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A METHOD WITH A CONTROLLABLE EXPONENTIAL CONVERGENCE RATE FOR NONLINEAR DIFFERENTIAL OPERATOR EQUATIONS

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Abstract — We propose a new analytical-numerical method with an embedded convergence control mechanism for solving nonlinear operator differential equations. The method provides the exponential convergence rate. A numerical example confirms the theoretical results.

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1. Introduction

There are various approaches to the construction of exponentially convergent approximations to solutions of nonlinear differential equations. The spectral methods (see [6] and the references therein) are based on spectral expansions or a special interpolation. One can obtain exponentially convergent methods for a wide class of differential equations with operator coefficients combining an appropriate quadrature formula for the Dunford — Cauchy representation of the operator exponential with a special interpolation of the nonlinearity (see [12] and the references cited therein). One more alternative which seems to be very promising is the use of the homotopy or perturbation idea (see e.g. [10, 11]) which is closely related to the Adomian decomposition method (ADM) [2–4]. One of the important elements of this approach are Adomian's polynomials [2–4, 33, 34]. The last decade has seen many publications devoted to the application of the Adomian decomposition method (ADM) for both the linear and the nonlinear operator and differential operator equations [1,7,11,19,20]. The class of problems for which these approximations provide the exponential convergence rate is restricted by the character of nonlinearity: there are nonlinearities which do not fulfill the known convergence conditions and the approximation methods do not contain some convergence control. In the present paper we propose a control mechanism which guarantees exponential convergence for a wide class of nonlinearities. The idea of such an approach for eigenvalue problems has been recently reported in [21].

Let us remind of the idea of an ADM that can be also interpreted as the FD-method proposed in [20] for the Sturm — Liouville problems and is very close to the homotopy or perturbation methods.

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If we have to solve the operator equation

$$u = -N(u) + F, \tag{1.1}$$

then we can imbed it into the family of equations

$$u(t) = -tN(u(t)) + F, \quad t \in [0,1], \tag{1.2}$$

and obtain explicitly

$$u(1) = u. \tag{1.3}$$

We look for the solution of (1.1) in the form

$$u(t) = \sum_{j=0}^{\infty} t^j u^{(j)}, \tag{1.4}$$

and represent

$$N\left(\sum_{j=0}^{\infty} t^j u^{(j)}\right) = \sum_{j=0}^{\infty} t^j A_j, \tag{1.5}$$

where

$$A_j = \frac{1}{j!} \frac{\partial^j N(\sum_{k=0}^{\infty} t^k u^{(k)})}{\partial t^j}\bigg|_{t=0}. \tag{1.6}$$

Substituting (1.4) into (1.1) we have

$$\sum_{j=0}^{\infty} t^j u^{(j)} = -tN\left(\sum_{j=0}^{\infty} t^j u^{(j)}\right) + F. \tag{1.7}$$

Applying to this equality successively the operator $\frac{1}{(j+1)!}\frac{d^{j+1}}{dt^{j+1}}$ and then setting $t = 0$, we obtain the following recurrence formulas:

$$\begin{aligned} u^{(j+1)} &= -A_j\left(N; u^{(0)}, u^{(1)}, \dots, u^{(j)}\right), \quad j = 0, 1, \dots, \\ A_0\left(N; u^{(0)}\right) &= N\left(u^{(0)}\right), \quad u^{(0)} = F. \end{aligned} \tag{1.8}$$

Here $A_j\left(N; u^{(0)}, u^{(1)}, \dots, u^{(j)}\right)$ are Adomian polynomials with the following explicit representation:

$$A_j\left(N; u^{(0)}, u^{(1)}, \dots, u^{(j)}\right) = \sum_{\alpha_1+\ldots+\alpha_j=j} N^{(\alpha_1)}\left(u^{(0)}\right) \frac{\left(u^{(1)}\right)^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!} \cdot \ldots \cdot \frac{\left(u^{(j-1)}\right)^{\alpha_{j-1}-\alpha_j}}{(\alpha_{j-1}-\alpha_j)!} \frac{\left(u^{(j)}\right)^{\alpha_j}}{(\alpha_j)!}, \tag{1.9}$$

where the sequence of natural indices α_i is not increasing, $N^{(i)}(u)$ is the i-th (Fréchet) derivative of the operator N.

The solution of (1.1) can be now given (provided that the convergence radius of series (1.4) is not less than 1) by

$$u = u(1) = \sum_{j=0}^{\infty} u^{(j)} \tag{1.10}$$

and the truncated sum

$$\overset{m}{u} = \sum_{j=0}^{m} u^{(j)} \tag{1.11}$$

represents an approximation to the exact solution.

Example 1.1. For the nonlinearity $N(u) = \sum_{i=1}^{q} a_i u^{2i}$ we have

$$A_0 = \sum_{i=1}^{q} a_i \left(\sum_{k=0}^{\infty} \right)^{2i} \Bigg|_{t=0} = \sum_{i=1}^{q} a_i \left[u^{(0)} \right]^{2i},$$

$$A_1 = \frac{1}{1!} \frac{\partial}{\partial t} \sum_{i=1}^{q} a_i \left(\sum_{k=0}^{\infty} t^k u^{(k)} \right)^{2i} \Bigg|_{t=0} = \left[\sum_{k=1}^{\infty} k \cdot t^{k-1} u^{(k)} \right] \cdot \sum_{i=1}^{q} (2i) a_i \left[\sum_{k=0}^{\infty} t^k u^{(k)} \right]^{2i-1} \Bigg|_{t=0} =$$

$$u^{(1)} \cdot \sum_{i=1}^{q} (2i) a_i [u^{(0)}]^{2i-1} = u^{(1)} \cdot \frac{d}{du^{(0)}} A_0,$$

$$A_2 = \frac{1}{2!} \frac{\partial^2}{\partial t^2} \sum_{i=1}^{q} a_i \left(\sum_{k=0}^{\infty} t^k u^{(k)} \right)^{2i} \Bigg|_{t=0} = \frac{1}{2} \left(\left[\sum_{k=2}^{\infty} k(k-1) t^{k-2} u^{(k)} \right] \cdot \left[\sum_{i=1}^{q} (2i) a_i \left(\sum_{k=0}^{\infty} t^k u^{(k)} \right)^{2i-1} \right] + \right.$$

$$\left. \left(\sum_{k=1}^{\infty} k t^{k-1} u^{(k)} \right)^2 \left[\sum_{i=1}^{q} (2i)(2i-1) a_i \left(\sum_{k=0}^{\infty} t^k u^{(k)} \right)^{2i-2} \right] \right) \Bigg|_{t=0} =$$

$$\frac{1}{2} \left[u^{(2)} \cdot \sum_{i=1}^{q} (2i) a_i \left(u^{(0)} \right)^{2i-1} + \left(u^{(1)} \right)^2 \cdot \sum_{i=1}^{q} (2i)(2i-1) a_i \left(u^{(0)} \right)^{2i-2} \right] =$$

$$\frac{1}{2} \left[u^{(2)} \frac{d}{du^{(0)}} A_0 + \left(u^{(1)} \right)^2 \cdot \frac{d^2}{du^{(0)2}} A_0 \right] \tag{1.12}$$

It is easy to see that the evaluation of the polynomial A_j for its fixed arguments needs $\mathcal{O}(qj)$ multiplications.

The following theorem from [1] gives some sufficient conditions for the convergence of (1.4) for all $t \in [0, 1]$.

Theorem 1.1. *Let H be a Banach space and $F \in H$. If the operator $N(u): H \to H$ is analytic in a ball $\|u - u_0\| < R$ with the center u_0 and if for all $n \geqslant 0$ $\|N^{(n)}(u_0)\| \leqslant n! M \alpha^n$ with some $M > 0$, $\alpha > 0$, holds then the conditions*
1) $4M\alpha \leqslant 1$, *for* $R = \infty$,
2) $5M\alpha \leqslant 1$, *for* $R < \infty$,
provide the convergence of (1.4) *for all* $t \in [0, 1]$ *and, therefore, the convergence of* (1.10).

In the recent paper [7] the following modification of the ADM was proposed. One looks for the summands of (1.10) in accordance with the recurrence formulas

$$u^{(j+1)} = -\overline{A_j} \left(u^{(0)}, u^{(1)}, \ldots, u^{(j)} \right), \quad j = 0, 1, \ldots,$$
$$u^{(0)} = F, \tag{1.13}$$

where $\overline{A_j} \left(u^{(0)}, u^{(1)}, \ldots, u^{(j)} \right)$ are the modified Adomian polynomials given by

$$\overline{A_j} \left(u^{(0)}, u^{(1)}, \ldots, u^{(j)} \right) = N \left(u^{(0)} + \ldots + u^{(j)} \right) - N \left(u^{(0)} + \ldots + u^{(j-1)} \right). \tag{1.14}$$

In [7], for the problem

$$\frac{d^k y(t)}{dt^k} + \beta(t) f(y(t)) = \kappa(t), \quad t \in (0,\ T), \quad \frac{d^p y(t)}{dt^p} = c_p, \quad p = \overline{0, k-1}, \tag{1.15}$$

with given c_p, $p = \overline{0, k-1}$, with $M = \max_{t\in[0,\ T]} \|\beta(t)|$ and with the right-hand side $f(y)$ satisfying the Lipschitz condition with a constant L it was shown that the modified Adomian method converges as a geometric progression with the quotient α and with the error estimate

$$\left|y(t) - \sum_{j=0}^{m} y_j(t)\right| \leqslant \frac{\alpha^m}{1-\alpha}\|y_1\|_\infty \tag{1.16}$$

provided that

$$\alpha = \frac{LMT^k}{k!} < 1. \tag{1.17}$$

Numerical experiments have shown that the modified ADM converges faster then the ordinary one. But it was ignored in [7] that in fact the modified ADM coincides with the usual fixed point iteration. Actually, relations (1.13), (1.14) imply

$$u^{m+1} = -N(u^m) + F, \quad m = 0, 1, \ldots,$$
$$u^0 = F. \tag{1.18}$$

Now the conclusions of [7] about the advantages of the modified ADM become understandable and have been well known since long ago (see, e.g., [25]).

The aim of the present paper is to construct an iteration method which converges whereas the fixed point iteration (1.18) can be divergent. The last Section 3 contains numerical examples which supports our theoretical results.

2. An iteration method for nonlinear problems with controllable exponential convergence

2.1. General algorithm. In this section we give a general description of the proposed algorithm. In order to avoid technical difficulties we justify this algorithm in the case of an ordinary differential equation.

Let us consider a nonlinear problem of the form

$$A_0 u + N(u)u = f, \tag{2.1}$$

in a Banach space B, where A_0 is an unbounded linear operator with the domain $D(A_0)$ and $N(u)$ for each fixed u is a linear operator in B. The associated linear problem

$$A_0 v + N(u)v = f \tag{2.2}$$

is supposed to have a unique solution.

Using the idea which is related to the ideas of homotopy, perturbation or FD-methods, we imbed problem (2.1) into the parametric family

$$A_0 u(t) + N\big(P_M\, u(t)\big)u(t) + t\Big[N(u(t)) - N\big(P_M u(t)\big)\Big]u(t) = f, \quad 0 \leqslant t \leqslant 1, \tag{2.3}$$

where $P_M :\ H \to B_M$ is a projector from B onto an M-dimensional subspace B_M of B.

It is clear that for $t = 1$ we have

$$u(1) = u. \tag{2.4}$$

Setting t=0 in (2.3) we obtain the base problem

$$A_0 u^{(0)} + N(P_M\, u^{(0)})u^{(0)} = f, \tag{2.5}$$

which is supposed to have a unique solution.

We look for the solution of the parametric problem (2.3) in the form

$$u(t) = \sum_{j=0}^{\infty} t^j u^{(j)}. \tag{2.6}$$

If this series is convergent in B with a convergence radius $R \geqslant 1$, then the solution of problem (2.1) can be represented in the form

$$u = \sum_{j=0}^{\infty} u^{(j)}. \tag{2.7}$$

Supposing additionally $N(u(t)) = \sum\limits_{j=0}^{\infty} t^j A_j(N; u^{(0)}, \dots, u^{(j)})$ and substituting this expansion and (2.6) into (2.3), we obtain for the coefficients $u^{(j)}$ the following recurrence equations:

$$A_0 u^{(j+1)} + N(P_M u^{(0)})u^{(j+1)} = A_{j+1}(N; P_M u^{(0)}, 0, \dots, 0, P_M u^{(j+1)})u^{(0)} - F^{(j+1)}, \quad j = 0, 1, \dots, \tag{2.8}$$

where

$$F^{(j+1)} = \sum_{p=1}^{j} A_{j+1-p}(N; P_M\, u^{(0)}, P_M\ u^{(1)}, \dots, P_M\ u^{(j+1-p)})\, u^{(p)} +$$

$$\sum_{p=0}^{j} \left[A_{j-p}(N; u^{(0)},\, u^{(1)}, \dots,\, u^{(j-p)}) - A_{j-p}(N; P_M\, u^{(0)},\, P_M\, u^{(1)}, \dots,\, P_M\, u^{(j-p)})\right]\, u^{(p)} +$$

$$A_{j+1}(N; P_M\, u^{(0)},\, P_M\, u^{(1)}, \dots,\, P_M\, u^{(j)}, 0)\, u^{(p)} \tag{2.9}$$

and $A_j(N; v_0, v_1, \dots, v_j)$ are Adomian's operator polynomials for the nonlinearity $N(v)$.

The approximate solution to (2.1) (of rank m or with the discreization parameter m) is then given by

$$\overset{m}{u} = \sum_{j=0}^{m} u^{(j)}. \tag{2.10}$$

A specific feature of (2.8) is the dependence of the right-hand side on the projection $P_M\, u^{(j+1)}$ of the unknown solution $u^{(j+1)}$.

Example 2.1. For the nonlinearity

$$N(u)v = \int_0^1 \frac{v(s)}{1 + u^2(s)}\, ds \tag{2.11}$$

using formula (1.6)(or (1.9)) we obtain the following Adomian operator polynomials

$$A_0(N; v_0)w = \int_0^1 \frac{w(s)}{1 + (v_0(s))^2}\, ds,$$

$$A_1(N; v_0, v_1)w = -\int_0^1 \frac{2v_0(s)v_1(s)}{(1+(v_0(s))^2)^2} w(s)\,ds,$$

$$A_2(N; v_0, v_1, v_2)w = \int_0^1 \left[\frac{4(v_0(s)\,v_1(s))^2}{\left(1+(v_0(s))^2\right)^3} - \frac{(v_1(s))^2}{\left(1+(v_0(s))^2\right)^2} - \frac{2v_0(s)v_2(s)}{\left(1+(v_0(s))^2\right)^2} \right] w(s)ds,$$

$$A_3(N; v_0, v_1, v_2, v_3)w = \int_0^1 \left[-\frac{8\big(v_0(s)\,v_1(s)\big)^3}{\left(1+(v_0(s))^2\right)^4} + \frac{4v_0(s)\big(v_1(s)\big)^3}{\left(1+(v_0(s))^2\right)^3} + \right.$$

$$\left. + \frac{8\big(v_0(s)\big)^2 v_1(s)\,v_2(s)}{\left(1+(v_0(s))^2\right)^3} - \frac{2v_0(s)v_2(s)}{\left(1+(v_0(s))^2\right)^2} - \frac{2v_0(s)v_3(s)}{\left(1+(v_0(s))^2\right)^2} \right] w(s)ds,$$

Example 2.2. The stationary Gross — Pitaevskii equation describes the structure of a Bose — Einstein condensate in various external potentials $V(x)$ (see, e.g., [24])

$$-\triangle u + (V(x) + g|u(x)|^2)u(x) = 0 \tag{2.12}$$

and (provided by proper boundary conditions) is an example of problem 2.1 with

$$A_0 u = -\triangle u, \quad N(u)u = (V(x) + g|u(x)|^2)u(x), \tag{2.13}$$

where $N(u)$ is of the type $N(u) = a_0 + a_1 u^2 + \cdots$ with bounded operators $a_0, a_1, \ldots$ such that $a_0 u = V(x) \cdot u, a_1 u = g|u|, a_i = 0, i = 2, 3, \ldots$

Example 2.3. The Korteveg-de-Vries equation [22]

$$u_t = u_{xxx} + u^2 \cdot u_x \tag{2.14}$$

is of the type (2.1) with $A_0 u = u_{xxx}$ (with corresponding boundary conditions), $N(u) = a_0 + a_1 u^2 + \cdots$ where the operator coefficients $a_0 = -\partial/\partial t, a_1 = \partial/\partial x$ are unbounded operators and $a_i = 0,\ i = 2, 3, \ldots$

2.2. Justification of the algorithm in the case of BVP for ODE. In this section we consider the following nonlinear model problem:

$$u''(x) - N(u(x))u(x) = -f(x), \quad x \in (0,1),$$
$$u(0) = u(1) = 0, \tag{2.15}$$

with a nonlinear function $N(u) : \mathbb{R}^1 \to \mathbb{R}^1$ satisfying the conditions

$$N(u) \geqslant 0, \quad [uN(u)]' \geqslant 0, \quad N''(u) \geqslant 0, \quad \forall\, u \in \mathbb{R}^1. \tag{2.16}$$

Using the Green function for the differential operator defined by

$$D(\mathcal{A}) = \{u : u \in W_2^2(0,1) : u(0) = u(1) = 0\}, \quad \mathcal{A}u = -\frac{d^2u}{dx^2}, \quad \forall u \in D(\mathcal{A}), \tag{2.17}$$

one can reduce problem (2.15) to the operator equation of kind (1.1). In the case where the fixed point iteration (1.18) is divergent, we propose the following method based on the idea of the FD-method or the homotopy method.

We introduce a grid

$$\overline{\hat{\omega}} = \left\{x_i \in [0,\ 1],\ i = \overline{1, K}:\ 0 = x_1 < x_2 < \ldots < x_{K-1} < x_K = 1\right\}$$

partitioning the interval $[0,\ 1]$ into subintervals $[x_{i-1},\ x_i]$, $i = \overline{1, K}$ of length $h_i = x_i - x_{i-1}$, $|h| = \max_i h_i$ and imbed problem (2.15) into the parametric family of problems

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \left\{N(u(x_{i-1},t)) + t\,[N(u(x,t)) - N(u(x_{i-1},t))]\right\} u(x,t) = -f(x), \quad x \in (0,\ 1),$$

$$[u(x)]_{x=x_i} = 0, \quad \left[\frac{du(x)}{dx}\right]_{x=x_i} = 0, \quad i = \overline{1, K-1},$$

$$u(0,t) = u(1,t) = 0, \quad t \in [0,\ 1]. \tag{2.18}$$

It is clear that for $t = 1$ the solution of problem (2.18) coincides with the solution of problem (2.15), i.e.,

$$u(x,1) = u(x),$$

For $t = 0$ we obtain the following base problem:

$$\frac{d^2 u^{(0)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1})\right) u^{(0)}(x) = -f(x), \quad x \in (x_{i-1},\ x_i), \quad i = \overline{1, K},$$

$$[u^{(0)}(x)]_{x=x_i} = 0, \quad \left[\frac{du^{(0)}(x)}{dx}\right]_{x=x_i} = 0, \quad i = \overline{1, K-1},$$

$$u^{(0)}(0) = u^{(0)}(1) = 0, \tag{2.19}$$

where $[v(x)]_{x=\xi} = v(\xi + 0) - v(\xi - 0)$ denotes the jump of the function $v(x)$ at the point $x = \xi$.

The last problem as well as problem (2.18) are representatives of the class of boundary value problems with a piecewise constant argument which have been the focus of attention of many researchers for some time (see e.g. [5] and the references therein).

We look for the solution of problem (2.18) in the form

$$u(x,t) = \sum_{j=0}^{\infty} t^j u^{(j)}(x). \tag{2.20}$$

Substituting (2.20) into (2.18) and comparing the coefficients in front of the powers of t, we obtain the following recurrence sequence of problems for $u^{(j)}(x)$ (with a piecewise constant argument):

$$\frac{d^2 u^{(j+1)}(x)}{dx^2} - N\left(u^{(0)}(x_{i-1})\right) u^{(j+1)}(x) = N'\left(u^{(0)}(x_{i-1})\right) u^{(j+1)}(x_{i-1}) u^{(0)}(x) + F^{(j+1)}(x),$$

$$x \in (x_{i-1}, x_i), \quad i = \overline{1, K}, \tag{2.21}$$

where

$$F^{(j+1)}(x) = \sum_{p=1}^{j} A_{j+1-p}\left(N; u^{(0)}(x_{i-1}), \ldots, u^{(j+1-p)}(x_{i-1})\right) u^{(p)}(x) +$$

$$\sum_{p=0}^{j}\left[A_{j-p}\left(N;u^{(0)}(x),\dots,u^{(j-p)}(x)\right)-A_{j-p}\left(N;u^{(0)}(x_{i-1}),\dots,u^{(j-p)}(x_{i-1})\right)\right]u^{(p)}(x)+$$

$$A_{j+1}\left(N;u^{(0)}(x_{i-1}),\dots,u^{(j)}(x_{i-1}),0\right)u^{(0)}(x),$$

$$[u^{(j+1)}(x)]_{x=x_i}=0,\quad \left[\frac{du^{(j+1)}(x)}{dx}\right]_{x=x_i}=0,\quad i=\overline{1,K-1},$$

$$u^{(j+1)}(0)=u^{(j+1)}(1)=0,\quad j=0,1,\dots,\tag{2.22}$$

$A_j(N;v_0,v_1,\dots,v_j)$ are Adomian's polynomials for the nonlinear function $N(v)$ given by the explicit formula (1.9). The solution of problem (2.15) is then given by

$$u(x)=\sum_{j=0}^{\infty}u^{(j)}(x)\tag{2.23}$$

(provided that the convergence radius of (2.20) is not less than 1) and the approximate solution by

$$u(x)\approx\overset{m}{u}(x)=\sum_{j=0}^{m}u^{(j)}(x),\tag{2.24}$$

where the exponential convergence will be controlled by the parameter $|h|$.

Let us consider the base problem (2.19). This problem is equivalent to the system of nonlinear equations

$$u^{(0)}(x_i)=\int_0^1 G\left(x_i,\xi,\overrightarrow{N}(u)\right)f(\xi)d\xi,\quad i=\overline{1,K-1},\tag{2.25}$$

where

$$\overrightarrow{N}(u)=\left(N\left(u^{(0)}(x_1)\right),\dots,N\left(u^{(0)}(x_{K-1})\right)\right)\tag{2.26}$$

and $G\left(x_i,\xi,\overrightarrow{N}\right)$ is the Green function of problem (2.19) provided that the vector $\overrightarrow{u}=\{u^{(0)}(x_1),u^{(0)}(x_2),\dots,u^{(0)}(x_{K-1})\}$ is known.

We introduce the operator

$$B(\overrightarrow{u})=\left(\int_0^1 G\left(x_i,\xi,\overrightarrow{N}(u)\right)f(\xi)d\xi\right)_{i=1}^{K-1}\tag{2.27}$$

which is continuous on a closed ball

$$\overline{S}=\left\{\overrightarrow{u}\in\mathbb{R}^{K-1}:\|\overrightarrow{u}\|_{0,\infty,\hat{\omega}_N}=\max_{1\leqslant i\leqslant K-1}|u^{(0)}(x_i)|\leqslant r\right\}$$

with r defined by $\|f\|_{0,\infty,[0,1]}$ and translate the ball $\overline{S}$ into itself. Therefore, by Brower's fixed point theorem (see, e.g., [15]) there exists a fixed point of this operator in $\overline{S}$, i.e., the system of equations (2.25) is solvable.

Remark 2.1. The fixed point iteration for equation (2.25) is equivalent to the solution of the sequence of the following problems:

$$\frac{d^2u^{(0),n+1}(x)}{dx^2} - N\left(u^{(0),n}(x_{i-1})\right)u^{(0),n+1}(x) = -f(x), \quad x \in (x_{i-1},\ x_i), \quad i = \overline{1,K},$$
$$u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0, \quad n = 0,1,\ldots, \tag{2.28}$$

where $\overrightarrow{u^{(0),0}} = \left(u^{(0),0}(x_i)\right)_{i=\overline{1,K}}$ is an arbitrary vector from the ball $\bar{S}$. For this problem there exists the following exact difference scheme [27]:

$$\left(a^n(x_i)u^{(0),n+1}(x_i)_{\bar{x}}\right)_x - d^n(x_i)u^{(0),n+1}(x_i) = -\varphi^n(x_i), \quad i = \overline{1,K},$$
$$u^{(0),n+1}(0) = u^{(0),n+1}(1) = 0, \tag{2.29}$$

with

$$a^n(x_i) = \left[\frac{\sinh(\sqrt{\mu_i^n}h_i)}{h_i\sqrt{\mu_i^n}}\right]^{-1}, \quad \mu_i^n = N\left(u^{(0),n}(x_{i-1})\right),$$
$$d^n(x_i) = \frac{\sqrt{\mu_i}}{\hbar_i}\tanh\frac{\sqrt{\mu_i}h_i}{2} + \frac{\sqrt{\mu_{i+1}}}{\hbar_i}\tanh\frac{\sqrt{\mu_{i+1}}h_{i+1}}{2}, \qquad \hbar_i = \frac{h_i + h_{i+1}}{2},$$
$$\varphi^n(x_i) = \frac{1}{\hbar_i}\sum_{\alpha=1}^{2}(-1)^\alpha\left[\frac{dW_\alpha^i(x_i)}{dx} - W_\alpha^i(x_i)(-1)^{\alpha+1}\sqrt{\mu_{i-1+\alpha}^n}\coth\sqrt{\mu_{i-1+\alpha}^n}\,h_{i-1+\alpha}\right],$$

where $W_\alpha^i(x)$, $\alpha = 1,2$, are solutions of the following two Cauchy problems:

$$\frac{d^2W_\alpha^j(x)}{dx^2} - N\left(u^{(0),n}(x_i)\right)W_\alpha^j(x) = -f(x), \quad x_{j-2+\alpha} < x < x_{j-1+\alpha},$$
$$W_\alpha^j(x_{j+(-1)^\alpha}) = \left.\frac{dW_\alpha^j(x)}{dx}\right|_{x=x_{j+(-1)^\alpha}} = 0, \quad \alpha = 1,2. \tag{2.30}$$

In order to compute the coefficients of the exact difference scheme for one iteration step, one has to solve $2(K-1)$ Cauchy problems by an IVP-solver, each on a small interval with the length of the corresponding step-size. Then the difference scheme with a tridiagonal matrix can be solved by the special elimination method (method of chasing, method "progonki") which in our case is stable [26].

Remark 2.2. Another algorithmic implementation of the fixed point iteration for (2.25) can be done by the multiple shooting method [30] in the following way. Let $u^{(0)}(x) = u^{(0)}(x; x_{i-1}, s_{i-1}, s_{i-1}^{(1)})$ and $\frac{d}{dx}u^{(0)}(x) = \frac{d}{dx}u^{(0)}(x; x_{i-1}, s_{i-1}, s_{i-1}^{(1)})$ be the solution and its derivative of the differential equation (2.19) on the subinterval (x_{i-1}, x_i) subject to the initial values s_{i-1} and $s_{i-1}^{(1)}$ (and obtained by some IVP-solver), i.e., we have

$$u^{(0)}(x_i; x_{i-1}, s_{i-1}, s_{i-1}^{(1)}) = s_i,$$
$$\frac{d}{dx}u^{(0)}(x_i; x_{i-1}, s_{i-1}, s_{i-1}^{(1)}) = s_i^{(1)}, \quad i = 1,\ldots,K,$$
$$s_0 = 0, \quad s_K = 0. \tag{2.31}$$

Analogously to the multiple shooting method this system of equations can be written in the form $s = F(s)$ where $s = (s_0^{(1)}, s_1, s_0^{(1)}, \ldots, s_{K-1}, s_{K-1}^{(1)}, s_K^{(1)})^T$ and $F(s)$ for an arbitrary s can be calculated using an IVP-solver. From the discussion above it follows that the fixed point iteration

$$\overset{m+1}{s} = F(\overset{m}{s}), \quad m = 0, 1, \dots \tag{2.32}$$

converges provided that $\overset{0}{s}$ was chosen within the corresponding ball.

Let us rewrite equations (2.21) in the form

$$\frac{d^2u^{(j+1)}(x)}{dx^2} - q(x)u^{(j+1)}(x) = N'\left(u^{(0)}(x_{i-1})\right)[u^{(j+1)}(x_{i-1})u^{(0)}(x) - u^{(j+1)}(x)u^{(0)}(x_{i-1})] +$$

$$F^{(j+1)}(x), \quad x \in (x_{i-1}, x_i), \quad i = \overline{1, K},$$

$$u^{(j+1)}(0) = u^{(j+1)}(1) = 0, \quad j = 0, 1, \dots, \tag{2.33}$$

with

$$q(x) = N\left(u^{(0)}(x_{i-1})\right) + N'\left(u^{(0)}(x_{i-1})\right)u^{(0)}(x_{i-1}), \quad x \in [x_{i-1}, x_i), \quad i = \overline{1, K}. \tag{2.34}$$

Given $u^{(0)}(x_i), i = 1, \dots, K-1$, let $G(x, \xi, q(\cdot))$ be the Green function corresponding to the operator on the left side of (2.33) with the Dirichlet boundary conditions. Then problem (2.33) can be transformed to

$$u^{(j+1)}(x) = \sum_{p=1}^{K} \int_{x_{p-1}}^{x_p} G\left(x_i, \xi, q(\cdot)\right) \int_{x_{p-1}}^{\xi} \frac{du^{(j+1)}(\eta)}{d\eta} d\eta \, u^{(0)}(\xi) \, d\xi \, N'(u^{(0)}(x_{k-1})) -$$

$$-\sum_{p=1}^{K} \int_{x_{p-1}}^{x_p} G(x_i, \xi, q(\cdot)) \int_{x_{p-1}}^{\xi} \frac{du^{(0)}(\eta)}{d\eta} d\eta \, u^{(j+1)}(\xi) \, d\xi \, N'(u^{(j+1)}(x_{p-1})) - \int_0^1 G(x_i, \xi, q(\cdot)) F^{(j+1)}(\xi) \, d\xi, \tag{2.35}$$

where $i = \overline{1, K}$.

In order to estimate $u^{(j+1)}(x)$ we need to estimate the Green function $G\left(x, \xi, q(\cdot)\right)$, which can be explicitly represented by the formula (see e.g. [26])

$$G\left(x_i, \xi, q(\cdot)\right) = \frac{1}{v_1(1)} \begin{cases} v_1(x)v_2(\xi), & x \leqslant \xi, \\ v_1(\xi)v_2(x), & \xi \leqslant x. \end{cases} \tag{2.36}$$

Here $v_\alpha(x)$, $\alpha = 1, 2$ are the so-called stencil functions which satisfy the equations

$$\frac{d^2}{dx^2} v_\alpha(x) - q(x) v_\alpha(x) = 0, \quad 0 < x < 1, \quad \alpha = 1, 2,$$

$$v_1(0) = 0, \quad v_1'(0) = 1, \quad v_2(1) = 0, \quad v_2'(1) = -1, \tag{2.37}$$

as well as the continuity conditions

$$[v_\alpha(x)]_{x=x_i} = 0, \quad [v_\alpha'(x)]_{x=x_i} = 0, \quad \alpha = 1, 2 \quad i = \overline{1, K-1}. \tag{2.38}$$

These functions possess the following properties:

1° $v_1(x)$ is a nondecreasing, nonnegative function on $[0, \ 1]$;

2° $v_2(x)$ is nonincreasing, nonnegative function on $[0,\ 1]$;
3° $v_1(1) = v_2(0)$;
4° $v_1'(x)v_2(x) - v_1(x)v_2'(x) \equiv v_1(1) = v_2(0)$.
These properties as well as the maximum principal imply the estimates

$$0 \leqslant G(x,\xi,q(\cdot)) \leqslant G(x,\xi,0), \qquad \left|\frac{\partial G(x,\xi,q(\cdot))}{\partial x}\right| \leqslant 1. \tag{2.39}$$

Using (2.39) as well as assumptions (2.16) we obtain from (2.35)

$$\|u^{(j+1)}\|_{1,\infty,[0,1]} \leqslant |h|\|u^{(0)}\|_{1,\infty,[0,1]}N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right)\|u^{(j+1)}\|_{1,\infty,[0,1]} + \|F^{(j+1)}\|_{1,\infty,[0,1]}. \tag{2.40}$$

For $|h|$ small enough this inequality can be transformed to

$$\|u^{(j+1)}\|_{1,\infty,[0,1]} \leqslant c_1\|F^{(j+1)}\|_{1,\infty,[0,1]} \tag{2.41}$$

with

$$c_1 = \left[1 - |h|\|u^{(0)}\|_{1,\infty,[0,1]}N'\left(\|u^{(0)}\|_{1,\infty,[0,1]}\right)\right]^{-1}. \tag{2.42}$$

and the norms

$$\|v\|_{0,\infty,[0,1]} = \max_{x\in[0,1]}|v(x)|, \qquad \|v\|_{1,\infty,[0,1]} = \max\left\{\max_{x\in[0,1]}|v(x)|, \max_{x\in[0,1]}|v'(x)|\right\}.$$

Further we will need the following two auxiliary statements.

Lemma 2.1. *Let $N(u)$ be represented by the power series $N(u) = \sum_{i=1}^{\infty} a_i u^{2i}$, $a_i \geqslant 0$, and $u^{(p)}(x) \in C^1[0,\ 1]$, $p = 0,1,\ldots,$ then*

$$\begin{aligned}&\|A_k\left(N(u);u^{(0)}(x),\ldots,u^{(k)}(x)\right) - A_k\left(N(u);u^{(0)}(x_{i-1}),\ldots,u^{(k)}(x_{i-1})\right)\|_{0,\infty,[0,1]} \leqslant\\ &2h\sum_{i=1}^{\infty} ia_iA_k\left(N(u);\|u^{(0)}\|_{1,\infty,[0,1]},\|u^{(1)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]}\right) =\\ &|h|A_k\left(N'(u);\|u^{(0)}\|_{1,\infty,[0,1]},\|u^{(1)}\|_{1,\infty,[0,1]},\ldots,\|u^{(k)}\|_{1,\infty,[0,1]}\right).\end{aligned} \tag{2.43}$$

Proof. Since the Adomian polynomials are linear operators with respect to the first argument (see (1.9)), i.e.,

$$A_k\left(N(u);u^{(0)}(x),\ldots,u^{(k)}(x)\right) = \sum_{i=1}^{\infty} a_iA_k\left(u^{2i};u^{(0)}(x),\ldots,u^{(k)}(x)\right)$$

it is sufficient to consider the case $N(u) = u^{2i}$ only. For this case we have

$$\begin{aligned}&A_k\left(u^{2i};u^{(0)}(x),\ldots,u^{(k)}(x)\right) =\\ &\sum_{\alpha_1+\ldots+\alpha_k=k} N^{(\alpha_1)}\left(u^{(0)}(x)\right)\frac{\left[u^{(1)}(x)\right]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdot\ldots\cdot\frac{\left[u^{(k-1)}(x)\right]^{\alpha_{k-1}-\alpha_k}}{(\alpha_{k-1}-\alpha_k)!}\frac{\left[u^{(k)}(x)\right]^{\alpha_k}}{(\alpha_k)!} =\\ &\sum_{\alpha_1+\ldots+\alpha_k=k} 2i(2i-1)\cdot\ldots\cdot(2i-\alpha_1+1)[u^{(0)}(x)]^{2i-\alpha_1}\frac{[u^{(1)}(x)]^{\alpha_1-\alpha_2}}{(\alpha_1-\alpha_2)!}\cdot\ldots\cdot\frac{[u^{(k-1)}(x)]^{\alpha_{k-1}-\alpha_k}}{(\alpha_{k-1}-\alpha_k)!}\frac{[u^{(k)}(x)]^{\alpha_k}}{(\alpha_k)!},\end{aligned}$$

$$\left\|A_k\left(u^{2i};u^{(0)}(x),\dots,u^{(k)}(x)\right)-A_k\left(u^{2i},u^{(0)}(x_{i-1}),\dots,u^{(k)}(x_{i-1})\right)\right\|\leqslant$$
$$\sum_{\alpha_1+\ldots+\alpha_k=k}2i(2i-1)\ldots(2i-\alpha_1+1)(2i-\alpha_1+\alpha_1-\alpha_2+\ldots+\alpha_{k-1}-\alpha_k+\alpha_k)\times$$
$$\|u^{(0)}\|^{2i-\alpha_1}_{1,\infty,[0,1]}\frac{\|u^{(1)}\|^{\alpha_1-\alpha_2}_{1,\infty,[0,1]}}{(\alpha_1-\alpha_2)!}\cdot\ldots\cdot\frac{\|u^{(k-1)}\|^{\alpha_{k-1}-\alpha_k}_{1,\infty,[0,1]}}{(\alpha_{k-1}-\alpha_k)!}\frac{\|u^{(k)}\|^{\alpha_k}_{1,\infty,[0,1]}}{(\alpha_k)!}|h|=$$
$$2i|h|A_k(u^{2i};\|u^{(0)}\|_{1,\infty,[0,1]},\|u^{(1)}\|_{1,\infty,[0,1]},\dots,\|u^{(k)}\|_{1,\infty,[0,1]})=$$
$$|h|A_k([u^{2i}]';\|u^{(0)}\|_{1,\infty,[0,1]},\|u^{(1)}\|_{1,\infty,[0,1]},\dots,\|u^{(k)}\|_{1,\infty,[0,1]}).$$

The lemma is proved. □

Lemma 2.2. *Let $N(u)$ be represented by the power series $N(u)=\sum_{j=1}^{\infty}a_ju^{2j}$ then*

$$A_{j+1}(N(u);V_0,\dots,V_j,0)=\frac{1}{(j+1)!}\left\{\frac{d^{j+1}}{dz^{j+1}}[N(f(z))-(f(z)-V_0)N'(V_0)]\right\}_{z=0},\quad j=0,1,\dots,\tag{2.44}$$

with $f(z)=\sum_{j=0}^{\infty}z^jV_j$.

Proof. The proof is obvious. □

Returning to (2.41) and taking into account (2.16), (2.17) we obtain

$$\|u^{(j+1)}\|_{1,\infty}\leqslant$$
$$c_1\Bigg\{\sum_{p=1}^{j}A_{j+1-p}(N(u);\|u^{(0)}\|_{1,\infty,[0,1]},\|u^{(1)}\|_{1,\infty,[0,1]},\dots,\|u^{(j+1-p)}\|_{1,\infty,[0,1]})\|u^{(p)}\|_{1,\infty,[0,1]}+$$
$$h\sum_{p=0}^{j}A_{j-p}(N'(u)u;\|u^{(0)}\|_{1,\infty,[0,1]},\dots,\|u^{(j-p)}\|_{1,\infty,[0,1]})\|u^{(p)}\|_{1,\infty,[0,1]}+$$
$$\frac{1}{(j+1)!}\left[\frac{d^{j+1}}{dz^{j+1}}\left(N\left(\sum_{s=0}^{\infty}z^s\|u^{(s)}\|_{1,\infty,[0,1]}\right)-\sum_{s=1}^{\infty}z^s\|u^{(s)}\|_{1,\infty,[0,1]}N'(\|u^{(0)}\|_{1,\infty,[0,1]})\right)\right]_{z=0}\Bigg\}.\tag{2.45}$$

Introducing in (2.45) the new variables by

$$|h|^{-j}\,\|u^{(j)}\|_{1,\infty}=v_j,\tag{2.46}$$

then changing v_j to V_j and the inequality sign to the equality one, we arrive at the following system of equations:

$$V_{j+1}=c_1\Bigg\{\sum_{p=1}^{j}A_{j+1-p}(N(u);V_0,\dots,V_{j+1-p})V_p+\sum_{p=0}^{j}A_{j-p}(N'(u)u;V_0,\dots,V_{j-p})V_p+$$
$$\frac{1}{(j+1)!}\frac{d^{j+1}}{dz^{j+1}}\left(N\left(\sum_{s=0}^{\infty}z^sV_s\right)\right)_{z=0}-V_{j+1}N'(V_0)\Bigg\},\quad j=0,1,\dots,\quad V_0=v_0=\|u^{(0)}\|_{1,\infty,[0,1]},\tag{2.47}$$

or

$$V_{j+1} = \frac{c_1}{1+c_1 N'(V_0)} \Bigg\{ \sum_{p=1}^{j} A_{j+1-p}(N(u); V_0, \dots, V_{j+1-p}) V_p +$$

$$\sum_{p=0}^{j} A_{j-p}(N'(u)u; V_0, \dots, V_{j-p}) V_p + \frac{1}{(j+1)!} \frac{d^{j+1}}{dz^{j+1}} \Bigg(N\Bigg(\sum_{s=0}^{\infty} z^s V_s \Bigg)\Bigg)_{z=0} \Bigg\}. \tag{2.48}$$

The solution of this system is a majorant for the solution of (2.45), i.e. $v_j \leqslant V_j$, $j = 0, 1, \dots$ Using the method of generating functions, we obtain from (2.48)

$$f(z) - V_0 = \frac{c_1}{1+c_1 N'(V_0)} \{[f(z) - V_0][N(f(z)) - N(V_0)] +$$

$$z f^2(z) N'(f(z)) + N(f(z)) - N(V_0)\}. \tag{2.49}$$

From this equation we can express z as a function of f

$$z = \frac{1}{f^2 N'(f)} \Bigg\{ \Bigg(\frac{1}{\tilde{C}} - N(f) + N(V_0) \Bigg)(f - V_0) - N(f) + N(V_0) \Bigg\},$$

$$V_0 \leqslant f, \quad \tilde{C} = \frac{c_1}{1+c_1 N'(V_0)}, \tag{2.50}$$

and then find f_m, for which z arrives at its maximum $z_m = R$. The condition

$$|h| V_0 \left[N'(V_0)\right]^2 < 1 \tag{2.51}$$

guarantees the existence of f_m because under assumption (2.51) we have

$$z(V_0) = 0, \quad \lim_{f \to \infty} z(f) = 0,$$

$$\frac{d}{df}\left[z(f) f^2 N'(f)\right]\Big|_{f=V_0} = \frac{1}{\tilde{c}} - N'(V_0) = \frac{1}{c_1} = 1 - |h| \, \|u^{(0)}\|_{1,\infty,[0,1]} N'(\|u^{(0)}\|_{1,\infty,[0,1]}) > 0.$$

The value z_m defines the convergence radius of series (2.49), i.e.,

$$R^j V_j = C \frac{1}{(j+1)^{1+\varepsilon}}, \tag{2.52}$$

with an arbitrarily small positive ε. Returning to the old notations, we have

$$\|u^{(j)}\|_{1,\infty,[0,1]} \leqslant \frac{C}{(j+1)^{1+\varepsilon}} \left(\frac{h}{R}\right)^j, \quad j = 0, 1, \dots, \tag{2.53}$$

which leads to the following sufficient convergence condition for the series $f(z) = \sum_{j=0}^{\infty} z^j V_j$:

$$h/R \leqslant 1. \tag{2.54}$$

Thus, we have proved the following assertion.

Theorem 2.1. *Under the assumptions of Lemma* 2.1 *the method* (2.24) *for problem* (2.49) *converges super-exponentially (converges) with the error estimate*

$$\|u - \overset{m}{u}\|_{1,\infty,[0,1]} \leqslant \frac{C}{(1+m)^{1+\varepsilon}} \frac{(h/R)^{m+1}}{1 - h/R} \quad \Bigg(\leqslant C \sum_{j=m+1}^{\infty} \frac{1}{(j+1)^{1+\varepsilon}} \Bigg) \tag{2.55}$$

provided that

$$h < R \quad (h = R). \tag{2.56}$$

3. Examples

Example 3.1. Let us consider the Dirichlet boundary value problem

$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0;\ 1), \quad u(0) = u(1) = 0, \tag{3.1}$$

with $f(x) = \pi^2 \sin \pi x + M(\sin \pi x)^3$ and a given constant $M \geqslant 0$. The exact solution of (3.1) is

$$u(x) = \sin \pi x. \tag{3.2}$$

Problem (3.1) is equivalent to the following nonlinear Fredholm integral equation:

$$u(x) = \int_0^1 G(x,\xi)[-Mu^3(\xi) + f(\xi)]\,d\xi, \quad G(x,\xi) = \begin{cases} x(1-\xi), & x \leqslant \xi \\ \xi(1-x), & \xi < x. \end{cases} \tag{3.3}$$

This equation is of the form (1.1) with $N(u) = M\int_0^1 G(x,\xi)u^3(\xi)\,d\xi$, $F = \int_0^1 G(x,\xi)f(\xi)\,d\xi$.

For $M = 20$ the usual fixed point iteration as well as the usual ADM are divergent. Due to symmetry we apply the FD-method with two steps to the modified problem (3.1):

$$u''(x) - Mu^3(x) = -f(x), \quad x \in (0,\ 0.5), \quad u(0) = 0, \quad u'(0.5) = 0. \tag{3.4}$$

Problem (2.18) for this example has the form

$$\frac{\partial^2 u(x,t)}{\partial x^2} - M\left\{u\left(\frac{1}{4},t\right)^2 + t\left[u(x,t)^2 - u\left(\frac{1}{4},t\right)^2\right]\right\}u(x,t) = -f(x), \quad x \in (0,\ 0.25),$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} - M\left\{\frac{1}{2}\left[u(\frac{1}{4},t) + u\left(\frac{1}{2},t\right)\right]^2 + \right.$$

$$\left. t\left[u(x,t)^2 - \frac{1}{2}\left(u(\frac{1}{4},t) + u(\frac{1}{2},t)\right)^2\right]\right\}u(x,t) = -f(x), \quad x \in (0.25, 0.5),$$

$$u(0,t) = 0, \quad \frac{\partial u(1/2,t)}{\partial x} = 0, \quad [u(x,t)]_{x=0.25} = 0, \quad \left[\frac{\partial u(x,t)}{\partial x}\right]_{x=0.25} = 0 \tag{3.5}$$

The numerical results obtained with Maple are presented in Table 3.1 and Table 3.2, where $\Delta^m(x) = u(x) - \overset{m}{u}(x)$.

Table 3.1. **Two subintervals ($K = 2$)**

x	0.25	0.5
$\Delta^0(x)$	.0109383495	.051202866
$\Delta^1(x)$	.0020822061	.0068667797

Table 3.2. **One subinterval ($K = 1$)**

x	0.25	0.5
$\Delta^0(x)$	.02842879773388	.1384486690937460
$\Delta^1(x)$	.01785603134623	.0287451291955546
$\Delta^2(x)$	.00133217977494	.0010162058541
$\Delta^3(x)$	.00455624274781	.0034293443153
$\Delta^4(x)$	.00003810934901	.00201582469655

Example 3.2. This example goes back to Troesch (see, e.g., [32]) and represents the well-known test problem for numerical software:

$$u'' = \lambda \sinh(\lambda u), \quad x \in (0,1), \quad \lambda > 0, \quad u(0) = 0, \quad u(1) = 1. \tag{3.6}$$

Due to the hyperbolic type of nonlinearity here, a moderate increase in λ leads to tremendous variations in the derivative of the nonlinear part and therefore of the solution. The exact solution of Troesch's test problem can be represented in the form (see, for example, [30])

$$u(x,s) = \frac{2}{\lambda}\operatorname{arcsinh}\left(\frac{s\cdot \operatorname{sn}(\lambda x,k)}{2\cdot \operatorname{cn}(\lambda x,k)}\right), \quad k^2 = 1-\frac{s^2}{4}, \tag{3.7}$$

where $\operatorname{sn}(\lambda x,k)$, $\operatorname{cn}(\lambda x,k)$ are elliptic Jacobi functions and the parameter s satisfies the equation

$$\frac{2}{\lambda}\operatorname{arcsinh}\left(\frac{s\cdot \operatorname{sn}(\lambda,k)}{2\cdot \operatorname{cn}(\lambda,k)}\right) = 1.$$

For example, for the parameter value $\lambda = 10$ $s = 0.35833778463\cdot 10^{-3}$ holds. To our best knowledge a numerical method working for the highest value of $\lambda = 62$ was presented in [8]. Furthermore, it turns out that modern numerical software (Maple, Mathematica) have difficulties even with a numerical solution of the above algebraic problem with respect to s for $\lambda > 62$.

Our goal here is to illustrate the presented approach rather than improve the result $\lambda = 62$. However, this approach in the theory allows one to treat problem (3.6) with any λ using an appropriate number of subdivisions K.

All computations presented in Table 3.3 have been performed using the QD library with quad-double precision (appx. 60 decimal digits of accuracy) and GSL templates for quadrature formulas and matrix calculations. To eliminate the influence of quadrature errors, we used a Runge type estimate with Eps $= 10^{-20}$. The final stop criterion in (2.24) was $\|u^{(j+1)}\|_1 < 10^{-10}$.

Table 3.3. **Troesch test with $K = 4$ and $K = 256$**

m	$\lambda = 1,\ K = 4$	$\lambda = 10,\ K = 256$
	$\|u - \overset{m}{u}\|_1$	$\|u - \overset{m}{u}\|_1$
0	.111608803790162471570291286e-2	.391955453562892425976436496e-1
1	.624289421129990539732878093e-4	.940937627128066975285517882e-3
2	.613274752600453001120360543e-4	.591438908394518928454089104e-4
3	.491115395175933661739635686e-7	.158072882785682348425691183e-5
4	.134193613660290576144888419e-8	.143796397730961836994428146e-5
5	.826036431341936136602905761e-11	.440376842230098521926768794e-7
7		.531844470733545661060519677e-9
8		.710761479514132417214021573e-11

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