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STABILITY OF THE TWO-PARAMETER SET OF NONLOCAL DIFFERENCE SCHEMES

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Abstract — The paper deals with difference schemes for the heat-conduction equation with nonlocal boundary conditions containing two real parameters, α and γ. Such schemes have been investigated for some special parameter values, but the general case was not considered previously. The eigenvalue problem arises as a result of variable division and is solved here explicitly. The so-called reality domains were selected on the (α, γ) plane for which all eigenvalues and eigenfunctions are real. It was demonstrated that the difference schemes in question are symmetrizable in reality domains, that is their transition operators are similar to self-adjoint ones. The necessary and sufficient stability conditions for difference schemes under consideration are obtained with respect to the initial data in the specially constructed norm. The equivalence of the above-mentioned norm to the grid L_2-norm has been proved.

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1. Introduction

The numerical stability problems occupy an outstanding place among A. A. Samarskii investigations on the theory of difference schemes for time-dependent mathematical physics equations, which were extended in monographs [1–4]. The first basic papers of A. A. Samarskii [5, 6] on the theory of difference schemes stability were published by Zh. Vychisl. Mat. i Mat. Fiz. (USSR Comput.Math.and Math.Phys.) in 1967. These papers were preceded by a short article [7] submitted by academician M. V. Keldysh and published in 1965 in Dokl. Akad. Nauk SSSR (Soviet.Math.Docl.). The necessary and sufficient stability conditions for operator-difference schemes were obtained in [8].

It is clear that the elements of the general theory of difference schemes theory presented in the above-mentioned papers, did not arise not by themselves but the result of Samarskii's colossal work in which he constructed wide classes of difference schemes and investigated their convergence. The comprehensive survey [9] made together with A. N. Tichonov is very representative in this respect. It is necessary to note that approximately at the same time some concepts of the general theory of difference schemes and iteration methods theory were advanced by G. I. Marchuk [10], E. G. D'yakonov [11] and other authors. A review of early papers on the stability theory of difference schemes was made in [12].

The tendencies of investigations in the field of difference schemes stability suggested in [5, 6] can be characterized as follows. The difference scheme is defined as an operator-difference

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equation in a finite dimensional space supplied by the Euclidean metric and is regarded as a self-contained subject of inquiry formally independent of some differential equations. The unified canonical form is introduced for all two-layer and three-layer difference schemes, and the necessary and sufficient stability conditions common for a given class of schemes are formulated in terms of operator inequalities. As a result, the stability investigation of each specific difference scheme is reduced to determination of its canonical form and to the check of the corresponding operator inequalities.

In the stability theorems obtained in [5–8], self-adjointness of the main difference operator is supposed. The development of the theory on not self-adjoint schemes, including nonlocal problems, meets certain difficulties connected mainly with the impossibility of representing stability criteria in the form of plain and easily examined operator inequalities.

Investigations of the stability and convergence of weighted difference schemes for the heat conduction equation with nonlocal boundary conditions were first constructed by N. I. Ionkin [13, 14]. The author obtained, under certain restrictions on the mesh size, sufficient stability conditions and in the grid L_2-norm constructed apriori estimates of the difference problem solution with respect to the initial data and the right-hand side. Here the expansion technique of the solution required in the biorthogonal sum on eigenfunctions and adjoint functions of the difference operator was used.

An attempt to embed the stability theory of nonlocal difference problems in the general Samarskii stability theory was made in [15]. A survey of further papers in this direction was made in [16].

The present paper considers the difference schemes approximating the following problem for the heat conduction equation with nonlocal boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad u(x,0) = u_0(x),$$

$$u(0,t) = \alpha u(1,t), \quad \gamma \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t). \tag{1.1}$$

Here the parameters α and γ are given real numbers. The following spectral problem arises after separation of the variables used in (1.1):

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1, \quad X(0) = \alpha X(1), \quad \gamma X'(0) = X'(1). \tag{1.2}$$

In the case that $\alpha = 0$, $\gamma = 1$ (Samarskii — Ionkin problem), the difference schemes for (1.1) were constructed and investigated in [13, 14]. It was demonstrated that in this case the system of eigenfunctions of (1.2) does not form the basis, but it can be complemented to the basis by adjoint functions. The necessary and sufficient stability conditions in the sense of the initial data found for weighted schemes in [15]. Here expansion on the basis from eigenfunctions and adjoint functions of corresponding difference operator was used.

The case $\alpha = 0$, $\gamma \in [0,\ 1)$ was considered in detail in [16]. It was shown that the system of eigenfunctions of the difference analog of problem (1.2) forms the basis in the space of grid functions, and therefore here the theory of symmetrizable difference schemes [17] is applicable. In [18], the case where $\alpha = 0$ and γ is an arbitrary complex number was considered. It was demonstrated that on the complex γ-plane there exist only two singular points, namely $\gamma = \pm 1$, for which the system of eigenfunctions of the difference analog of (1.2) does not form the basis. The difference scheme stability for $\alpha = 0$, $\gamma = -1$ was investigated in [19].

2. The difference scheme

Let us consider the difference scheme for problem (1.1). We introduce the grid $\omega_h = \{x_i = ih\}_{i=0}^{N}$ with a mesh size $h = 1/N$ in the space variable and the time grid $\omega_\tau = \{t_n = n\tau\}_{n=0}^{K}$ with a mesh size $\tau > 0$. For the grid functions $y_i = y(x_i)$ defined on ω_h, let us denote

$$y_{\bar{x},i} = \frac{y_i - y_{i-1}}{h}, \quad y_{x,i} = \frac{y_{i+1} - y_i}{h}, \quad y_{\bar{x}x,i} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}.$$

For the functions $y_i^n = y(x_i, t_n)$ defined on the grid $\omega_h \times \omega_\tau$, let us denote

$$y_{t,i} = (y_i^{n+1} - y_i^n)/\tau, \quad y_i^{(\sigma)} = \sigma y_i^{n+1} + (1-\sigma) y_i^n.$$

We associate with problem (1.1) the weighted difference scheme

$$y_{t,i} - y_{\bar{x}x,i}^{(\sigma)} = 0, \quad i = 1, 2, \ldots, N-1, \quad y_i^0 = u_0(x_i), \quad y_0^{n+1} - \alpha y_N^{n+1} = 0,$$

$$(1 + \alpha\gamma) y_{t,N} + \frac{2}{h}\left(y_{\bar{x},N}^{(\sigma)} - \gamma y_{x,0}^{(\sigma)}\right) = 0. \tag{2.1}$$

Let us represent the difference scheme (2.1) in the canonical form (see [1])

$$B\frac{y_{n+1} - y_n}{\tau} + Ay_n = 0, \quad n = 0, 1, \ldots \tag{2.2}$$

For this purpose we eliminate from (2.1) the values y_0^{n+1} with the aid of the condition $y_0^{n+1} = \alpha y_N^{n+1}$, and consider (2.1) as a system of equations with respect to the unknowns y_i^{n+1}, $i = 1, 2, \ldots, N$. Let us consider the set H of grid functions $y = y(x)$, where $x = x_i = ih$, $i = 1,\ 2,\ \ldots,\ N$. The set H consists of real vectors $y = (y_1\, y_2\, \ldots\, y_N)^{\mathrm{T}}$ and is a linear space of dimension N with coordinate-wise addition and multiplication by the number.

For the functions $y,\ v \in H$, let us denote

$$(y, v]_{\alpha,\gamma} = \sum_{i=1}^{N-1} h y_i v_i + 0,5h(1 + \alpha\gamma) y_N v_N,$$

$$(y, v] = \sum_{i=1}^{N-1} h y_i v_i + 0,5h y_N v_N, \quad (y, v) = \sum_{i=1}^{N} h y_i v_i. \tag{2.3}$$

Note that the bilinear form $(y, v]_{\alpha,\gamma}$ does not defin in general the inner product in H as far as the factor $1 + \alpha\gamma$ is able to be negative.

Let us denote through $y_n = (y_1^n\, y_2^n\, \ldots\, y_N^n)^{\mathrm{T}}$ the functions of the discrete argument $t_n = n\tau$ with values in H. The difference scheme (2.1) has the canonical form (2.2), where the operators A and B act in H and are defined by the following rules:

$$(Ay)_i = -y_{\bar{x}x,i}, \quad i = 1, 2, \ldots, N-1, \quad y_0 = \alpha y_N, \quad (Ay)_N = \frac{2}{h}\left(y_{\bar{x},N} - \gamma y_{x,0}\right), \tag{2.4}$$

$$(By)_i = y_i - \sigma\tau y_{\bar{x}x,i}, \quad i = 1, 2, \ldots, N-1, \quad y_0 = \alpha y_N,$$

$$(By)_N = (1 + \alpha\gamma) y_N + \frac{2}{h}\sigma\tau\left(y_{\bar{x},N} - \gamma y_{x,0}\right). \tag{2.5}$$

The operators A and B are related by the equality $B = \tilde{E} + \sigma\tau A$, where

$$(\tilde{E}y)_i = y_i, \quad i = 1, 2, \ldots, N-1, \quad (\tilde{E}y)_N = (1 + \alpha\gamma) y_N. \tag{2.6}$$

Further we shall identify and denote by the same letter the difference operator and its matrix in the unit basis.

3. The main spectral problem

Let us formulate the eigenproblem arising as a result of the separation of variables in the difference scheme (2.1). The difference scheme (2.2), where $B = \tilde{E} + \sigma\tau A$, we represent as the operator-difference equation

$$\tilde{E}\frac{y_{n+1} - y_n}{\tau} + Ay^{(\sigma)} = 0. \tag{3.1}$$

According to the method of separation of variables, we seek a solution to the equation as a product $y_n = w^n \mu$, where $w^n = w(t_n)$ is the grid function depending only on n, and $\mu \in H$ is an unknown n-independent vector. Taking into account the n-independence of the operators $\tilde{E}$ and A, we arrive at the equality $w_t \tilde{E}\mu + w^{(\sigma)} A\mu = 0$, which is correct if the vector μ and the function w^n satisfy, respectively, the equations

$$A\mu = \lambda \tilde{E}\mu, \tag{3.2}$$

and

$$w_t + \lambda w^{(\sigma)} = 0, \tag{3.3}$$

where λ is a separation constant. The eigenvalue problem (3.2) for operators (2.4) and (2.6) can be written in the difference form

$$\mu_{\bar{x}x,i} + \lambda \mu_i = 0, \quad i = 1, 2, \ldots, N-1, \quad \mu_0 = \alpha \mu_N,$$
$$\frac{2}{h}\left(\gamma \mu_{x,0} - \mu_{\bar{x},N}\right) + (1 + \alpha\gamma)\lambda \mu_N = 0. \tag{3.4}$$

Further we shall construct the solution of the problem (3.4) in explicit form. Note first that problem (3.4) has a zero eigenvalue if and only if at least one of the parameters, α or γ, is equal to 1. The eigenfunction $\mu_i = \alpha + (1 - \alpha)x_i$ corresponds to the zero eigenvalue.

Let us seek the solution of problem (3.4) in the form

$$\mu_j = c_1 \sin(\nu x_j) + c_2 \cos(\nu x_j). \tag{3.5}$$

The direct verification shows that the basic equation holds if

$$\lambda = \frac{4}{h^2}\sin^2\frac{\nu h}{2}. \tag{3.6}$$

The boundary conditions reduce to the following system of equations for coefficients c_1 and c_2:

$$\begin{aligned}&(\alpha \sin\nu)c_1 + (\alpha\cos\nu - 1)c_2 = 0, \quad [(\gamma - \cos\nu)\sin(\nu h) - \alpha\gamma\sin\nu\cos(\nu h)]\, c_1 + \\ &+ [\sin\nu\sin(\nu h) + \gamma(1 - \alpha\cos\nu)\cos(\nu h)]\, c_2 = 0.\end{aligned} \tag{3.7}$$

Making the determinant of this system vanish, we get the characteristic equation $\sin(\nu h)[\alpha + \gamma - (1 + \alpha\gamma)\cos\nu] = 0$. The equality $\sin(\nu h) = 0$ is possible only for a zero eigenvalue λ. Hence for $\lambda \neq 0$ the characteristic equation is equivalent to the following one:

$$(1 + \alpha\gamma)\cos\nu = \alpha + \gamma. \tag{3.8}$$

If $1 + \alpha\gamma = \alpha + \gamma = 0$, i.e., $\gamma = \pm 1$, $\alpha = \mp 1$, equation (3.8) holds for all ν. If $1 + \alpha\gamma = 0$ and $\alpha + \gamma \neq 0$, equation (3.8) has not any solution. Further we eliminate from consideration all parameter values situated on the hyperbola $1+\alpha\gamma = 0$. Then the characteristic equation (3.8) can be written in the form

$$\cos\nu = \frac{\alpha+\gamma}{1+\alpha\gamma}. \tag{3.9}$$

Equation (3.8) has only real roots under the condition

$$\left|\frac{\alpha+\gamma}{1+\alpha\gamma}\right| \leqslant 1, \tag{3.10}$$

and it has complex roots under the condition $|(\alpha+\gamma)/(1+\alpha\gamma)| > 1$. It is easy to see that inequality (3.10) is equivalent to the conditions

$$(1-\gamma^2)(1-\alpha^2) \geqslant 0. \tag{3.11}$$

In turn inequality (3.11) holds if and only if either $(1-\gamma^2) \geqslant 0, \quad (1-\alpha^2) \geqslant 0$ or $(1-\gamma^2) \leqslant 0, \quad (1-\alpha^2) \leqslant 0$. Thus, *the characteristic equation* (3.9) *has real roots if and only if*

$$\text{either } |\gamma| \leqslant 1, \quad |\alpha| \leqslant 1 \text{ or } |\gamma| \geqslant 1, \quad |\alpha| \geqslant 1. \tag{3.12}$$

Equation (3.9) *has complex roots if and only if either* $|\gamma| > 1$, $|\alpha| < 1$ *or* $|\gamma| < 1$, $|\alpha| > 1$.

In Fig. 3.1 the set of (γ,α)-plane points, for which equation (3.9) has real roots is shaded. All the other points correspond to complex roots. The curve represents the function $\alpha = -1/\gamma$. We shall call the reality domains the sets of parameters (γ,α) corresponding to the real roots of the characteristic equation. Let us enumerate the reality domains as in Fig. 3.1, namely domain 0 is $\{|\gamma| < 1, \quad |\alpha| < 1\}$, domain 1 is $\{\gamma > 1, \quad \alpha > 1\}$, domain 2 is $\{\gamma < -1, \quad \alpha > 1\}$, domain 3 is $\{\gamma < -1, \quad \alpha < -1\}$, domain 4 is $\{\gamma > 1, \quad \alpha < -1\}$. In turn, domain 0 is divided into subdomains $0a\,\{0 \leqslant \gamma < 1,\ 0 \leqslant \alpha < 1\}$, $0b\,\{-1 < \gamma \leqslant 0,\ 0 \leqslant \alpha < 1\}$, $0c\,\{-1 < \gamma \leqslant 0,\ -1 < \alpha \leqslant 0\}$ and $0d\,\{0 \leqslant \gamma < 1, -1 < \alpha \leqslant 0\}$.

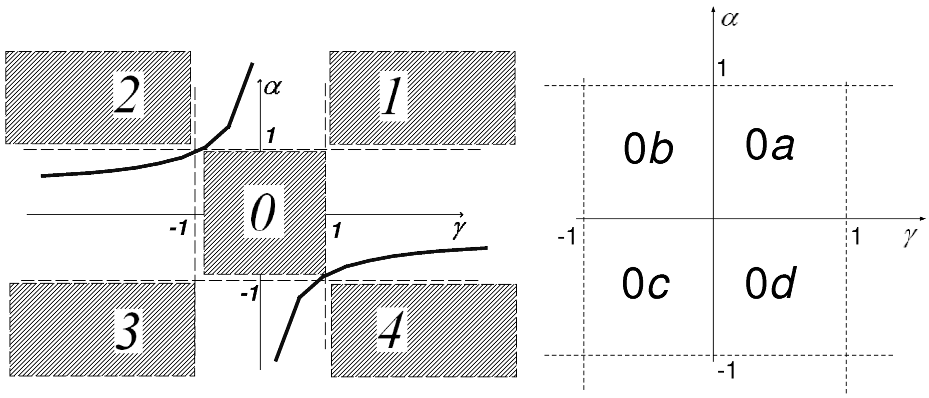

Fig. 3.1. Reality domains

4. Real case

Suppose now that the reality conditions are fulfilled with signs of the strict inequality,

$$\text{either } \ |\gamma| < 1, \quad |\alpha| < 1 \quad \text{or} \quad |\gamma| > 1, \quad |\alpha| > 1. \tag{4.1}$$

Then $1+\alpha\gamma \neq 0$ and

$$\left|\frac{\alpha+\gamma}{1+\alpha\gamma}\right| < 1, \quad (1-\alpha^2)(1-\gamma^2) > 0, \quad \frac{1-\gamma^2}{1-\alpha^2} > 0. \tag{4.2}$$

All the roots of Eq. (3.9) are real, as defined by the formulas

$$\nu_{2k-1} = 2\pi k - \psi, \quad \nu_{2k} = 2\pi k + \psi, \quad \psi = \arccos \frac{\alpha + \gamma}{1 + \alpha\gamma}. \tag{4.3}$$

Let us turn now to system (3.7) which defines the constants c_1 and c_2. We shall first consider the case $\nu = 2\pi k - \psi$. Then $\cos\nu = \cos\psi$ and $\sin\nu = -\sin\psi$. Because $\psi \in (0,\ \pi)$, we have $\sin\nu < 0$.

For $\alpha = 0$ we get that c_1 is arbitrary and $c_2 = 0$. The eigenfunction (3.5) can be represented in the form

$$\mu_j = c_1 \sin((2\pi n - \psi)x_j), \quad \psi = \arccos\gamma, \quad \alpha = 0.$$

For $\alpha \neq 0$, the first equation of system (3.7) can be written as

$$c_1 = \frac{1 - \alpha\cos\nu}{\alpha\sin\nu} c_2$$

hence

$$\mu_j = c_3 \left[\frac{1 - \alpha^2}{1 + \alpha\gamma} \sin(\nu x_j) + \alpha \sin\nu \cos(\nu x_j) \right].$$

Thus, in the case $\nu = 2\pi k - \psi$ we get within a constant

$$\mu_j = \frac{1 - \alpha^2}{1 + \alpha\gamma} \sin(\nu x_j) - \alpha \sin\psi \cos(\nu x_j). \tag{4.4}$$

Lemma 4.1. *The following representation is valid within the constant multiplier:*

$$\mu_j = \tilde{a}\cos(\nu x_j) + \tilde{b}\sin(\nu x_j) \tag{4.5}$$

where $\nu = 2\pi k - \psi$ *and*

$$\tilde{a} = -\alpha \sqrt{\frac{1 - \gamma^2}{1 - \alpha^2\gamma^2}}, \quad \tilde{b} = \frac{1 - \alpha^2}{1 + \alpha\gamma} \sqrt{\frac{1 + \alpha\gamma}{(1 - \alpha^2)(1 - \alpha\gamma)}}, \tag{4.6}$$

so that $\tilde{a}^2 + \tilde{b}^2 = 1$.

Proof. Note first of all that in the reality domains all subradical expressions, including those in (4.6), are positive. Further, let us write (4.4) in the form

$$\mu_j = a\cos(\nu x_j) + b\sin(\nu x_j),$$

where

$$a = \alpha\sin\nu = -\alpha\sin\psi, \quad b = \frac{1 - \alpha^2}{1 + \alpha\gamma}. \tag{4.7}$$

From the equalities

$$\cos\psi = \frac{\alpha + \gamma}{1 + \alpha\gamma}, \quad \sin^2\psi = \frac{(1 - \alpha^2)(1 - \gamma^2)}{(1 + \alpha\gamma)^2}$$

we get

$$a^2 + b^2 = \frac{(1 - \alpha^2)(1 - \alpha\gamma)}{1 + \alpha\gamma}, \qquad \frac{1}{\sqrt{a^2 + b^2}} = \sqrt{\frac{1 + \alpha\gamma}{(1 - \alpha^2)(1 - \alpha\gamma)}}.$$

From here and (4.7) we get within the constant multiplier that

$$\mu_j = \frac{a}{\sqrt{a^2+b^2}}\cos(\nu x_j) + \frac{b}{\sqrt{a^2+b^2}}\sin(\nu x_j) =$$

$$-\alpha\sin\psi\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}}\cos(\nu x_j) + \left(\frac{1-\alpha^2}{1+\alpha\gamma}\right)\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}}\sin(\nu x_j).$$

Further, as far as $0<\psi<\pi$, it follows from the formula for $\sin^2\psi$ that

$$\sin\psi = \sqrt{\frac{(1-\alpha^2)(1-\gamma^2)}{(1+\alpha\gamma)^2}}. \tag{4.8}$$

Hence

$$\mu_j = -\alpha\sqrt{\frac{(1-\alpha^2)(1-\gamma^2)}{(1+\alpha\gamma)^2}}\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}}\cos(\nu x_j)+$$

$$\left(\frac{1-\alpha^2}{1+\alpha\gamma}\right)\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}}\sin(\nu x_j) =$$

$$-\alpha\sqrt{\frac{1-\gamma^2}{1-\alpha^2\gamma^2}}\cos(\nu x_j) + \left(\frac{1-\alpha^2}{1+\alpha\gamma}\right)\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}}\sin(\nu x_j),$$

at was to be proved. □

Remark 4.1. In the case $\nu = 2\pi k + \psi$ instead of 4.1 the following statement is valid. *The representation $\mu_j = \tilde{a}\cos(\nu x_j) + \tilde{b}\sin(\nu x_j)$ with $\nu = 2\pi k + \psi$ and*

$$\tilde{a} = \alpha\sqrt{\frac{1-\gamma^2}{1-\alpha^2\gamma^2}}, \quad \tilde{b} = \frac{1-\alpha^2}{1+\alpha\gamma}\sqrt{\frac{1+\alpha\gamma}{(1-\alpha^2)(1-\alpha\gamma)}},$$

is valid within the constant multiplier so that $\tilde{a}^2 + \tilde{b}^2 = 1$.

Thus, only a change of the sign of the coefficient $\tilde{a}$ occurs when we pass from $\nu = 2\pi k - \psi$ to $\nu = 2\pi k + \psi$. Let us introduce the angle

$$\varphi = \arccos\sqrt{\frac{1-\alpha^2}{1-\alpha^2\gamma^2}} \tag{4.9}$$

and write with its aid the eigenfunctions of problem (3.4).

Lemma 4.2. *Let $\nu = 2\pi k - \psi$ and the angle φ be determinated by* (4.9). *Then the eigenfunctions of problem* (3.4) *within the constant multiplier are defined by the formulas*

$$\mu_j = \begin{cases} \sin(\nu x_j - \varphi) & \text{in domains } 0a, 0b, 2, 3, \\ \sin(\nu x_j + \varphi) & \text{in domains } 0c, 0d, 1, 4. \end{cases}$$

Proof. It follows from (4.9) that

$$\cos\varphi = \sqrt{\frac{1-\alpha^2}{1-\alpha^2\gamma^2}} > 0, \quad 0 < \varphi \leqslant \pi/2, \quad \sin\varphi = |\alpha|\sqrt{\frac{1-\gamma^2}{1-\alpha^2\gamma^2}} > 0. \tag{4.10}$$

From here and (4.6) we get $\tilde{a} = -\frac{\alpha}{|\alpha|}\sin\varphi = -\operatorname{sgn}\alpha\cdot\sin\varphi$. Further,

$$\tilde{b} = \frac{1-\alpha^2}{1+\alpha\gamma}\sqrt{\frac{(1-\alpha^2\gamma^2)}{(1-\alpha^2)(1-\alpha\gamma)^2}} = \frac{(1-\alpha^2)(1-\alpha\gamma)}{1-\alpha^2\gamma^2}\sqrt{\frac{1}{\cos^2\varphi(1-\alpha\gamma)^2}} =$$

$$(1-\alpha\gamma)\cos^2\varphi\sqrt{\frac{1}{\cos^2\varphi(1-\alpha\gamma)^2}} = \operatorname{sgn}(1-\alpha\gamma)\cdot\cos\varphi.$$

Hence, the following equalities are proved:

$$\tilde{a} = -\operatorname{sgn}\alpha\cdot\sin\varphi, \quad \tilde{b} = \operatorname{sgn}(1-\alpha\gamma)\cdot\cos\varphi. \tag{4.11}$$

Let us examine expressions (4.11) separately in each reality domain (see Fig. 3.1). In the subdomains of the domain 0 the coefficients are as follows:

	$\tilde{a}$	$\tilde{b}$
$0a$	$-\sin\varphi$	$\cos\varphi$
$0b$	$-\sin\varphi$	$\cos\varphi$
$0c$	$\sin\varphi$	$\cos\varphi$
$0d$	$\sin\varphi$	$\cos\varphi$

Consequently, the eigenfunction can be written in the form

	μ_j
$0a$	$-\sin\varphi\cos(\nu x_j) + \cos\varphi\sin(\nu x_j) = \sin(\nu x_j - \varphi)$
$0b$	$-\sin\varphi\cos(\nu x_j) + \cos\varphi\sin(\nu x_j) = \sin(\nu x_j - \varphi)$
$0c$	$\sin\varphi\cos(\nu x_j) + \cos\varphi\sin(\nu x_j) = \sin(\nu x_j + \varphi)$
$0d$	$\sin\varphi\cos(\nu x_j) + \cos\varphi\sin(\nu x_j) = \sin(\nu x_j + \varphi)$

In domain 1 we get $\alpha > 0$, $1-\alpha\gamma < 0$, hence $\tilde{a} = -\sin\varphi$, $\tilde{b} = -\cos\varphi$ $\mu_j = -\sin(\nu x_j+\varphi)$. Analogously, $\mu_j = \sin(\nu x_j - \varphi)$ in domain 2, $\mu_j = -\sin(\nu x_j - \varphi)$ in domain 3, and $\mu_j = \sin(\nu x_j + \varphi)$ in domain 4. □

Remark 4.2. In the case $\nu = 2\pi k + \psi$ instead of Lemma 4.2 the following statement is valid. *Let $\nu = 2\pi k + \psi$ and the angle φ be defined as* (4.9). *Then the eigenfunctions of* (3.4) *are defined within the constant multiplier as*

$$\mu_j = \begin{cases} \sin(\nu x_j + \varphi) & \text{in domains } 0a, 0b, 2, 3, \\ \sin(\nu x_j - \varphi) & \text{in domains } 0c, 0d, 1, 4. \end{cases}$$

Let us formulate the main result of this section. *Under the restrictions*

$$\textit{either } |\gamma| < 1, \quad |\alpha| < 1 \quad \textit{or} \quad |\gamma| > 1, \quad |\alpha| > 1$$

let us introduce angles ψ *and* φ *by the rules*

$$\psi = \arccos\left(\frac{\alpha+\gamma}{1+\alpha\gamma}\right), \quad \varphi = \arccos\sqrt{\frac{1-\alpha^2}{1-\alpha^2\gamma^2}}. \tag{4.12}$$

Then the eigenvalues and eigenfunctions of problem (2.2) *are defined by the following formulas:*

$$\lambda_0 = \frac{4}{h^2}\sin^2\frac{\psi h}{2}, \quad \mu^{(0)}(x_j) = \begin{cases} \sin(\psi x_j + \varphi) & \text{in domains } 0a,\ 0b,\ 2,\ 3, \\ \sin(\psi x_j - \varphi) & \text{in domains } 0c,\ 0d,\ 1,\ 4, \end{cases} \tag{4.13}$$

$$\begin{cases} \lambda_{2k-1} = \dfrac{4}{h^2}\sin^2\dfrac{(2\pi k-\psi)h}{2}, \\ \mu^{(2k-1)}(x_j) = \begin{cases} \sin((2\pi k-\psi)x_j - \varphi) & \text{in domains } 0a,\ 0b,\ 2,\ 3, \\ \sin((2\pi k-\psi)x_j + \varphi) & \text{in domains } 0c,\ 0d,\ 1,\ 4, \end{cases} \end{cases} \tag{4.14}$$

$$\begin{cases} \lambda_{2k} = \dfrac{4}{h^2}\sin^2\dfrac{(2\pi k+\psi)h}{2}, \\ \mu^{(2k)}(x_j) = \begin{cases} \sin((2\pi k+\psi)x_j + \varphi) & \text{in domains } 0a,\ 0b,\ 2,\ 3, \\ \sin((2\pi k+\psi)x_j - \varphi) & \text{in domains } 0c,\ 0d,\ 1,\ 4. \end{cases} \end{cases} \tag{4.15}$$

5. Symmetrizability

The strong inequalities $0 < \psi < \pi$ hold in the reality domains. It follows that problem (3.4) has not multiple eigenvalues. The inequalities $\lambda_{2k-1} < \lambda_{2k} < \lambda_{2k+1}$ hold. The following statement defines more exactly the domain of realizability of the inequalities in question.

Lemma 5.1. *The following inequalities hold:*

$$\lambda_{2k-1} < \lambda_{2k},$$

where $k = 1, 2, \ldots, (N-1)/2$, *if* N *is odd,* $k = 1, 2, \ldots, N/2 - 1$, *if* N *is even,*

$$\lambda_{2k} < \lambda_{2k+1},$$

where $k = 0, 1, \ldots, (N-3)/2$, *if* N *is odd,* $k = 0, 1, \ldots, N/2 - 1$, *if* N *is even.*

Proof. It is sufficient to prove the inequalities $\lambda_{2k-1} < \lambda_{2k} < \lambda_{2k+1}$. Let us prove primarily the first one. We shall use the identity

$$\lambda_{2k} - \lambda_{2k-1} = \frac{2}{h^2}\left(\cos(2\pi kh - \psi h) - \cos(2\pi kh + \psi h)\right) = \frac{4}{h^2}\sin(2\pi kh)\sin(\psi h).$$

In the reality domains we have $0 < \psi < \pi$, hence $0 < \psi h < \pi$ and $\sin(\psi h) > 0$. Further, denote by $m = (N-1)/2$ if N is odd, and by $m = N/2 - 1$ if N is even. For N odd and

$0 < k \leqslant m$ we have $0 < 2\pi kh \leqslant 2\pi mh = (N-1)\pi h = \pi - \pi h < \pi$, hence $\sin(2\pi kh) > 0$ and $\lambda_{2k} > \lambda_{2k-1}$. For N even and $0 < k \leqslant m$ we have

$$0 < 2\pi kh \leqslant 2\pi mh = (N-2)\pi h = \pi - 2\pi h < \pi,$$

hence $\sin(2\pi kh) > 0$ and $\lambda_{2k} > \lambda_{2k-1}$. Let us prove now the inequality $\lambda_{2k+1} > \lambda_{2k}$. Here the identity

$$\lambda_{2k+1} - \lambda_{2k} = \frac{4}{h^2} \sin((2k+1)\pi h) \sin((\pi - \psi)h)$$

can be used. Since $0 < \psi < \pi$, the inequalities $0 < (\pi - \psi)h < \pi$s hold, hence $\sin((\pi-\psi)h) > 0$. Further, for N odd and $0 \leqslant k < m$ we have

$$0 < \pi h \leqslant (2k+1)\pi h) < N\pi h = \pi, \quad \sin((2k+1)\pi h) > 0,$$

so $\lambda_{2k+1} > \lambda_{2k}$. For N even we require $0 \leqslant k \leqslant m$. Then

$$0 < \pi h \leqslant (2k+1)\pi h) \leqslant (N-1) = \pi - h < \pi,$$

hence $\sin((2k+1)\pi h) > 0$ and $\lambda_{2k+1} > \lambda_{2k}$. □

It follows from Lemma 5.1 that under conditions (4.1) all eigenvalues of problem (3.4) are distinct, therefore the set of eigenfunctions is a basis in H. Let us form the matrix

$$M = \left[\mu^{(0)}\ \mu^{(1)}\ \cdots\ \mu^{(N-1)}\right], \tag{5.1}$$

whose columns are the eigenvectors of problem (3.4). As far as the system of eigenvectors is linearly independent, the matrix M is invertible.

Lemma 5.2. *Under conditions* (4.1) *the difference scheme* (2.1) *is symmetrizable.*

Proof. Let us write the spectral problem (3.4) in the matrix form $A\mu^{(k)} = \lambda_k \tilde{E}\mu^{(k)}$, or $\left(\tilde{E}^{-1}A\right)\mu^{(k)} = \lambda_k \mu^{(k)}$, $k = 1, 2, \ldots, N-1$. The system of equations, presented here is equivalent to the matrix equation

$$\left(\tilde{E}^{-1}A\right) M = M\Lambda, \tag{5.2}$$

where M is matrix (5.1) and $\Lambda = \operatorname{diag}(\lambda_0,\ \lambda_1,\ \cdots,\ \lambda_{N-1})$ is a diagonal matrix with eigenvalues of problem (3.4) on the main diagonal. Relation (5.2) can be written as the equality $\tilde{E}^{-1}A = M\Lambda M^{-1}$, which means that the matrix $\tilde{E}^{-1}A$ is similar to the diagonal matrix Λ. It follows that the transition operator $S = E - \tau B^{-1}A$ of difference scheme (2.1) is similar to some diagonal operator, hence the difference scheme (2.1) is symmetrizable. In fact, as far as $B = \tilde{E} + \sigma\tau A$ the transition operator is equal to

$$S = E - \tau\left(\tilde{E} + \sigma\tau A\right)^{-1} A = E - \tau\left(E + \sigma\tau\left(\tilde{E}^{-1}A\right)\right)^{-1}\left(\tilde{E}^{-1}A\right).$$

Substituting here instead $\tilde{E}^{-1}A$ the equal operator $M\Lambda M^{-1}$, we find $S = M\tilde{S}M^{-1}$, where $\tilde{S} = E - \tau(E + \sigma\tau\Lambda)^{-1}\Lambda$ is a diagonal operator. Note that $\tilde{S}$ is a self-adjoint operator in the sense of the inner product $(y, v]$ defined as (2.3). So, there exists a self-adjoint operator $\tilde{S}$ for which the equality $M^{-1}SM = \tilde{S}$ holds. This means that the difference scheme (2.1) is symmetrizable. □

6. Stability

Let H be a linear space of the real functions $y = y(x_i)$ where $x = x_i = ih$, $i = 1, 2, \ldots, N$. Let us introduce in H the scalar product

$$(y, v] = \sum_{i=1}^{N-1} hy_i v_i + 0,5hy_N v_N \tag{6.1}$$

and the norm $||y|| = \sqrt{(y, y]}$. Suppose that the self-adjoint positive operator $D : H \to H$ is prescribed. By the space H_D is meant the linear space H with norm $|y]|_D = \sqrt{(Dy, y]}$. The difference scheme (2.1) is said to be *stable in* H_D if the inequalities $||y_{n+1}]|_D \leqslant ||y_n]|_D$, $n = 0,\ 1,\ \ldots$ hold for its solution satisfying arbitrary initial data $y_0 \in H$. It was demonstrated above that under conditions (4.1) the eigensystem $\left\{\mu^{(k)}\right\}_{k=0}^{N-1}$ of problem (3.4) forms a basis in H. Any element $y \in H$ can be represented uniquely as a linear combination

$$y = \sum_{k=0}^{N-1} c_k \mu^{(k)}. \tag{6.2}$$

For a given $y \in H$ the coefficients c_k can be reestablished in the following way. Let us introduce the vector $c = (c_0\, c_1\, \ldots\, c_{N-1})^{\mathrm{T}}$ and write expansion (6.2) in vector form as $y = Mc$, where M is matrix (5.1). Then $c = M^{-1}y$.

The matrix M defines in H a linear operator, which we also denote by M. By M^* is meant the operator, conjugate to M in the sense of the scalar product (6.1). We shall use the energetic norm $||y]|_D = \sqrt{(Dy, y]}$ generated by the self-adjoint positive operator

$$D = (hMM^*)^{-1}. \tag{6.3}$$

Note that for such an operator the equalities

$$h(Dy, y] = (M^{*-1}M^{-1}y, y] = (M^{-1}y, M^{-1}y] = (c, c],$$

hold, consequently,

$$(Dy, y] = c_0^2 + c_1^2 + \cdots + c_{N-2}^2 + 0,5c_{N-1}^2. \tag{6.4}$$

The following theorem shows that condition (6.5) is necessary and sufficient for stability of scheme (2.1) if the parameters α and γ satisfy the strong reality conditions (4.1).

Theorem 6.1. *Let conditions* (4.1) *be fulfilled. If the difference scheme* (2.1) *is stable in some linear space* H_D*, then the inequality*

$$\sigma \geqslant \frac{1}{2} - \frac{h^2}{4\tau} \tag{6.5}$$

holds. Conversely, if conditions (6.5) *and* (4.1) *are fulfilled, then scheme* (2.1) *is stable in* H_D*, where the operator* D *is defined by* (6.3) *and* M *is matrix* (5.1).

Proof. Let us write equation (3.3) as $w^{n+1} = qw^n$, where $q = (1 - (1-\sigma)\tau\lambda)/(1 + \sigma\tau\lambda)$. It is necessary for the scheme stability that $|q| \leqslant 1$ for all eigenvalues λ. In particular, for N even and $\lambda = \lambda_{N/2} = 4/h^2$ the condition $|q| \leqslant 1$ coincides with inequality (6.5). For sufficient proof we shall use the fact of symmetrizability of the difference scheme (2.1). It was proved in [17] that if for the difference scheme (2.2) the equality $\tilde{S} = KSK^{-1}$, where $\tilde{S}$ is a self-adjoint operator, holds, then the scheme is stable in H_{K^*K} under the condition $-E \leqslant \tilde{S} \leqslant E$. In the case of scheme (2.1), we have by Lemma 5.2 $K = M^{-1}$ and $\tilde{S} = E - \tau\left(E + \sigma\tau\Lambda\right)^{-1}\Lambda$. Consequently, scheme (2.1) is stable in H_D with $D = (MM^*)^{-1}$ or, which is the same, with $D = (hMM^*)^{-1}$ under the condition

$$-E \leqslant E - \tau\left(E + \sigma\tau\Lambda\right)^{-1}\Lambda \leqslant E,$$

which is equivalent to

$$-1 \leqslant 1 - \tau\frac{\lambda_k}{1 + \sigma\tau\lambda_k} \leqslant 1$$

for all eigenvalues λ_k of problem (3.4). As far as $\lambda_{N/2} = 4/h^2$ is a maximal eigenvalue, all the other inequalities are a corollary of (6.5). □

7. The conjugate spectral problem

Let us construct the problem conjugate in a certain sense to (3.4). First of all we rewrite (3.4) as $\tilde{A}\mu = \lambda\mu$, where $\mu = (\mu_1\ \mu_2\ \cdots\ \mu_N)^{\mathrm{T}}$ and the operator $\tilde{A}$ is defined as

$$(\tilde{A}\mu)_j = -\mu_{\bar{x}x,j}, \quad j = 1,\ 2,\ \ldots\ N-1, \quad \mu_0 = \alpha\mu_N,$$

$$(\tilde{A}\mu)_N = -\frac{2}{h(1+\alpha\gamma)}(\gamma\mu_{x,0} - \mu_{\bar{x},N}). \tag{7.1}$$

It follows from the definition of the bilinear form $(y, v]_{\alpha,\gamma}$ (see (2.3)) that

$$(\tilde{A}\mu, v]_{\alpha,\gamma} = -\sum_{i=1}^{N-1} h\mu_{\bar{x}x,i}v_i - (\gamma\mu_{x,0}v_N - \mu_{\bar{x},N}v_N) \tag{7.2}$$

for all $\mu, v \in H$. For the transformation of the first sum we use the difference analog of the Green formula (see [2, p. 101])

$$-\sum_{i=1}^{N-1} h\mu_{\bar{x}x,i}v_i = -\sum_{i=1}^{N-1} h\mu_i v_{\bar{x}x,i} - v_N\mu_{\bar{x},N} + v_{\bar{x},N}\mu_N + v_0\mu_{x,0} - v_{x,0}\mu_0. \tag{7.3}$$

Substituting (7.3) into (7.2), we derive

$$(\tilde{A}\mu, v]_{\alpha,\gamma} = -\sum_{i=1}^{N-1} h\mu_i v_{\bar{x}x,i} + \mu_N(v_{\bar{x},N} - \alpha v_{x,0}) + \mu_{x,0}(v_0 - \gamma v_N).$$

As far as $v_0 = \gamma v_N$, the previous equality can be written in the form

$$(\tilde{A}\mu, v]_{\alpha,\gamma} = -\sum_{i=1}^{N-1} h\mu_i v_{\bar{x}x,i} + 0,5h(1+\alpha\gamma)\mu_N\left(\frac{2}{h(1+\alpha\gamma)}(v_{\bar{x},N} - \alpha v_{x,0})\right).$$

Consequently, the operator $\tilde{A}^*$ acting in H and defined by the rule

$$(\tilde{A}^*v)_j = -v_{\bar{x}x,j}, \quad j = 1,\ 2,\ \ldots,\ N, \quad v_0 = \gamma v_N,$$

$$(\tilde{A}^*v)_N = -\frac{2}{h(1+\alpha\gamma)}\left(\alpha v_{x,0} - v_{\bar{x},N}\right) \tag{7.4}$$

is conjugate in the sense of the bilinear form $(y, v]_{\alpha,\gamma}$ to the operator $\tilde{A}$.

The eigenvalue problem $\tilde{A}^* v = \lambda v$ or in the difference form

$$-v_{\bar{x}x,j} = \lambda v_j, \quad j = 1, 2, \ldots, N-1, \quad v_0 = \gamma v_N,$$

$$\frac{2}{h}\left(v_{\bar{x},N} - \alpha v_{x,0}\right) = (1 + \alpha\gamma)\lambda v_N, \tag{7.5}$$

is said to be *the conjugate eigenvalue problem*. Therefore *changing the places of the parameters* α *and* γ *in the problem* (3.4), *we obtain the conjugate problem*.

According to what has been said in the section 4, we can draw the following conclusions about the eigenvalues and eigenfunctions of the conjugate problem. As a result of the simultaneous change of α to γ and γ to α, the reality conditions are still valid. Since the expression for the angle ψ contains the parameters α and γ symmetrically, the spectrum of the conjugate problem coincides with the spectrum of the original problem (3.4). Further, to describe the eigenfunctions of the conjugate problem, we introduce the angles

$$\psi = \arccos\left(\frac{\alpha + \gamma}{1 + \alpha\gamma}\right), \quad \tilde{\varphi} = \arccos\sqrt{\frac{1 - \gamma^2}{1 - \alpha^2\gamma^2}}. \tag{7.6}$$

In accordance with (4.13) — (4.15) under the reality conditions (4.1) the eigenvalues and the corresponding eigenfunctions of the conjugate problem (7.5) are defined as follows:

$$\lambda_0 = \frac{4}{h^2}\sin^2\frac{\psi h}{2}, \quad v_0(x_j) = \begin{cases} \sin(\psi x_j + \tilde{\varphi}) & \text{in domains } 0_a,\ 0_d,\ 3,\ 4, \\ \sin(\psi x_j - \tilde{\varphi}) & \text{in domains } 0_b,\ 0_c,\ 1,\ 2 \end{cases} \tag{7.7}$$

$$\begin{cases} \lambda_{2l-1} = \dfrac{4}{h^2}\sin^2\dfrac{(2\pi l - \psi)h}{2}, \\ v_{2l-1}(x_j) = \begin{cases} \sin((2\pi l - \psi)x_j - \tilde{\varphi}) & \text{in domains } 0_a,\ 0_d,\ 3,\ 4, \\ \sin((2\pi l - \psi)x_j + \tilde{\varphi}) & \text{in domains } 0_b,\ 0_c,\ 1,\ 2, \end{cases} \end{cases} \tag{7.8}$$

$$\begin{cases} \lambda_{2l} = \dfrac{4}{h^2}\sin^2\dfrac{(2\pi l + \psi)h}{2}, \\ v_{2l}(x_j) = \begin{cases} \sin((2\pi l + \psi)x_j + \tilde{\varphi}) & \text{in domains } 0_a,\ 0_d,\ 3,\ 4, \\ \sin((2\pi l + \psi)x_j - \tilde{\varphi}) & \text{in domains } 0_b,\ 0_c,\ 1,\ 2. \end{cases} \end{cases} \tag{7.9}$$

Formulas (7.7)–(7.9) were derived from (4.13)–(4.15) by changing α to γ and γ to α.

It is convenient for furthest to eliminate the angle $\tilde{\varphi}$ in (7.7)–(7.9) and express the eigenfunctions in terms of the parameters ψ and φ defined by (4.12). It is easy to show that in all reality domains the equalities

$$\cos\psi = \frac{\alpha + \gamma}{1 + \alpha\gamma}, \quad \cos\varphi = \sqrt{\frac{1 - \alpha^2}{1 - \alpha^2\gamma^2}}, \quad \cos\tilde{\varphi} = \sqrt{\frac{1 - \gamma^2}{1 - \alpha^2\gamma^2}},$$

$$\sin\psi = \frac{\sqrt{(1 - \alpha^2)(1 - \gamma^2)}}{|1 + \alpha\gamma|}, \quad \sin\varphi = |\alpha|\sqrt{\frac{1 - \gamma^2}{1 - \alpha^2\gamma^2}}, \quad \sin\tilde{\varphi} = |\gamma|\sqrt{\frac{1 - \alpha^2}{1 - \alpha^2\gamma^2}}$$

hold. From here after simple calculations we get

$$\cos(\psi+\tilde\varphi)=\begin{cases}\sin\varphi \ \text{ in domain } \ 0a,\\ -\sin\varphi \text{ in domains } 0d,3,4,\end{cases} \quad \sin(\psi+\tilde\varphi)=\begin{cases}\cos\varphi \ \text{ in domains } \ 0a,0d,4,\\ -\cos\varphi \ \text{ in domain } \ 3,\end{cases}$$

$$\cos(\psi-\tilde\varphi)=\begin{cases}\sin\varphi \ \text{ in domains } \ 0b,1,2,\\ -\sin\varphi \ \text{ in domain } \ 0c,\end{cases} \quad \sin(\psi-\tilde\varphi)=\begin{cases}\cos\varphi \ \text{ in domains } \ 0b,0c,2,\\ -\cos\varphi \ \text{ in domain } \ 1.\end{cases}$$

Next, using (7.7)–(7.9), we arrive at the desired representation for the eigenfunctions of the conjugate problem

$$v^{(0)}(x_j)=\begin{cases}a_0\cos(\psi x_j-(\psi+\varphi)) \ \text{ in domains } \ 0a,0b,2,3,\\ a_0\cos(\psi x_j-(\psi-\varphi)) \ \text{ in domains } \ 0c,0d,1,4,\end{cases} \tag{7.10}$$

$$v^{(2l-1)}(x_j)=\begin{cases}a_{2l-1}\cos((2\pi l-\psi)x_j+(\psi+\varphi)) \ \text{ in domains } \ 0a,0b,2,3,\\ a_{2l-1}\cos((2\pi l-\psi)x_j+(\psi-\varphi)) \ \text{ in domains } \ 0c,0d,1,4,\end{cases} \tag{7.11}$$

$$v^{(2l)}(x_j)=\begin{cases}a_{2l}\cos((2\pi l+\psi)x_j-(\psi+\varphi)) \ \text{ in domains } \ 0a,0b,2,3,\\ a_{2l}\cos((2\pi l+\psi)x_j-(\psi-\varphi)) \ \text{ in domains } \ 0c,0d,1,4.\end{cases} \tag{7.12}$$

The constants a_{2l-1} and a_{2l} will be specified later from normalization consideartions.

It can be proved by direct calculations that the systems $\{\mu^{(k)}\}_{k=0}^{N-1}$ and $\{v^{(l)}\}_{l=0}^{N-1}$ are biorthogonal, i. e., $(\mu^{(k)},v^{(l)}]_{\alpha,\gamma}=0$ for $k\neq l$. Let us consider, for example, the sum $(\mu^{(2k-1)},v^{(2l-1)}]_{\alpha,\gamma}$ in domains $0a$, $0b$, 2, 3. In these domains

$$\mu_j^{(2k-1)}=\sin((2\pi k-\psi)x_j-\varphi), \quad v_j^{(2l-1)}=\cos((2\pi l-\psi)x_j+(\psi+\varphi))$$

for $j=0,1,\dots,N$, with

$$\mu_0^{(2k-1)}=-\sin\varphi, \quad \mu_N^{(2k-1)}=-\sin(\psi+\varphi), \quad v_0^{(2l-1)}=\cos(\psi+\varphi), \quad v_N^{(2l-1)}=\cos\varphi,$$

so $0,5h(\mu_0^{(2k-1)}v_0^{(2l-1)}+\mu_N^{(2k-1)}v_N^{(2l-1)})=-0,5h\sin(\psi+2\varphi)$. Let us find now the sum

$$\sum_{j=1}^{N-1}h\mu_j^{(2k-1)}v_j^{(2l-1)}=\sum_{j=1}^{N-1}h\sin\alpha_j\cos\beta_j,$$

where $\alpha_j=(2\pi k-\psi)x_j-\varphi, \quad \beta_j=(2\pi l-\psi)x_j+(\psi+\varphi)$.

It follows from the identity $\sin\alpha_j\cos\beta_j=0,5(\sin(\alpha_j+\beta_j)+\sin(\alpha_j-\beta_j))$ that the sum required is equal to $0,5(S_1+S_2)$, where

$$S_1=\sum_{j=1}^{N-1}h\sin((2\pi(k+l)-2\psi)x_j+\psi), \quad S_2=\sum_{j=1}^{N-1}h\sin((2\pi(k-l))x_j-(\psi+2\varphi)).$$

To calculate these sums, we use the well-known equality

$$\sum_{j=1}^{N-1}h\sin(ax_j+b)=\frac{h\sin(0,5a(1-h))}{\sin(0,5ah)}\sin(b+0,5a), \quad a\neq 0.$$

As a result, we find that $S_1 = 0$ for all k and l, and

$$S_2 = h\sin(\psi + 2\varphi) \text{ for } k \neq l, \quad S_2 = -(1-h)\sin(\psi + 2\varphi) \text{ for } k = l.$$

Thus,

$$(\mu^{(2k-1)}, v^{(2l-1)}]_{\alpha,\gamma} = 0 \text{ if } k \neq l, \quad (\mu^{(2k-1)}, v^{(2l-1)}]_{\alpha,\gamma} = -0,5\sin(\psi + 2\varphi) \text{ if } k = l.$$

Note that

$$\sin(\psi + 2\varphi) = \operatorname{sgn}(1+\alpha\gamma)\frac{\sqrt{(1-\alpha^2)(1-\gamma^2)}}{1-\alpha\gamma} \quad \text{in domains } \; 0a, 0b, 2, 3.$$

In a similar manner, the scalar products $(\mu^{(2k-1)}, v^{(2l)}]_{\alpha,\gamma}$, $(\mu^{(2k)}, v^{(2l-1)}]_{\alpha,\gamma}$ and $(\mu^{(2k)}, v^{(2l)}]_{\alpha,\gamma}$ can be calculated. The normalized basis $\left\{v^{(l)}\right\}_{l=0}^{N-1}$ is defined by (7.10)–(7.12), where

$$a_0 = a_{2l-1} = -a_{2l} = 2\operatorname{sgn}(1+\alpha\gamma)\frac{1-\alpha\gamma}{\sqrt{(1-\alpha^2)(1-\gamma^2)}}.$$

It is important in what follows that

$$a_0^2 = a_{2l-1}^2 = a_{2l}^2 = \frac{4(1-\alpha\gamma)^2}{(1-\alpha^2)(1-\gamma^2)}$$

in all reality domains.

8. Orthonormal basis

Let H be a linear space of the real vectors $y = (y_0\, y_1\, \cdots\, y_N)^\top$ with a scalar product $(y, v) = \sum_{i=1}^N h y_i v_i$. For definiteness, assume that N is odd and denote $m = (N-1)/2$. The system of grid functions $\left\{w^{(n)}\right\}_{n=0}^N$, given by the formulas

$$w^{(0)}(x_i) = 1, \quad w^{(2k-1)}(x_i) = \sqrt{2}\cos(2\pi k x_i), \quad w^{(2k)}(x_i) = \sqrt{2}\sin(2\pi k x_i) \tag{8.1}$$

for $k = 1, 2, \ldots, m$, $i = 1, 2, \ldots, N$ forms an orthonormal basis in H. It follows from (4.13)–(4.15) that the eigenfunctions of the main spectral problem are related to functions (8.1) by the following equalities:

$$\begin{aligned} \mu^{(0)}(x_j) &= \sin(\psi x_j + \varphi) w^{(0)}(x_j), \\ \mu^{(2k-1)}(x_j) &= -\frac{1}{\sqrt{2}}\sin(\psi x_j + \varphi) w^{(2k-1)}(x_j) + \frac{1}{\sqrt{2}}\cos(\psi x_j + \varphi) w^{(2k)}(x_j), \\ \mu^{(2k)}(x_j) &= \frac{1}{\sqrt{2}}\sin(\psi x_j + \varphi) w^{(2k-1)}(x_j) + \frac{1}{\sqrt{2}}\cos(\psi x_j + \varphi) w^{(2k)}(x_j) \end{aligned} \tag{8.2}$$

in domains $0a$, $0b$, 2, 3 for $k = 1, 2, \ldots, m$, $j = 1, 2, \ldots, N$ and

$$\mu^{(0)}(x_j) = \sin(\psi x_j - \varphi) w^{(0)}(x_j),$$

$$\mu^{(2k-1)}(x_j) = -\frac{1}{\sqrt{2}}\sin(\psi x_j - \varphi) w^{(2k-1)}(x_j) + \frac{1}{\sqrt{2}}\cos(\psi x_j - \varphi) w^{(2k)}(x_j),$$

$$\mu^{(2k)}(x_j) = \frac{1}{\sqrt{2}} \sin(\psi x_j - \varphi) w^{(2k-1)}(x_j) + \frac{1}{\sqrt{2}} \cos(\psi x_j - \varphi) w^{(2k)}(x_j) \tag{8.3}$$

in domains $0c$, $0d$, 1, 4. In just the same way it follows from (7.10) — (7.12) that

$$v^{(0)}(x_j) = a_0 \cos(\psi(1 - x_j) + \varphi) w^{(0)}(x_j),$$

$$v^{(2l-1)}(x_j) = \frac{a_{2l-1}}{\sqrt{2}} \left[\cos((1 - x_j)\psi + \varphi) w^{(2l-1)}(x_j) - \sin((1 - x_j)\psi + \varphi) w^{(2l)}(x_j)\right],$$

$$v^{(2l)}(x_j) = \frac{a_{2l}}{\sqrt{2}} \left[\cos((1 - x_j)\psi + \varphi) w^{(2l-1)}(x_j) + \sin((1 - x_j)\psi + \varphi) w^{(2l)}(x_j)\right] \tag{8.4}$$

in domains $0a$, $0b$, 2, 3 for $l = 1, 2, \ldots, m$, $j = 1, 2, \ldots, N$ and

$$v^{(0)}(x_j) = a_0 \cos(\psi(1 - x_j) - \varphi) w^{(0)}(x_j),$$

$$v^{(2l-1)}(x_j) = \frac{a_{2l-1}}{\sqrt{2}} \left[\cos((1 - x_j)\psi - \varphi) w^{(2l-1)}(x_j) - \sin((1 - x_j)\psi - \varphi) w^{(2l)}(x_j)\right],$$

$$v^{(2l)}(x_j) = \frac{a_{2l}}{\sqrt{2}} \left[\cos((1 - x_j)\psi - \varphi) w^{(2l-1)}(x_j) + \sin((1 - x_j)\psi - \varphi) w^{(2l)}(x_j)\right] \tag{8.5}$$

in domains $0c$, $0d$, 1, 4.

Let us introduce the diagonal matrices $\sin(\psi x + \varphi)$, $\cos(\psi x + \varphi)$, $\sin((1 - x)\psi + \varphi)$, $\cos((1 - x)\psi + \varphi)$ order N, with the following elements on the main diagonals respectively:

$$\sin(\psi x_j + \varphi), \quad \cos(\psi x_j + \varphi), \quad \sin((1 - x_j)\psi + \varphi), \quad \cos((1 - x_j)\psi + \varphi), \quad j = 1, 2, \ldots, N.$$

Then equalities (8.2) — (8.5) can be written in a matrix form. Namely, in domains $0a$, $0b$, 2, 3 for $k = 1, 2, \ldots, m$ the equalities

$$\mu^{(0)} = \sin(\psi x + \varphi) w^{(0)},$$

$$\mu^{(2k-1)} = \frac{1}{\sqrt{2}} \left[-\sin(\psi x + \varphi) w^{(2k-1)} + \cos(\psi x + \varphi) w^{(2k)}\right],$$

$$\mu^{(2k)} = \frac{1}{\sqrt{2}} \left[\sin(\psi x + \varphi) w^{(2k-1)} + \cos(\psi x + \varphi) w^{(2k)}\right],$$

$$v^{(0)} = a_0 \cos(\psi(1 - x) + \varphi) w^{(0)},$$

$$v^{(2l-1)} = \frac{a_{2l-1}}{\sqrt{2}} \left[\cos((1 - x)\psi + \varphi) w^{(2l-1)} - \sin((1 - x)\psi + \varphi) w^{(2l)}\right],$$

$$v^{(2l)} = \frac{a_{2l}}{\sqrt{2}} \left[\cos((1 - x)\psi + \varphi) w^{(2l-1)} + \sin((1 - x)\psi + \varphi) w^{(2l)}\right]$$

hold. The same equalities after substituting φ on $(-\varphi)$ hold in domains $0c$, $0d$, 1, 4.

9. Spectrum boundaries of the norm operator

Stability of the difference scheme (2.1) in the space H_D with the norm $||y]|_D = \sqrt{(Dy, y]}$, prescribed by the operator $D = (hMM^*)^{-1}$, where the matrix M is defined by (5.1), has been established in Theorem 6.1. Let us demonstrate now that the norm $||y]|_D$ is equivalent to the grid L_2-norm, i. e., there exist such positive constants κ_1 and κ_2 independent of h that $\kappa_1||y]|^2 \leqslant (Dy, y] \leqslant \kappa_2||y]|^2$ for all $y \in H$. According to (6.4), the fulfillment of such estimates is equivalent to the above and below boundedness of the sum of squares of coefficients of biorthogonal expansion.

For the sake of presentation's simplicity we shall suppose hereinafter that N is odd, and parameters α and γ are situated in one of the domains $0a$, $0b$, 2, 3. Let $y \in H$ be an arbitrary element. Let us consider the expansion $y = \sum_{k=0}^{N-1} c_k\mu^{(k)}$ and let us estimate above the sum

$$S = c_0^2 + \sum_{l=1}^{m}(c_{2l-1}^2 + c_{2l}^2). \tag{9.1}$$

We shall evaluate S through the sum of squares of the Fourier coefficients of $y \in H$ by the system $\{w^{(k)}\}_{k=0}^{N-1}$. This system is orthonormal in the sense of the scalar product $(y, z) = \sum_{i=1}^{N} hy_iz_i$. Let us keep the system $\{\mu^{(k)}\}_{k=0}^{N-1}$ without modification, but construct a system $\{z^{(k)}\}_{k=0}^{N-1}$, biorthonormal to $\{\mu^{(k)}\}_{k=0}^{N-1}$ in the the sense of the scalar product (y, z). Note that

$$(\mu^{(k)}, v^{(l)}]_{\alpha,\gamma} = \sum_{i=1}^{N-1} h\mu_i^{(k)}v_i^{(l)} + h\mu_N^{(k)}(\rho v_N^{(l)}),$$

where $\rho = 0,5(1 + \alpha\gamma) \neq 0$. Let us introduce the functions $\{z^{(k)}\}_{k=0}^{N-1}$ in the following way:

$$z_i^{(l)} = v_i^{(l)}, \quad i = 1, 2, \ldots, N-1, \quad z_N^{(l)} = \rho v_N^{(l)}, \quad l = 0, 1, \ldots, N-1. \tag{9.2}$$

Then we get $(\mu^{(k)}, v^{(l)}]_{\alpha,\gamma} = (\mu^{(k)}, z^{(l)})$, consequently, $(\mu^{(k)}, z^{(l)}) = \delta_{k,l}$ for $k, l = 0, 1, \ldots, N-1$, i. e., the system $\{z^{(k)}\}_{k=0}^{N-1}$ possesses the desired property.

The systems $\{z^{(k)}\}_{k=0}^{N-1}$ and $\{w^{(k)}\}_{k=0}^{N-1}$ are related by the equalities

$$\begin{gathered} z^{(0)} = a_0 \cos_\rho((1-x)\psi + \varphi)w^{(0)}, \quad \rho = 0,5(1+\alpha\gamma), \\ z^{(2l)} + z^{(2l-1)} = \sqrt{2}a_0 \sin_\rho((1-x)\psi + \varphi)w^{(2l)}, \\ z^{(2l)} - z^{(2l-1)} = \sqrt{2}a_0 \cos_\rho((1-x)\psi + \varphi)w^{(2l-1)}, \quad l = 0, 1, \ldots, N-1, \end{gathered} \tag{9.3}$$

where

$$\cos_\rho((1-x)\psi+\varphi) = \mathrm{diag}(\cos((1-x_1)\psi+\varphi), \cos((1-x_2)\psi+\varphi), \ldots, \cos((1-x_{N-1})\psi+\varphi), \rho\cos\varphi),$$

$$\sin_\rho((1-x)\psi+\varphi) = \mathrm{diag}(\sin((1-x_1)\psi+\varphi), \sin((1-x_2)\psi+\varphi), \ldots, \sin((1-x_{N-1})\psi+\varphi), \rho\sin\varphi).$$

Lemma 9.1. *Let the strong reality conditions* (4.1) *be fulfilled. Then the equality*

$$S = a_0^2\left\{\left(\hat{y}_0^{(1)}\right)^2 + \sum_{l=1}^{m}\left[\left(\hat{y}_{2l-1}^{(1)}\right)^2 + \left(\hat{y}_{2l}^{(2)}\right)^2\right]\right\}, \tag{9.4}$$

where

$$\hat{y}_0^{(1)} = (y^{(1)}, w^{(0)}), \quad \hat{y}_{2l}^{(2)} = (y^{(2)}, w^{(2l)}), \quad \hat{y}_{2l-1}^{(1)} = (y^{(1)}, w^{(2l-1)}),$$

$$y^{(1)} = \cos_\rho((1-x)\psi + \varphi)y, \quad y^{(2)} = \sin_\rho((1-x)\psi + \varphi)y, \quad a_0^2 = \frac{4(1-\alpha\gamma)^2}{(1-\alpha^2)(1-\gamma^2)}\textit{holds.}$$

Proof. We shall use the expansions $y = \sum_{k=0}^{N-1} c_k \mu^{(k)}$, $y = \sum_{l=0}^{N-1} d_l z^{(l)}$ for arbitrary $y \in H$, where the systems $\{\mu^{(k)}\}_{k=0}^{N-1}$ and $\{z^{(l)}\}_{l=0}^{N-1}$ are biorthonormal in the sense of the scalar product $(\cdot,\cdot)$. Let us transform the sum S to the form

$$S = c_0^2 + \frac{1}{2}\left(\sum_{l=1}^{m}\left[(c_{2l}+c_{2l-1})^2 + (c_{2l}-c_{2l-1})^2\right]\right), \quad m = \frac{N-1}{2}.$$

As a consequence of the biorthonormality, we obtain for $l = 1, 2, \ldots, m$ that

$$c_0 = (y, z^{(0)}), \quad c_{2l}+c_{2l-1} = (y, z^{(2l)} + z^{(2l-1)}), \quad c_{2l}-c_{2l-1} = (y, z^{(2l)} - z^{(2l-1)}).$$

Further, we obtain by using (9.3) and the self-adjointness of the operator $\cos_\rho((1-x)\psi+\varphi)$ that

$$c_0 = (y, a_0 \cos_\rho((1-x)\psi+\varphi)w^{(0)}) = a_0(\cos_\rho((1-x)\psi+\varphi)y, w^{(0)}) = a_0(y^{(1)}, w^{(0)}).$$

Consequently, $c_0 = a_0 \hat{y}_0^{(1)}$. Analogously, we get $c_{2l}+c_{2l-1} = \sqrt{2}a_0\hat{y}_{2l}^{(2)}$ $c_{2l}-c_{2l-1} = \sqrt{2}a_0\hat{y}_{2l-1}^{(1)}$. From this equality (9.4) follows. □

Lemma 9.2. *Let the strong reality conditions* (4.1) *be fulfilled. Then the estimate*

$$S \leqslant a_0^2\left(\sum_{i=1}^{N-1} hy_i^2 + h\rho^2 y_N^2\right), \quad a_0^2 = \frac{4(1-\alpha\gamma)^2}{(1-\alpha^2)(1-\gamma^2)}, \quad \rho = \frac{1+\alpha\gamma}{2} \tag{9.5}$$

is valid.

Proof. The estimate

$$S \leqslant a_0^2\left\{\sum_{k=0}^{N-1}\left(\hat{y}_k^{(1)}\right)^2 + \sum_{k=0}^{N-1}\left(\hat{y}_k^{(2)}\right)^2\right\}$$

follows from (9.4). Using this estimate and the equalities

$$\sum_{k=0}^{N-1}\left(\hat{y}_k^{(1)}\right)^2 = (y^{(1)}, y^{(1)}), \quad \sum_{k=0}^{N-1}\left(\hat{y}_k^{(2)}\right)^2 = (y^{(2)}, y^{(2)}),$$

we obtain the inequality $S \leqslant a_0^2\left\{(y^{(1)}, y^{(1)}) + (y^{(2)}, y^{(2)})\right\}$. Further, by definition we get

$$(y^{(1)}, y^{(1)}) + (y^{(2)}, y^{(2)}) = \sum_{i=1}^{N-1} h\left[(\cos((1-x_i)\psi+\varphi)y_i)^2 + (\sin((1-x_i)\psi+\varphi)y_i)^2\right] +$$

$$h\left[(\rho\cos((1-x_N)\psi+\varphi)y_N)^2 + (\rho\sin((1-x_N)\psi+\varphi)y_N)^2\right] = \sum_{i=1}^{N-1} hy_i^2 + h\rho^2 y_N^2,$$

which was to be proved. □

We turn now to the obtaining of the lower estimate for sum (9.1). For this purpose, we shall first obtain the upper estimate for the sum $Z = \sum_{l=0}^{N-1} d_l^2$, where d_l are the expansion coefficients $y = \sum_{l=0}^{N-1} d_l z^{(l)}$. We shall need in the next relation between the bases $\left\{\mu^{(k)}\right\}_{k=0}^{N-1}$ and $\left\{w^{(k)}\right\}_{k=0}^{N-1}$:

$$\mu^{(0)} = \sin(\psi x + \varphi) w^{(0)}, \quad \mu^{(2k-1)} = -\frac{\sin(\psi x + \varphi)}{\sqrt{2}} w^{(2k-1)} + \frac{\cos(\psi x + \varphi)}{\sqrt{2}} w^{(2k)},$$

$$\mu^{(2k)} = \frac{\sin(\psi x + \varphi)}{\sqrt{2}} w^{(2k-1)} + \frac{\cos(\psi x + \varphi)}{\sqrt{2}} w^{(2k)}. \tag{9.6}$$

Here

$$\sin(\psi x + \varphi) = \operatorname{diag}\left[\sin(\psi x_1 + \varphi),\ \sin(\psi x_2 + \varphi),\ \cdots\ \sin(\psi x_N + \varphi)\right],$$
$$\cos(\psi x + \varphi) = \operatorname{diag}\left[\cos(\psi x_1 + \varphi),\ \cos(\psi x_2 + \varphi),\ \cdots\ \cos(\psi x_N + \varphi)\right].$$

Lemma 9.3. *Let the strong reality conditions* (4.1) *be fulfilled. Then the estimate*

$$Z = \sum_{k=0}^{N-1} d_k^2 \leqslant (y, y) \tag{9.7}$$

is valid.

Proof. The equalities $d_l = (y, \mu^{(l)})$ hold by virtue of the bases biorthonormality. Let us represent the sum Z in the form

$$Z = d_0^2 + \frac{1}{2}\left\{\sum_{l=1}^{m}\left[(d_{2l} + d_{2l-1})^2 + (d_{2l} - d_{2l-1})^2\right]\right\}.$$

Denoting

$$\hat{y}_{2l}^{(1)} = (y^{(1)}, w^{(2l)}), \quad \hat{y}_0^{(2)} = (y^{(2)}, w^{(0)}), \quad \hat{y}_{2l-1}^{(2)} = (y^{(2)}, w^{(2l-1)}),$$
$$y^{(1)} = \cos(\psi x + \varphi) y, \quad y^{(2)} = \sin(\psi x + \varphi) y,$$

we get from (9.6) that

$$d_0 = \hat{y}_0^{(2)}, \quad \hat{y}_0^{(2)} = (y^{(2)}, w^{(0)}), \quad y^{(2)} = \sin(\psi x + \varphi) y,$$
$$d_{2l} + d_{2l-1} = \sqrt{2}\hat{y}_{2l}^{(1)}, \quad \hat{y}_{2l}^{(1)} = (y^{(1)}, w^{(2l)}), \quad y^{(1)} = \cos(\psi x + \varphi) y,$$
$$d_{2l} - d_{2l-1} = \sqrt{2}\hat{y}_{2l-1}^{(2)}, \quad \hat{y}_{2l-1}^{(2)} = (y^{(2)}, w^{(2l-1)}).$$

So, the equality

$$Z = \left(\hat{y}_0^{(2)}\right)^2 + \sum_{l=1}^{m}\left[\left(\hat{y}_{2l}^{(1)}\right)^2 + \left(\hat{y}_{2l-1}^{(2)}\right)^2\right],$$

holds. From this equality the desired estimate follows:

$$Z \leqslant \sum_{k=0}^{N-1}\left[\left(\hat{y}_k^{(1)}\right)^2 + \left(\hat{y}_k^{(2)}\right)^2\right] = (y^{(1)}, y^{(1)}) + (y^{(2)}, y^{(2)}) =$$

$$\sum_{i=1}^{N} h\left[\left(y_i^{(1)}\right)^2 + \left(y_i^{(2)}\right)^2\right] = \sum_{i=1}^{N} h\left[(\cos(\psi x_i + \varphi) y_i)^2 + (\sin(\psi x_i + \varphi) y_i)^2\right] =$$

$$\sum_{i=1}^{N} h y_i^2 = (y, y).$$

□

The two-sided estimate of the sum S contains

Lemma 9.4. *Let the strong reality conditions* (4.1) *are fulfilled. Then for arbitrary* $y \in H$ *the inequalities*

$$(y, y) \leqslant S \leqslant \kappa a_0^2 (y, y) \tag{9.8}$$

where $\kappa = \max\left(1,\ \rho^2\right),\ \rho = 0,5(1+\alpha\gamma),\ a_0^2 = \dfrac{4(1-\alpha\gamma)^2}{(1-\alpha^2)(1-\gamma^2)}$

hold.

Proof. The proof as in [13]. The equality $(y, y) = \sum_{k=0}^{N-1} c_k d_k$ follows from the expansions $y = \sum_{k=0}^{N-1} c_k \mu^{(k)}$, $y = \sum_{l=0}^{N-1} d_l z^{(l)}$ and the biorthonormality of the systems $\{\mu^{(k)}\}_{k=0}^{N-1}$ and $\{z^{(l)}\}_{l=0}^{N-1}$. By using the Cauchy — Schwarz — Bunyakovskii inequalities we obtain from here that

$$(y, y) \leqslant \left(\sum_{k=0}^{N-1} c_k^2\right)^{1/2} \left(\sum_{k=0}^{N-1} d_k^2\right)^{1/2}.$$

Let us use the estimate from lemma 9.3 $(\sum_{k=0}^{N-1} d_k^2)^{1/2} \leqslant (y, y)^{1/2}$. Then we arrive at the inequality $(y, y)^{1/2} \leqslant (\sum_{k=0}^{N-1} c_k^2)^{1/2}$. Squaring provides the desired lower estimate for sum S. Further, using inequality (9.5), we obtain the two-sided estimate

$$(y, y) \leqslant S \leqslant a_0^2 \left(\sum_{i=1}^{N-1} h y_i^2 + h\rho^2 y_N^2\right).$$

The right-hand side of the second inequality can be majorized by the scalar product (y, y) multiplied by some positive factor. Indeed, $\sum_{i=1}^{N-1} h y_i^2 + h\rho^2 y_N^2 \leqslant \kappa \sum_{i=1}^{N} h y_i^2$, where $\kappa = \max\left(1,\ \rho^2\right)$. □

From lemma 9.4 the theorem about the equivalence of the norm $||y||_D = \sqrt{[Dy, y]}$, for which the stability of the difference scheme (2.1) is guaranteed, to the grid L_2-norm follows.

Theorem 9.1. *Suppose that the strong reality conditions* (4.1) *hold, the matrix* M *is specified as* (5.1)*, and the operator* D *is defined according to* (6.3)*. Then for arbitrary* $y \in H$ *the inequalities* $0,5(y, y) \leqslant (Dy, y] \leqslant \kappa a_0^2 (y, y)$, *where*

$$\kappa = \max\left(1,\ \rho^2\right), \quad \rho = 0,5(1+\alpha\gamma), \quad a_0^2 = \frac{4(1-\alpha\gamma)^2}{(1-\alpha^2)(1-\gamma^2)},$$

hold.

Proof. According to (6.4), we get $(Dy, y] = c_0^2 + c_1^2 + \cdots + c_{N-2}^2 + 0,5 c_{N-1}^2$, from which we derive $0,5S \leqslant (Dy, y] \leqslant S$. From here and lemma 9.4 the statement of theorem 9.1 follows. □

Corollary 9.1. *Under conditions* (4.1) *for arbitrary* $y \in H$ *the inequalities* $0,5(y, y] \leqslant (Dy, y] \leqslant 2\kappa a_0^2 (y, y]$ *hold.*

The statement follows from theorem 9.1 and the obvious inequalities $(y, y] \leqslant (y, y) \leqslant 2(y, y]$.

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