective for the solution of difference boundary-value problems for second-order elliptic equations with highly varying discontinuous coefficients both in rectangular domains and in domains of arbitrary shape [30, 34]. For the number of iterations one obtains the estimate

$$n(\epsilon) = O\Big(h^{-\gamma} \ln \frac{2}{\epsilon}\Big), \; \gamma = 0.5,$$

where h is the mesh-size. The domain shape does not seriously influence the number of iterations (for an irregular domain this number equals the number of iterations of the alternately-triangular iterative method applied to the same problem in a square whose side equals the diameter of the domain).

4.3. Iterative methods for cluster aggregation. The theory of iterative methods for the solution of systems of linear equations is developed in various directions. On modern parallel computers, success is achieved by the use of classical block iterative methods and multicolor sorting of unknown variables. When solving elliptic boundary-value problems, approaches based on domain decomposition into subdomains with and without overlapping are considered. An example is the classical alternating Schwarz method. Domain decomposition methods on the matrix level can be considered as special iterative methods of block type. Following [35], we select a class of iterative methods with a special organization of calculations that is typical of classical block methods. The individual equations of the system are combined into groups after pre-treatment (for example, after scaling). Moreover the same equation can be included in different groups; these groups are called clusters. For symmetric systems of linear equations the convergence of iterative methods of cluster aggregation is proved. Among them, we mention methods that are associated with pointwise and block-wise relaxation and with domain decomposition methods such as the alternating Schwarz method.

In the finite-dimensional Hilbert space H we consider the solution of equation (4.1) where A is a positive and self-adjoint operator. Suppose that from one equation of (4.1) one obtains a system of p equations (p clusters). The construction of the system of equations is formalized by the introduction of aggregation operators $G^{(\alpha)}$, $\alpha = 1, 2, \ldots, p$. In general, suppose that

$$G^{(\alpha)} = (G^{(\alpha)})^* \geqslant 0, \quad \alpha = 1, 2, \ldots, p, \quad \bar{G} = \sum_{\alpha=1}^{p} G^{(\alpha)} > 0. \tag{4.6}$$

We obtain the individual equations by multiplying the original equation (4.1) by $G^{(\alpha)}$, $\alpha = 1, 2, \ldots, p$:

$$G^{(\alpha)} A y = G^{(\alpha)} f, \quad \alpha = 1, 2, \ldots, p.$$

Thus the equation (4.1) is reduced to the system of equations

$$A^{(\alpha)} y = f^{(\alpha)}, \quad \alpha = 1, 2, \ldots, p, \tag{4.7}$$

where

$$A^{(\alpha)} = G^{(\alpha)} A, \quad f^{(\alpha)} = G^{(\alpha)} f, \quad \alpha = 1, 2, \ldots, p. \tag{4.8}$$

Let y be any vector satisfying (4.7), (4.8). Then summing over all $\alpha = 1, 2, \ldots, p$ we get

$$\bar{G} A y = \bar{G} f.$$

In our assumptions, $\bar{G} > 0$ and consequently any solution of (4.7), (4.8) is just the unique solution of the original system of equations (4.1) that we desired.

To solve approximately the system of equations (4.1), we use an iterative method based on cluster aggregation, i.e., on a transformation to a system of equations described by (4.6)–(4.8). Methods of these class are therefore called *iterative methods of cluster aggregation*.

The transition from each iterate y_k to the new iterate y_{k+1} is based on solving p problems that correspond to the cluster splitting (4.7), (4.8). Denote the approximate solution associated with cluster (equation) number α by $y_{k+\alpha/p}$. First consider a method where the $y_{k+\alpha/p}$ are determined sequentially one after another as α increases. This case is referred to as a simultaneous iteration method.

The approximate solution $y_{k+\alpha/p}$ will be determined from the computed solution $y_{k+(\alpha-1)/p}$. For a stationary iterative method we put

$$(\mu E + A^{(\alpha)}) \frac{y_{k+\alpha/p} - y_{k+(\alpha-1)/p}}{\tau} + A^{(\alpha)} y_{k+(\alpha-1)/p} = f^{(\alpha)}, \quad \alpha = 1, 2, \ldots, p. \tag{4.9}$$

Here μ is a positive constant (an iteration parameter). The iterative method (4.9) can be associated with an application of the additive scheme of component-wise splitting when solving the corresponding Cauchy problem for the equation

$$\mu \frac{dy}{dt} + \sum_{\alpha=1}^{p} A^{(\alpha)} y = f, \quad t > 0.$$

Theorem 4.5. *The iterative method of cluster aggregation* (4.6)–(4.9) *converges in* H_A *for any* $0 < \tau < 2$.

When considering the construction of computational algorithms for modern parallel computers, asynchronous iterative methods deserve special attention. Here the approximate solution is computed from separate subproblems which can be solved independently of each other. Let us outline some possibilities in this research direction.

Define the vectors $\tilde{y}_{k+\alpha/p}$ from the equations

$$(\mu E + A^{(\alpha)}) \frac{\tilde{y}_{k+\alpha/p} - y_k}{\tau} + A^{(\alpha)} y_k = \varphi^{(\alpha)}, \quad \alpha = 1, 2, \ldots, p. \tag{4.10}$$

For the iterate on level $k+1$ we use the expression

$$y_{k+1} = \frac{1}{p} \sum_{\alpha=1}^{p} \tilde{y}_{k+\alpha/p}. \tag{4.11}$$

The principal difference between the algorithm (4.10), (4.11) and (4.9) is that the computations of the $\tilde{y}_{k+\alpha/p}$ can be performed independently (asynchronously) of each other using only the iterate y_k. The convergence of this method is established under the same conditions as those considered above for the synchronous variant of the iterative method of cluster aggregation.

One also has the opportunity of constructing iterative methods of cluster aggregation from a somewhat different perspective. Above we consider some variants that reduce the system of equations (aggregation of equations). A second approach [24] splits the unknown solution (aggregation of unknown variables). Write the solution of equation (4.1) in the form

$$y = \sum_{\alpha=1}^{p} G^{(\alpha)} y^{(\alpha)}. \tag{4.12}$$

Substituting this into (4.1) we get

$$\sum_{\alpha=1}^{p} A G^{(\alpha)} y^{(\alpha)} = f. \tag{4.13}$$

Instead of one unknown y in equation (4.1), one now has p unknowns $y^{(\alpha)}, \quad \alpha = 1, 2, \ldots, p$, in (4.13). The function y that is determined by (4.13) can be interpreted as the solution of original problem (4.1). It is clear that the new problem (4.13) has many solutions; to obtain any of them we use an iterative method.

Denote by $y_k^{(\alpha)}$ the approximate solution of $y^{(\alpha)}$ at the kth iteration. The next iterate is found from the system of equations

$$(\mu E + A G^{(\alpha)}) \frac{y_{k+1}^{(\alpha)} - y_k^{(\alpha)}}{\tau} + \sum_{\beta=1}^{p} A G^{(\beta)} y_k^{(\beta)} = f. \tag{4.14}$$

This iterative process of cluster aggregation of unknown variables resembles the organization of computations in the iterative method of cluster aggregation of equations in the form (4.10), (4.11). The iterative method of cluster aggregation (4.6), (4.14) converges for any $0 < \tau < 2/p$.

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