

Energy Principles and Finite Element Methods for Pure Traction Linear Elasticity

Pavel Bochev · Richard Lehoucq

Abstract — A conforming finite element discretization of the pure traction elasticity boundary value problem results in a singular linear system of equations. The singularity of the linear system is removed through various methods. In this report, we consider an alternative approach in which discrete finite element formulations are derived directly from the principle of minimum potential energy, rather than from the weak form of the differential equations. This point of view turns out to be particularly well suited to handle the rigid body modes, which are the source of the singularity in the finite element linear system. Our approach allows us to formulate a regularized potential energy principle and prove that the weak problem corresponding to its first-order optimality condition is coercive in $H^1(\Omega)$. This guarantees nonsingular algebraic problems, enables simplified solution algorithms and leads to more efficient and robust preconditioners for the iterative solution linear equations.

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1. Introduction

We consider an isotropic elastic body occupying a bounded open region Ω in \mathbf{R}^n for $n = 2, 3$. The surface Γ of the body is assumed to be Lipschitz continuous and consisting of two disjoint pieces Γ_D and Γ_T . For small deformations the static behavior of this elastic body is governed by the linear *equilibrium equation* [7, pp.200-206]

$$-\nabla \cdot (\sigma(\mathbf{u})) = \mathbf{f} \quad \text{in } \Omega, \quad (1.1)$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D, \quad (1.2)$$

$$\sigma(\mathbf{u}) \cdot \mathbf{n} = \mathbf{t} \quad \text{on } \Gamma_T. \quad (1.3)$$

In (1.1)–(1.3) \mathbf{u} is the *displacement*, and

$$\sigma(\mathbf{u}) = 2\mu\varepsilon(\mathbf{u}) + \lambda\text{tr}(\varepsilon(\mathbf{u}))\mathbf{I} \quad (1.4)$$

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Pavel Bochev

Numerical Analysis and Applications, Sandia National Laboratories, P.O. Box 5800, MS 1320, Albuquerque, NM 87185, USA

E-mail: pbboche@sandia.gov.

Richard Lehoucq

Multiphysics Simulation Technologies, Sandia National Laboratories, P.O. Box 5800, MS 1320, Albuquerque, NM 87185, USA

E-mail: rblehou@sandia.gov.

is the *stress*, where

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (1.5)$$

is the *infinitesimal strain*. The vector functions \mathbf{g} and \mathbf{t} prescribe the surface displacement and the surface traction, respectively, and the scalar functions λ and μ are the Lamé moduli. If the material is homogeneous λ and μ are constants. When $\Gamma_D = \emptyset$, or $\Gamma_T = \emptyset$, the problem is called a pure displacement or pure traction problem, respectively.

When Γ_D is not empty the partial differential equations (PDEs) (1.1)–(1.3) have a unique solution. In contrast, the pure traction problem has a nontrivial nullspace consisting of all *infinitesimal rigid displacements*. The dimension of this nullspace is 3 in \mathbf{R}^2 and 6 in \mathbf{R}^3 . As a result, the pure traction problem is solvable if and only if the body force \mathbf{f} and the surface traction \mathbf{t} satisfy the compatibility relations

$$\int_{\Omega} \mathbf{f} + \int_{\Gamma} \mathbf{t} = \mathbf{0}, \quad (1.6)$$

$$\int_{\Omega} \mathbf{r} \times \mathbf{f} + \int_{\Gamma} \mathbf{r} \times \mathbf{t} = \mathbf{0}, \quad (1.7)$$

where $\mathbf{r} = [\rho_1, \rho_2, \rho_3]^T$ is position vector in \mathbf{R}^3 .

Equations (1.6)–(1.7) express the equilibrium of all external forces applied to an elastic body. For the remainder of our report, we consider pure traction problems so that $\Gamma_D = \emptyset$.

A standard Galerkin procedure applied directly to the pure traction boundary value problem leads to a linear system with a non-trivial nullspace. To ensure consistency of this singular linear system, a discrete version of (1.6)–(1.7) must hold. However, in finite precision arithmetic discrete consistency may become problematic. For instance, if the discrete load vector is computed using numerical quadrature, then quadrature error introduces components that do not discretely satisfy (1.6)–(1.7).

In this paper, we focus on the finite element solution of the pure traction boundary value problem. The motivation for this work is two-fold. The first is to understand carefully the relationship between the variational form and the ensuing algebraic equations. The second is that this information is important for the construction of preconditioners and/or when the linear system is solved with a preconditioned iterative method. For example, the dual Schur domain-decomposition and algebraic multigrid preconditioners [5, 10, 9] involve floating structures containing rigid body motions. Moreover, when an iterative method such as the conjugate gradient algorithm is used, knowledge of the matrix is only assumed through a matrix vector product.

We are not aware of any other study that carefully links variational methods and the resulting algebraic equations for linear elasticity for use with preconditioned iterative methods. For example, the recent papers [6, 8] instead consider and propose techniques for computing a basis for the rigid body modes via algebraic and geometric techniques. The connection with an energy minimization principle is never discussed.

Our approach utilizes the existing connection between the PDE (1.1)–(1.3) and unconstrained minimization of a quadratic functional expressing the potential energy of an elastic body. This minimization problem, known as the *minimum potential energy principle*, is the starting point of our finite element development. We use it to derive three other minimization problems that possess unique minimizers. The first two utilize non-standard spaces of kinematically admissible states but retain the original potential energy functional. The

third energy principle uses the original admissible space but augments the potential energy functional by terms characterizing the rigid body deformations.

When restricted to a finite element subspace, each one of those problems gives rise to a quadratic program whose solution represents the finite element approximant. Thus, the principal difference between a conventional Galerkin scheme and our approach is that the former treats finite element approximants as solutions of linear systems, while here they are considered as minimizers of a potential energy functional.

The reasons to place the minimization problem, rather than the PDE, at the core of our approach are not merely pedagogical. The minimum potential energy principle is the primal mathematical model describing admissible states of an elastic body. The PDE (1.1)–(1.3) gives the strong form of the necessary optimality condition for this principle. Consequently, our approach emphasizes finite element methods for elasticity as an instance of a Rayleigh-Ritz principle. Characterization of finite element approximants as solutions of a minimization problem offers the additional flexibility of utilizing ideas and solution methods from numerical optimization. Of course, our approach does not rule out conventional solution techniques applied to linear systems representing the necessary condition of the quadratic functional. Third, for conforming finite element subspaces, the coefficient matrix is guaranteed to be symmetric and positive definite because of the coercivity of the corresponding bilinear form. This precludes the need to maintain a discrete form of the compatibility relations and allows for effortless convergence of iterative methods such as conjugate gradients.

In this paper, we use the standard notation $H^s(\Omega)$ for a Sobolev space of order s with norm and inner product given by $\|\cdot\|_s$ and $(\cdot, \cdot)_s$, respectively. Semi-norms on $H^s(\Omega)$ will be denoted by $|\cdot|_k$, $0 \leq k \leq s$. For example, $|u|_1 = \int_{\Omega} |\nabla u|^2 dx$. For $s = 0$ we write $L^2(\Omega)$ instead of $H^0(\Omega)$ and denote the resulting inner product by (\cdot, \cdot) .

Because our study also makes use of matrix theory, we introduce some useful notation. Let $\{\mathbf{e}_i\}_{i=1}^N$ denote the canonical basis of \mathbf{R}^N and \mathbf{I}_N be the identity matrix of order N . The ordering of the eigenvalues of the standard stiffness matrix \mathbf{A} for (1.1) is $0 = \lambda_1 = \dots = \lambda_6 \leq \lambda_7 \leq \dots \leq \lambda_N$ in three-dimensions and $0 = \lambda_1 = \dots = \lambda_3 \leq \lambda_4 \leq \dots \leq \lambda_N$ in two-dimensions.

We call attention to our specific use of bold font for tensors, vectors and the associated Sobolev spaces (denoted by $\mathbf{H}^s(\Omega)$). Elements of matrices and vectors will be denoted by lower-case Greek letters. The symbols \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 , and \mathbf{I}_3 denote the oriented Cartesian basis, and the unit matrix in \mathbf{R}^3 , respectively.

Our paper is organized as follows. In the next Section we review the original principle of minimum potential energy. Section 3 deals with several schemes of purging the rigid displacement modes from the pure traction elasticity. These are later used in Section 4 to formulate three alternative potential energy principles. The quadratic programs defining finite element methods for these principles are derived in Section 5. We conclude with a discussion of properties of algebraic problems related to each quadratic program.

2. Minimum potential energy principle

The potential energy of an elastic isotropic body Ω under a given load \mathbf{f} and a surface traction \mathbf{t} is a quadratic function of the displacement \mathbf{v}

$$\mathcal{J}(\mathbf{v}; \mathbf{f}, \mathbf{t}) = \frac{1}{2} \int_{\Omega} \sigma(\mathbf{v}) : \varepsilon(\mathbf{v}) d\Omega - \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\Omega - \int_{\Gamma_T} \mathbf{t} \cdot \mathbf{v} d\Gamma. \quad (2.1)$$

The load \mathbf{f} and the traction \mathbf{t} are assumed to satisfy the force equilibrium constraint (1.6)–(1.7). The first term in $\mathcal{J}(\cdot)$ represents the strain energy $\mathcal{E}(\mathbf{v})$ of the body. Because our main focus is on approaches for handling the rigid body modes without loss of generality we set $\mathbf{t} = \mathbf{0}$ for the rest of this paper and denote the resulting energy functional $\mathcal{J}(\mathbf{v}; \mathbf{f})$.

The classical *principle of minimum potential energy* [7, p.208] postulates that among all *kinematically admissible states* \mathbf{v} the state of the elastic body Ω is the one that minimizes the potential energy $\mathcal{J}(\mathbf{v})$. Kinematically admissible states for (2.1) can be identified with the Sobolev space $\mathbf{H}^1(\Omega)$. Thus, a formal mathematical statement of the minimum energy principle is

$$\inf_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \mathcal{J}(\mathbf{v}; \mathbf{f}). \quad (2.2)$$

The minimum energy principle also states that if \mathbf{u}_1 and \mathbf{u}_2 are two minimizers, then $\mathbf{u}_1 - \mathbf{u}_2$ is a *rigid displacement* [7, p.55] of Ω , that is a vector field \mathbf{u} with constant and skew gradient terms. We denote the set of all rigid displacements by \mathcal{N} . In \mathbf{R}^3

$$\mathcal{N} = \{\mathbf{u} \in \mathbf{H}^1(\Omega) \mid \mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{r}; \quad \mathbf{a}, \mathbf{b} \in \mathbf{R}^3\}. \quad (2.3)$$

Formula (2.3) remains valid in \mathbf{R}^2 with the tacit understanding that $\mathbf{a} \in \mathbf{R}^2$, $\mathbf{b} = b$ is a constant, and that the vector product $b \times \mathbf{r}$ is the defined by $b\mathbf{e}_3 \times (\rho_1\mathbf{e}_1 + \rho_2\mathbf{e}_2)$. Note that $\dim \mathcal{N}$ is 3 in \mathbf{R}^2 and 6 in \mathbf{R}^3 . This corresponds to the fact that in \mathbf{R}^3 there are three rotational and translational rigid displacements, while in \mathbf{R}^2 there's only one rotational and two translational modes. The compatibility conditions for the pure traction problem in two and three dimensions can be expressed by the compact statement

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{w} = \mathbf{0} \quad \text{for all } \mathbf{w} \in \mathcal{N}; \quad \text{that is, } \mathbf{f} \in \mathcal{N}^{\perp}. \quad (2.4)$$

For the remainder of this paper we confine the discussion to the case of \mathbf{R}^3 . Specialization of all results to two-dimensions is straightforward. The set

$$\{\mathbf{w}_i\}_{i=1}^6 = \{\mathbf{e}_i\}_{i=1}^3 \oplus \{\mathbf{e}_i \times \mathbf{r}\}_{i=1}^3 \quad (2.5)$$

is a basis for \mathcal{N} ; the first three vectors span the translational modes, whereas the last three span the rotational modes. In mathematical terms, existence of the nontrivial set \mathcal{N} means that the minimization problem (2.2) has multiple minimizers.

Our plan is to formulate finite elements for the pure traction problem by directly using the minimum energy principle, albeit in a modified form. To carry out this plan we will introduce three alternative versions of (2.2). Two of these new energy principles will use the same energy functional but a different space of minimizers, while the third one will use a different, regularized energy functional but the original minimizer space. In contrast to (2.2), where minimizers are determined up to a rigid displacement, all three alternative energy principles will possess unique minimizers. The key to our approach is the proper handling of the rigid modes of the pure traction problem. The next section considers this issue.

Before leaving this section, we define several matrices and vectors that we will use in the remainder of the paper. Let

$$\mathbf{W} = [\mathbf{w}_1 \quad \cdots \quad \mathbf{w}_6] \quad (2.6)$$

be the matrix of nullspace vectors (2.5) with three rows and six columns. Any $\mathbf{w} \in \mathcal{N}$ has the form

$$\mathbf{w} = \mathbf{W}\mathbf{x} = \sum_{i=1}^6 \xi_i \mathbf{w}_i$$

where $\mathbf{x} = [\xi_1, \dots, \xi_6]$ are the coordinates of \mathbf{w} with respect to the basis (2.5). Define

$$\mathbf{B}(\mathbf{u}) \equiv \begin{bmatrix} \int_{\Omega} \Theta_1 \mathbf{u} \\ \int_{\Omega} \Theta_2 \nabla \times \mathbf{u} \end{bmatrix}, \quad (2.7)$$

where the entries of the diagonal matrices

$$\Theta_1 = \text{diag}(\theta_{1,1}, \theta_{1,2}, \theta_{1,3}) \quad \text{and} \quad \Theta_2 = \text{diag}(\theta_{2,1}, \theta_{2,2}, \theta_{2,3}) \quad (2.8)$$

are elements of $H^{-1}(\Omega)$ satisfying the hypothesis

$$\int_{\Omega} \theta_{1,i} \mathbf{w}_i \neq 0 \quad \text{and} \quad \int_{\Omega} \theta_{2,i} \mathbf{w}_{i+3} \neq 0$$

for $i = 1, 2, 3$. These dual functions serve the useful purpose of allowing us to enforce the mean and mean of the curl of the displacement along a portion of Ω . For the remainder of the paper, we will assume that $\Theta_1 = \mathbf{I}_3 = \Theta_2$ unless otherwise stated. Let $|\mathbf{B}(\mathbf{u})| \equiv \sqrt{\mathbf{B}(\mathbf{u})^T \mathbf{B}(\mathbf{u})}$. Finally, define the 6×6 matrix

$$\mathbf{S} = \mathbf{B}(\mathbf{W}). \quad (2.9)$$

A simple calculation shows that the above matrix is

$$\mathbf{S} = \begin{pmatrix} \mu(\Omega) \mathbf{I}_3 & \mathbf{S}_{12} \\ \mathbf{0} & 2\mu(\Omega) \mathbf{I}_3 \end{pmatrix}, \quad (2.10)$$

where $\mu(\Omega)$ is the volume of the region Ω and

$$\mathbf{S}_{12} \equiv \begin{pmatrix} 0 & \int_{\Omega} \rho_3 \mathbf{e}_3 & -\int_{\Omega} \rho_2 \mathbf{e}_2 \\ -\int_{\Omega} \rho_3 \mathbf{e}_3 & 0 & \int_{\Omega} \rho_1 \mathbf{e}_1 \\ \int_{\Omega} \rho_2 \mathbf{e}_2 & -\int_{\Omega} \rho_1 \mathbf{e}_1 & 0 \end{pmatrix}.$$

Therefore, \mathbf{S} is nonsingular.

3. Elimination of rigid displacements

A rigid displacement cannot increase the strain energy of an elastic body. This can be verified by showing that the strain energy functional $\mathcal{E}(\mathbf{w})$ vanishes for all $\mathbf{w} \in \mathcal{N}$. Mathematically, this means that $\mathcal{E}(\cdot)$ is only a seminorm on the space $\mathbf{H}^1(\Omega)$. Our main goal will be to elevate the status of $\mathcal{E}(\cdot)$ to that of a true norm. Once this is accomplished, the finite element solution of the pure traction problem can be treated within the classical Rayleigh-Ritz variational framework.

There are two distinct ways of turning $\mathcal{E}(\cdot)$ into an equivalent norm. Both eliminate the rigid displacements from the pure traction problem, however they differ in the manner in how this is accomplished.

The first one is to purge the rigid modes from the admissible space $\mathbf{H}^1(\Omega)$ while retaining the original definition of the strain energy. Two cases will be discussed, one uses a factor space and the other uses a complement space. The second method retains the original admissible space, but regularizes the strain energy in a manner that flushes out rigid displacements. We will pay special attention to the second approach because it leads to a class of finite elements that has not yet been carefully examined in the literature.

3.1. Complement space

Define the space

$$\mathbf{H}_C^1(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{B}(\mathbf{v}) = \mathbf{0}\}. \quad (3.1)$$

Evidently, $\mathbf{H}_C^1(\Omega)$ is a closed subspace of $\mathbf{H}^1(\Omega)$. The next lemma explains why $\mathbf{H}_C^1(\Omega)$ is an algebraic complement of \mathcal{N} in $\mathbf{H}^1(\Omega)$.

Lemma 3.1. *For every $\mathbf{u} \in \mathbf{H}^1(\Omega)$ there exist unique $\mathbf{u}_C \in \mathbf{H}_C^1(\Omega)$ and $\mathbf{u}_\mathcal{N} \in \mathcal{N}$, such that*

$$\mathbf{u} = \mathbf{u}_C + \mathbf{u}_\mathcal{N}.$$

Proof. We prove the theorem in \mathbf{R}^3 , the proof in the planar case is identical. Let $\mathbf{u} \in \mathbf{H}^1(\Omega)$ be arbitrary but fixed function. We seek $\mathbf{u}_\mathcal{N} = \mathbf{W}\mathbf{x}$ as solution of the 6×6 system of equations

$$\int_{\Omega} \mathbf{W}\mathbf{x} = \int_{\Omega} \mathbf{u} \quad \text{and} \quad \int_{\Omega} \nabla \times (\mathbf{W}\mathbf{x}) = \int_{\Omega} \nabla \times \mathbf{u}. \quad (3.2)$$

In terms of the matrix defined in (2.9) and the functional \mathbf{B} system (3.2) has the following algebraic representation:

$$\mathbf{S}\mathbf{x} = \mathbf{B}(\mathbf{u}). \quad (3.3)$$

Because \mathbf{S} is nonsingular (3.3), resp., (3.2) has a unique solution \mathbf{x} , which defines the component $\mathbf{u}_\mathcal{N} = \mathbf{W}\mathbf{x} \in \mathcal{N}$. Set $\mathbf{u}_C = \mathbf{u} - \mathbf{u}_\mathcal{N}$. Then

$$\int_{\Omega} (\mathbf{u} - \mathbf{u}_\mathcal{N}) = \int_{\Omega} \nabla \times (\mathbf{u} - \mathbf{u}_\mathcal{N}) = 0,$$

that is, $\mathbf{u}_C \in \mathbf{H}_C^1(\Omega)$. The conclusion of the lemma is established. \square

We remark that $\mathcal{N}^\perp \neq \mathbf{H}_C^1(\Omega)$ and so $(\mathbf{u}_C, \mathbf{u}_\mathcal{N}) \neq 0$ in general.

Corollary 3.1. *The complement space $\mathbf{H}_C^1(\Omega)$ has a trivial intersection with \mathcal{N} , $\mathcal{N} \cap \mathbf{H}_C^1(\Omega) = \{\mathbf{0}\}$.*

Proof. Assume that $\mathbf{u} \in \mathcal{N} \cap \mathbf{H}_C^1(\Omega)$. Because $\mathbf{u} \in \mathcal{N}$ it has the form $\mathbf{u} = \mathbf{W}\mathbf{x}$ where \mathbf{x} solves the linear system (3.3). Because $\mathbf{u} \in \mathbf{H}_C^1(\Omega)$ there holds $\mathbf{B}(\mathbf{u}) = \mathbf{0}$, i.e., the linear system (3.3) has homogenous right hand side. It follows that $\mathbf{x} = 0$ and $\mathbf{u} = \mathbf{W}\mathbf{x} = 0$ as well. \square

This corollary asserts that rigid body modes have been purged from $\mathbf{H}_C^1(\Omega)$, i.e., the complement space contains only a trivial rigid displacement. For us, the most notable property of $\mathbf{H}_C^1(\Omega)$ is that strain energy is an equivalent norm on $\mathbf{H}_C^1(\Omega)$. This follows from a result known as Korn's second inequality that we quote for future reference.

Theorem 3.1. [3, p.222] *There exists a positive constant C such that*

$$C\|\mathbf{v}\|_1 \leq \|\varepsilon(\mathbf{v})\|_0 \quad \text{for all } \mathbf{v} \in \mathbf{H}_C^1(\Omega). \quad (3.4)$$

Using Korn's second inequality it is a simple matter to show that $\mathcal{E}(\cdot)$ is an equivalent norm on $\mathbf{H}_C^1(\Omega)$. We note that a second Korn's inequality can be established on other subspaces as well. For example, it is valid on the orthogonal complement $\mathcal{N}_{H^1}^\perp$ of \mathcal{N} in the space $\mathbf{H}^1(\Omega)$, $\mathcal{N}_{H^1}^\perp = \{\mathbf{v} \in \mathbf{H}^1(\Omega) \mid (\mathbf{v}, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathcal{N}\}$.

3.2. Factor space

A standard way of removing an undesirable subspace is to factor it from the respective function space. In the present context this leads to the factor space

$$\mathbf{H}_{\mathcal{N}}^1(\Omega) = \{\hat{\mathbf{u}} \subset \mathbf{H}^1(\Omega) \mid \mathbf{u}_1, \mathbf{u}_2 \in \hat{\mathbf{u}} \Leftrightarrow \mathbf{u}_1 - \mathbf{u}_2 \in \mathcal{N}\}. \quad (3.5)$$

When equipped with the factor norm

$$\|\hat{\mathbf{u}}\|_{\mathbf{H}_{\mathcal{N}}^1(\Omega)} \equiv \inf_{\mathbf{v} \in \hat{\mathbf{u}}} \|\mathbf{v}\|_1 \quad (3.6)$$

$\mathbf{H}_{\mathcal{N}}^1(\Omega)$ is a Hilbert space. We recall that a factor space does not contain usual functions, instead its members are equivalence classes of functions. For instance, the subspace of rigid displacements \mathcal{N} is confined to the zero element $\hat{\mathbf{0}}$ of the factor space $\mathbf{H}_{\mathcal{N}}^1(\Omega)$. This is the trick that turns $\mathcal{E}(\cdot)$ into an equivalent norm on $\mathbf{H}_{\mathcal{N}}^1(\Omega)$.

Lemma 3.2. *The mapping $\mathbf{H}_{\mathcal{N}}^1(\Omega) \ni \hat{\mathbf{u}} \mapsto \|\varepsilon(\mathbf{u})\|_0$, where $\mathbf{u} \in \hat{\mathbf{u}}$ is arbitrary, defines an equivalent norm on $\mathbf{H}_{\mathcal{N}}^1(\Omega)$.*

Proof. Let $\hat{\mathbf{u}}$ be a given equivalence class and $\mathbf{u} \in \hat{\mathbf{u}}$ an arbitrary but fixed function in that class. Using this fixed function the right hand side in (3.6) can be expressed as

$$\inf_{\mathbf{v} \in \hat{\mathbf{u}}} \|\mathbf{v}\|_1 = \inf_{\mathbf{w} \in \mathcal{N}} \|\mathbf{u} - \mathbf{w}\|_1$$

because for $\mathbf{u}, \mathbf{v} \in \hat{\mathbf{u}}$ we have that $\mathbf{u} - \mathbf{v} = \mathbf{w} \in \mathcal{N}$. Therefore, we need to prove that there are positive constants C_1 and C_2 , independent of \mathbf{u} and such that

$$C_1 \|\varepsilon(\mathbf{u})\|_0 \leq \inf_{\mathbf{w} \in \mathcal{N}} \|\mathbf{u} - \mathbf{w}\|_1 \leq C_2 \|\varepsilon(\mathbf{u})\|_0$$

The lower bound is straightforward because $\varepsilon(\mathbf{u}) = \varepsilon(\mathbf{u} - \mathbf{w})$. To prove the upper bound we use Korn's second inequality (3.4) and Lemma 3.1. According to the lemma we can write the function $\mathbf{u} \in \hat{\mathbf{u}}$ as $\mathbf{u} = \mathbf{u}_{\mathcal{C}} + \mathbf{u}_{\mathcal{N}}$ where $\mathbf{u}_{\mathcal{C}} \in \mathbf{H}_{\mathcal{C}}^1(\Omega)$ and $\mathbf{u}_{\mathcal{N}} \in \mathcal{N}$. Because $\varepsilon(\mathbf{u}) = \varepsilon(\mathbf{u} - \mathbf{u}_{\mathcal{N}})$ and $\mathbf{u} - \mathbf{u}_{\mathcal{N}} = \mathbf{u}_{\mathcal{C}} \in \mathbf{H}_{\mathcal{C}}^1(\Omega)$, the Korn's second inequality yields the statement of the lemma:

$$\|\varepsilon(\mathbf{u})\|_0 = \|\varepsilon(\mathbf{u} - \mathbf{u}_{\mathcal{N}})\|_0 \geq C \|\mathbf{u} - \mathbf{u}_{\mathcal{N}}\|_1 \geq \inf_{\mathbf{w} \in \mathcal{N}} C \|\mathbf{u} - \mathbf{w}\|_1.$$

□

3.3. Regularized strain energy

If we wish to retain the original admissible space $\mathbf{H}^1(\Omega)$, removal of the rigid displacements must necessarily be accomplished by regularization of the strain energy. To motivate the regularization process, we prove a modified version of Korn's inequality.

Theorem 3.2. *There exists a positive C such that*

$$\|\varepsilon(\mathbf{v})\|_0 + |\mathbf{B}(\mathbf{v})| \geq C \|\mathbf{v}\|_1 \quad (3.7)$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$.

Proof. We use standard compactness argument. Suppose that (3.7) does not hold for any positive C . Then, there exists a sequence $\{\mathbf{v}_n\} \in \mathbf{H}^1(\Omega)$ such that

$$\|\mathbf{v}_n\|_1 = 1 \quad \text{and} \quad \|\varepsilon(\mathbf{v}_n)\|_0 + |\mathbf{B}(\mathbf{v}_n)| < \frac{1}{n}. \quad (3.8)$$

From Lemma 3.1, given \mathbf{v}_n there are $\mathbf{w}_n \in \mathcal{N}$ and $\mathbf{z}_n \in \mathbf{H}_c^1(\Omega)$ such that $\mathbf{v}_n = \mathbf{w}_n + \mathbf{z}_n$. The function $\mathbf{w}_n = \mathbf{W}\mathbf{x}$ where the coefficient vector \mathbf{x} solves the linear system (3.3), i.e.,

$$\mathbf{x} = \mathbf{S}^{-1}\mathbf{B}(\mathbf{v}_n).$$

Because the right hand side $\mathbf{B}(\mathbf{v}_n)$ involves at most integrals of the first derivatives of \mathbf{v}_n it easily follows that there exists a positive constant C such that

$$\|\mathbf{w}_n\|_1 \leq C\|\mathbf{v}_n\|_1$$

for all integers n . Therefore,

$$\|\mathbf{z}_n\|_1 + \|\mathbf{w}_n\|_1 = \|\mathbf{v}_n - \mathbf{w}_n\|_1 + \|\mathbf{w}_n\|_1 \leq \|\mathbf{v}_n\|_1 + 2\|\mathbf{w}_n\|_1 \leq C\|\mathbf{v}_n\|_1. \quad (3.9)$$

The Korn's second inequality (3.4) holds for all functions in $\mathbf{H}_c^1(\Omega)$, in particular it is valid for \mathbf{z}_n . In combination with the fact that the strain energy vanishes for \mathbf{w}_n and the hypothesis on \mathbf{v}_n we find that

$$C\|\mathbf{z}_n\|_1 \leq \|\varepsilon(\mathbf{z}_n)\|_0 = \|\varepsilon(\mathbf{z}_n + \mathbf{w}_n)\|_0 = \|\varepsilon(\mathbf{v}_n)\|_0 < \frac{1}{n}.$$

Therefore,

$$\lim_{n \rightarrow \infty} \mathbf{z}_n = 0 \quad \text{in } \mathbf{H}^1(\Omega).$$

This, and (3.9) imply that $\{\mathbf{w}_n\}$ is bounded in $\mathbf{H}^1(\Omega)$. Compactness of the imbedding $\mathbf{H}^1(\Omega) \hookrightarrow L^2(\Omega)$ means that a subsequence of $\{\mathbf{w}_n\}$, denoted by the same symbol, converges in $L^2(\Omega)$. Since \mathcal{N} is finite dimensional, this subsequence must also converge in $\mathbf{H}^1(\Omega)$. Let $\mathbf{w} \in \mathcal{N}$ be its limit. Then

$$\lim_{n \rightarrow \infty} \mathbf{v}_n = \lim_{n \rightarrow \infty} \mathbf{z}_n + \mathbf{w}_n = \mathbf{w} \in \mathcal{N}.$$

Therefore,

$$\|\mathbf{w}\|_1 = \lim_{n \rightarrow \infty} \|\mathbf{v}_n\|_1 = 1.$$

On the other hand, taking the limit in (3.8) and noting that \mathbf{w} has vanishing strain gives

$$\mathbf{B}(\mathbf{w}) = \mathbf{0},$$

that is $\mathbf{w} \in \mathcal{N} \cap \mathbf{H}_c^1(\Omega)$. But Corollary 3.1 asserts that the intersection of these two spaces is trivial and so we must have $\mathbf{w} = \mathbf{0}$, a contradiction. \square

Remark 3.1. The modified Korn inequality in Theorem 3.2 plays the same role for the pure traction elasticity problem as the modified Poincare-Friedrichs inequality [2, Lemma 2.2]

$$\|u\|_0 \leq C \left(|u|_1 + \left| \int_{\Omega} u dx \right| \right)$$

does for the pure Neumann problem.

The new Korn inequality prompts regularization of the strain energy $\mathcal{E}(\cdot)$ by adding the mean and the mean of the curl of the kinematic state. The new, regularized strain energy functional is

$$\mathcal{E}_R(\mathbf{v}) = \frac{1}{2} \int_{\Omega} \sigma(\mathbf{v}) : \varepsilon(\mathbf{v}) + \frac{1}{2} \mathbf{B}(\mathbf{v})^T \Upsilon \mathbf{B}(\mathbf{v}), \quad (3.10)$$

where Υ is a diagonal matrix of order six. The regularization parameters are the diagonal elements of Υ . The regularization terms serve to increase the frequency of the rigid body modes.

Together, the last term in (3.10) represent an “energy” measure of the rigid displacements. As a result, the modified strain energy does not vanish on \mathcal{N} .

4. Alternative energy principles

The factor space, the complement space and the regularized strain energy give rise to three functionals for the minimum potential energy principle. The first one is to minimize the original potential energy $\mathcal{J}(\cdot)$ over the factor space $\mathbf{H}_{\mathcal{N}}^1(\Omega)$ leading to the minimization problem

$$\inf_{\hat{\mathbf{v}} \in \mathbf{H}_{\mathcal{N}}^1(\Omega)} \mathcal{J}(\hat{\mathbf{v}}; \mathbf{f}). \quad (4.1)$$

The second alternative is to minimize the same functional over the complement space $\mathbf{H}_c^1(\Omega)$ and so the minimization problem now is

$$\inf_{\mathbf{v} \in \mathbf{H}_c^1(\Omega)} \mathcal{J}(\mathbf{v}; \mathbf{f}). \quad (4.2)$$

A third alternative is to minimize a regularized potential energy functional over the original admissible space $\mathbf{H}^1(\Omega)$. The regularized minimum potential energy principle is

$$\inf_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \mathcal{J}_R(\mathbf{v}, \mathbf{f}) = \inf_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \mathcal{E}_R(\mathbf{v}) - \int_{\Omega} \mathbf{f} \cdot \mathbf{v}. \quad (4.3)$$

The remainder of this section demonstrates that each one of the three alternative energy principles is a well-posed minimization problem with a unique minimizer.

4.1. Factor Space minimization

Theorem 4.1. *If the load $\mathbf{f} \in L^2(\Omega)$ satisfies compatibility condition (2.4) then the minimization problem (4.1) has a unique minimizer $\hat{\mathbf{u}}_f$ from $\mathbf{H}_{\mathcal{N}}^1(\Omega)$.*

Proof. The energy principle (4.1) is a well-defined minimization problem for any load \mathbf{f} that satisfies the force equilibrium (2.4). In the context of a factor space well-defined means that the value of $\mathcal{J}(\cdot)$ does not depend on the particular class member $\mathbf{u} \in \hat{\mathbf{u}}$. To show this, let \mathbf{u}_1 and \mathbf{u}_2 be two functions from the same class $\hat{\mathbf{u}} \in \mathbf{H}_{\mathcal{N}}^1(\Omega)$. Then, $\mathbf{u}_1 - \mathbf{u}_2 = \mathbf{w} \in \mathcal{N}$; $\mathcal{E}(\mathbf{u}_1) = \mathcal{E}(\mathbf{u}_2)$ and so

$$\mathcal{J}(\mathbf{u}_1) - \mathcal{J}(\mathbf{u}_2) = \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_1 - \mathbf{u}_2) = 0$$

because of (2.4). Let us now prove that (4.1) has a unique minimizer. The first-order optimality condition (Euler-Lagrange equation) for (4.1) is represented by the variational problem: seek $\hat{\mathbf{u}} \in \mathbf{H}_{\mathcal{N}}^1(\Omega)$ such that

$$\mathcal{A}(\hat{\mathbf{u}}; \hat{\mathbf{v}}) = \mathcal{F}(\hat{\mathbf{v}}) \quad \text{for all } \hat{\mathbf{v}} \in \mathbf{H}_{\mathcal{N}}^1(\Omega), \quad (4.4)$$

where

$$\mathcal{A}(\widehat{\mathbf{u}}; \widehat{\mathbf{v}}) = \int_{\Omega} \sigma(\widehat{\mathbf{u}}) : \varepsilon(\widehat{\mathbf{v}}) \quad \text{and} \quad \mathcal{F}(\widehat{\mathbf{v}}) = \int_{\Omega} \mathbf{f} \cdot \widehat{\mathbf{v}}. \quad (4.5)$$

The bilinear form $\mathcal{A}(\cdot; \cdot)$ is well-defined in the same sense—its value does not depend on the members of the classes used to evaluate it. Same holds true for $\mathcal{F}(\cdot)$ provided (2.4) is valid.

Furthermore, Lemma 3.2 implies that $\mathcal{A}(\cdot; \cdot)$ is coercive on $\mathbf{H}_{\mathcal{N}}^1(\Omega) \times \mathbf{H}_{\mathcal{N}}^1(\Omega)$. Continuity of the form and the right hand side functional is obvious and from the Lax-Milgram theorem it follows that (2.1) has a unique minimizer out of $\mathbf{H}_{\mathcal{N}}^1(\Omega)$. \square

4.2. Complement space minimization

Theorem 4.2. *The minimization problem (4.2) has a unique minimizer from $\mathbf{H}_{\mathcal{C}}^1(\Omega)$ for any load $\mathbf{f} \in L^2(\Omega)$.*

Proof. Since the complement space contains only the trivial rigid displacement, $\mathcal{J}(\cdot)$ is well-defined on $\mathbf{H}_{\mathcal{C}}^1(\Omega)$ without any compatibility condition on the data. To prove that (4.2) has a unique minimizer consider again the necessary condition: seek $\mathbf{u} \in \mathbf{H}_{\mathcal{C}}^1(\Omega)$ such that

$$\mathcal{A}(\mathbf{u}; \mathbf{v}) = \mathcal{F}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_{\mathcal{C}}^1(\Omega). \quad (4.6)$$

The bilinear form $\mathcal{A}(\cdot; \cdot)$ and functional $\mathcal{F}(\cdot)$ are the same as in (4.5), except that the arguments are drawn from $\mathbf{H}_{\mathcal{C}}^1(\Omega)$ and hence are functions.

From Korn's second inequality (3.4) it follows that this form is coercive on $\mathbf{H}_{\mathcal{C}}^1(\Omega)$. It is easy to see that both the form and the functional are also continuous. Existence of a unique minimizer thus follows from Lax-Milgram theorem. \square

Solutions of (4.2) are implicitly constrained by the definition of $\mathbf{H}_{\mathcal{C}}^1(\Omega)$ to have a zero mean and zero mean curl. These constraints can also be imposed explicitly producing an equivalent, but constrained, minimization problem:

$$\inf_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \mathcal{J}(\mathbf{v}) \quad \text{subject to} \quad \mathbf{B}(\mathbf{v}) = \mathbf{0}. \quad (4.7)$$

Using a Lagrange multiplier $\mathbf{y} \in \mathbf{R}^6$ to enforce the constraints gives another problem that is equivalent to (4.2) but is of the saddle-point type:

$$\sup_{\mathbf{y} \in \mathbf{R}^6} \inf_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \mathcal{J}(\mathbf{v}; \mathbf{f}) + \mathbf{y}^T \mathbf{B}(\mathbf{v}). \quad (4.8)$$

The first-order optimality system for (4.8) is to find $(\mathbf{u}, \mathbf{y}) \in \mathbf{H}^1(\Omega) \times \mathbf{R}^6$ such that

$$\begin{aligned} \mathcal{A}(\mathbf{u}; \mathbf{v}) + \mathbf{B}(\mathbf{v})^T \mathbf{y} &= \mathcal{F}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{z}^T \mathbf{B}(\mathbf{u}) &= 0 \quad \forall \mathbf{z} \in \mathbf{R}^6 \end{aligned} \quad (4.9)$$

Theorem 4.3. *The saddle point system (4.9) is well-posed; a unique solution (\mathbf{u}, \mathbf{y}) exists for any $\mathbf{f} \in L^2(\Omega)$.*

Proof. Equation (4.9) is a saddle-point problem and is well-posed if and only if an inf-sup condition holds. For (4.9), we need to demonstrate that there exists a positive constant γ such that for any $\mathbf{y} \in \mathbf{R}^6$

$$\sup_{\mathbf{v} \in \mathbf{H}^1(\Omega)} \frac{\mathbf{y}^T \mathbf{B}(\mathbf{v})}{\|\mathbf{v}\|_1} \geq \gamma |\mathbf{y}|. \quad (4.10)$$

Given $\mathbf{y} \in \mathbf{R}^6$ the proof of Lemma 3.1 allows us to construct $\mathbf{u}_{\mathcal{N}} = \mathbf{W}\mathbf{x} \in \mathcal{N}$ such that $\mathbf{B}(\mathbf{u}_{\mathcal{N}}) = \mathbf{y}$ by solving the system $\mathbf{S}\mathbf{x} = \mathbf{y}$. Furthermore,

$$\|\mathbf{u}_{\mathcal{N}}\|_1 = \|\mathbf{W}\mathbf{x}\|_1 \leq \sum_{i=1}^6 |\xi_i| \|\mathbf{w}_i\|_1 \leq |\mathbf{x}| \sqrt{\sum_{i=1}^6 \|\mathbf{w}_i\|_1^2} \leq \frac{1}{\gamma} |\mathbf{y}|$$

where $1/\gamma = \|\mathbf{S}^{-1}\| \sqrt{\sum_{i=1}^6 \|\mathbf{w}_i\|_1^2}$. Therefore,

$$\frac{\mathbf{y}^T \mathbf{B}(\mathbf{u}_{\mathcal{N}})}{\|\mathbf{u}_{\mathcal{N}}\|_1} = \frac{\mathbf{y}^T \mathbf{y}}{\|\mathbf{u}_{\mathcal{N}}\|_1} \geq \gamma |\mathbf{y}|,$$

which proves the inf-sup condition. \square

4.3. Regularized minimization

Theorem 4.4. *The regularized energy functional (4.3) has a unique minimizer from $\mathbf{H}^1(\Omega)$ for any $\mathbf{f} \in L^2(\Omega)$.*

Proof. The Euler-Lagrange equation for (2.1) reads: find $\mathbf{u} \in \mathbf{H}^1(\Omega)$ such that

$$\mathcal{A}_R(\mathbf{u}; \mathbf{v}) = \mathcal{F}(\mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}^1(\Omega) \quad (4.11)$$

where

$$\mathcal{A}_R(\mathbf{u}; \mathbf{v}) = \int_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) + \mathbf{B}(\mathbf{u}) \Upsilon \mathbf{B}(\mathbf{v})^T \quad (4.12)$$

and the functional $\mathcal{F}(\cdot)$ is the same as in (4.5). Korn's modified inequality (3.7) implies that form $\mathcal{A}_R(\cdot; \cdot)$ is coercive on $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$. Therefore, from the Lax-Milgram lemma we can infer that the regularized problem has a unique minimizer out of $\mathbf{H}^1(\Omega)$. \square

4.4. Relationship between Complement and Regularized Minimization

We point out that the results we established for the unique minimizers of the complement and regularized energy principles did not require the force equilibrium condition (2.4). For (4.2) this was possible because the complement space does not contain any nontrivial rigid displacements. In contrast, for the regularized energy principle (4.3), this was possible because the functional was augmented so that rigid displacements have positive energy. Hence, (4.2) and (4.3) will always generate nonsingular linear systems. However, the fact that (4.2) and (4.3) possess solutions even for incompatible loads raises the question about the proper interpretation of their minimizers as solutions of the pure traction problem. To answer this question we need an auxiliary lemma.

Lemma 4.1. *Given $\mathbf{f} \in L^2(\Omega)$ there exists a unique decomposition $\mathbf{f} = \mathcal{P}\mathbf{f} + \mathcal{Q}\mathbf{f}$, where $\mathcal{P}\mathbf{f}$ is the L^2 projection onto \mathcal{N} and $\mathcal{Q}\mathbf{f}$ is the L^2 projections onto \mathcal{N}^\perp . $\mathcal{Q}\mathbf{f}$ satisfies the force equilibrium condition (2.4).*

Proof. The orthogonal projection $\mathcal{P}\mathbf{f}$ onto \mathcal{N} is defined by the variational equation

$$(\mathcal{P}\mathbf{f}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{N}. \quad (4.13)$$

The elements in \mathcal{N} have the form $\mathbf{W}\mathbf{x}$ where \mathbf{W} is the rigid displacement basis defined in (2.6). Using the ansatz $\mathcal{P}\mathbf{f} = \mathbf{W}\mathbf{x}$ it is easy to see that (4.13) is a 6×6 linear system

$$\left(\int_{\Omega} \mathbf{W}^T \mathbf{W} \right) \mathbf{x} = (\mathbf{f}, \mathbf{W}) \quad (4.14)$$

for the unknown coefficient vector \mathbf{x} . The matrix $\int_{\Omega} \mathbf{W}^T \mathbf{W}$ is the Gram matrix of the rigid displacement basis \mathbf{W} , and is nonsingular. Therefore, (4.14), resp. (4.13), has a unique solution \mathbf{x} which defines the unique projection $\mathcal{P}\mathbf{f} = \mathbf{W}\mathbf{x}$. Let $\mathcal{Q}\mathbf{f} = \mathbf{f} - \mathcal{P}\mathbf{f}$. Then from (4.13)

$$(\mathcal{Q}\mathbf{f}, \mathbf{w}) = (\mathbf{f} - \mathcal{P}\mathbf{f}, \mathbf{w}) = (\mathbf{f}, \mathbf{w}) - (\mathcal{P}\mathbf{f}, \mathbf{w}) = 0 \quad \forall \mathbf{w} \in \mathcal{N},$$

Therefore, $\mathcal{Q}\mathbf{f} \in \mathcal{N}^{\perp}$, i.e., it satisfies the equilibrium condition. \square

The next result characterizes the minimizer of the regularized energy principle.

Theorem 4.5. *If \mathbf{u}_R denotes the unique minimizer of the regularized energy principle (4.3) for some given $\mathbf{f} \in L^2(\Omega)$, then*

$$\mathbf{B}(\mathbf{u}_R) = \Upsilon^{-1} \mathbf{S}^{-T} (\mathcal{P}\mathbf{f}, \mathbf{W}) \quad (4.15)$$

where \mathbf{S} is given by (3.3) and $\mathcal{P}\mathbf{f}$ is the projection defined in Lemma 4.1.

Proof. A minimizer of the regularized energy principle (4.3) satisfies the weak equation (4.11), which is the necessary optimality condition for (4.3). Choose the test function in (4.11) to be a rigid displacement $\mathbf{w} \in \mathcal{N}$. Using that $\varepsilon(\mathbf{w}) = 0$ for $\mathbf{w} \in \mathcal{N}$ and the definition of $\mathcal{P}\mathbf{f}$, the weak equation reduces to

$$\mathbf{B}(\mathbf{u}_R) \Upsilon \mathbf{B}(\mathbf{w})^T = (\mathcal{P}\mathbf{f}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathcal{N}.$$

Because \mathbf{W} spans \mathcal{N} , and $\mathbf{B}(\mathbf{W})$ defines the matrix \mathbf{S} in (2.9), this equation is equivalent to the linear system

$$\mathbf{B}(\mathbf{u}_R) \Upsilon \mathbf{S}^T = (\mathcal{P}\mathbf{f}, \mathbf{W}).$$

Formula (4.15) gives the unique solution of this linear system. \square

The following corollary provides further information about the regularized minimizer.

Corollary 4.1. *If \mathbf{f} satisfies the force equilibrium condition (2.4), then $\mathbf{u}_R \in \mathbf{H}_C^1(\Omega)$.*

Proof. Assume that the load \mathbf{f} satisfies (2.4), i.e., $\mathbf{f} \in \mathcal{N}^{\perp}$. Then, $\mathcal{P}\mathbf{f} = \mathbf{0}$ and from (4.15) in Theorem 4.5 it follows that $\mathbf{B}(\mathbf{u}_R) = \mathbf{0}$, i.e., $\mathbf{u}_R \in \mathbf{H}_C^1(\Omega)$. \square

Theorem 4.5 and Corollary 4.1 indicate that strain energy principle (3.10) represents a regularization by selection rather than by penalty. For compatible loads, \mathbf{u}_R does not depend on the regularization matrix Υ . When the load is incompatible, the regularization terms help to select a unique solution from $\mathbf{H}^1(\Omega)$.

Consider next the potential energy principle (4.2) posed on the complement space.

Theorem 4.6. *Let \mathbf{u}_C denote the unique minimizer of (4.2) for some given $\mathbf{f} \in L^2(\Omega)$. If $\mathbf{f} = \mathcal{Q}\mathbf{f} + \mathcal{P}\mathbf{f}$ where the decomposition is given by Lemma 4.1, then $\mathbf{u}_C = \mathbf{u}_{C,Q} + \mathbf{u}_{C,P}$ where*

1. $\mathbf{u}_{C,Q} \in \mathbf{H}_{\mathcal{N}}^1(\Omega)$ with respect to the compatible load $\mathcal{Q}\mathbf{f}$;

2. $\mathbf{u}_{C,\mathcal{P}}$ solves the weak equation: seek $\mathbf{u}_{C,\mathcal{P}} \in \mathbf{H}_C^1(\Omega)$ such that

$$\mathcal{A}(\mathbf{u}_{C,\mathcal{P}}; \mathbf{v}) = (\mathcal{P}\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_C^1(\Omega) \quad (4.16)$$

Proof. After the ansatz $\mathbf{f} = \mathcal{Q}\mathbf{f} + \mathcal{P}\mathbf{f}$, the weak equation (4.6) reads:

$$\mathcal{A}(\mathbf{u}_C; \mathbf{v}) = (\mathcal{P}\mathbf{f}, \mathbf{v}) + (\mathcal{Q}\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_C^1(\Omega).$$

Define $\mathbf{u}_{C,\mathcal{Q}} \in \mathbf{H}_C^1(\Omega)$ to be the solution of the weak equation

$$\mathcal{A}(\mathbf{u}_{C,\mathcal{Q}}; \mathbf{v}) = (\mathcal{Q}\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_C^1(\Omega).$$

Because $\mathcal{Q}\mathbf{f}$ satisfies the force equilibrium condition (2.4), it follows that $\mathbf{u}_{C,\mathcal{Q}}$ satisfies the weak equation (4.4) with $\mathcal{F}(\mathbf{v}) = (\mathcal{Q}\mathbf{f}, \mathbf{v})$. Thus, $\mathbf{u}_{C,\mathcal{Q}}$ belongs to a minimizing class of a potential energy principle set with respect to $\mathcal{Q}\mathbf{f}$. The difference $\mathbf{u}_C - \mathbf{u}_{C,\mathcal{Q}}$ satisfies the weak equation

$$\mathcal{A}(\mathbf{u}_C - \mathbf{u}_{C,\mathcal{Q}}; \mathbf{v}) = (\mathcal{P}\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}_C^1(\Omega),$$

and so it gives the second solution component $\mathbf{u}_{C,\mathcal{P}}$ in the statement of the theorem. \square

Corollary 4.2. *If \mathbf{f} satisfies the force equilibrium condition (2.4), then*

1. \mathbf{u}_C belongs to the minimizing class of (4.1);
2. Minimizers of the regularized problem (4.3) and the complement space problem (4.2) coincide: $\mathbf{u}_C = \mathbf{u}_R$.

The significance of this corollary is that the relationship between minimizers of the two alternative potential energy principles and the original pure traction problem is revealed. When \mathbf{f} satisfies (2.4), minimizers of (4.2) and (4.3) coincide with the unique member of $\hat{\mathbf{u}}$ associated with \mathbf{f} . Thus, both problems lead to a solution of the pure traction problem. If the data is not compatible, then (4.2) and (4.3) have distinct minimizers. These minimizers are solutions of a perturbed pure traction problem.

5. Finite element methods

Throughout this section we use standard finite element notations. \mathcal{T}_h denotes a uniformly regular partition of Ω into finite elements. For simplicity we consider only simplicial triangulations and standard C^0 Lagrangian finite element spaces P^k , where $k \geq 1$ stands for the polynomial degree employed. The standard nodal basis for P^k will be denoted by $\{\phi_i^h\}_{i=1}^N$. Vector fields are approximated by the product space

$$\mathbf{X}^h = \prod_{i=1}^d P^k,$$

where $d = 2$ or $d = 3$ is the space dimension. We present details for $d = 3$, the planar case is very similar. The basis of \mathbf{X}^h can be expressed in terms of the nodal basis in P^k as $\{\phi_i^h \mathbf{e}_1, \phi_j^h \mathbf{e}_2, \phi_k^h \mathbf{e}_3\}_{i,j,k=1}^N$. We will also use the compact notation $\{\phi_i^h\}$ with the tacit

understanding that the nodes and unknowns have been ordered in some manner. We denote the finite element approximant

$$\mathbf{u}^h \equiv \sum_{i,j,k=1}^N v_{i,1} \phi_i^h \mathbf{e}_1 + v_{j,2} \phi_j^h \mathbf{e}_2 + v_{k,3} \phi_k^h \mathbf{e}_3.$$

With some abuse of notation we will use \mathbf{u} to denote the coefficient vector of the finite element vector function \mathbf{u}^h . The coefficient sets corresponding to finite element expansions of the components of \mathbf{u}^h will be denoted by \mathbf{u}_i . With I_h we denote the standard nodal interpolation operator $H^1(\Omega) \cap C^0(\Omega) \mapsto P^k$, while i_h is the operator $P^k \mapsto \mathbf{R}^N$ that returns the nodal coefficients of a finite element function.

We now discuss the formulation of finite element methods for the pure traction linear elasticity. Let us first point out that the linear equilibrium equation (1.1) and the minimum potential energy principle (2.2) are equivalent in the sense that the differential equation problem is a strong form of the necessary condition for the minimization problem. Traditionally, to derive a discrete problem finite element methods have relied exclusively upon the differential equation description of the elastic body state. The common Galerkin approach is to write (1.1) in weak form and then restrict this variational problem to finite element subspaces. The result is a linear algebraic system whose solution defines the finite element approximation. Thus, Galerkin finite elements use the extant optimization setting implicitly.

In contrast, here we adopt an alternative approach in which discrete problems are obtained directly from a minimum potential energy principle by restricting its admissible states to a finite element subspace. This results in a quadratic programming problem and so solution techniques from numerical optimization may be used.

Another reason to favor the quadratic programming as a starting point in the determination of the finite element solution has to do with the spaces used in (4.1) and (4.2). When rigid displacements are purged from the admissible space or confined to a zero class, the result is a nonstandard space of kinematic states. By nonstandard here we mean spaces whose members are either classes of functions, as in (3.5), or are subject to constraints as in (3.1). The use of nonstandard spaces does not render these settings unusable for finite element methods. However, it does require more careful handling of the ensuing algebraic problems. In what follows we will see that (4.2) is related to an equality constrained quadratic program, (4.1) will give a quadratic program with semi-definite matrix, while (4.3) leads to a problem with symmetric and positive definite matrix.

To state the quadratic programs and discuss their properties we shall need some additional vectors and matrices. Since we consider at least piecewise linear elements, the functions 1, x_1 , x_2 and x_3 are all represented exactly by P^k . Let \mathbf{c} , \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_3 denote the nodal coefficients of their finite element expansions on the mesh \mathcal{T}_h , that is $\mathbf{c} = i_h(1)$, $\mathbf{x}_1 = i_h(x_1)$, $\mathbf{x}_2 = i_h(x_2)$ and $\mathbf{x}_3 = i_h(x_3)$. For example, it is easy to see that $\mathbf{c} = (1, \dots, 1)^T$ is the constant vector in \mathbf{R}^N . The coefficients of the other three functions can also be easily determined once \mathcal{T}_h is given. Next, we introduce the vectors

$$\mathbf{c}_1 = i_h(\mathbf{e}_1); \quad \mathbf{c}_2 = i_h(\mathbf{e}_2); \quad \mathbf{c}_3 = i_h(\mathbf{e}_3);$$

and

$$\mathbf{r}_1 = i_h(\mathbf{e}_1 \times \mathbf{r}); \quad \mathbf{r}_2 = i_h(\mathbf{e}_2 \times \mathbf{r}); \quad \mathbf{r}_3 = i_h(\mathbf{e}_3 \times \mathbf{r}).$$

The first three vectors contain the nodal coefficients of the three translational rigid displacements. We have $\mathbf{c}_1 = \mathbf{e}_1 \otimes \mathbf{c}$, $\mathbf{c}_2 = \mathbf{e}_2 \otimes \mathbf{c}$ and $\mathbf{c}_3 = \mathbf{e}_3 \otimes \mathbf{c}$. The \mathbf{r}_i 's represent nodal coefficients

of the three rotational modes. For example,

$$\mathbf{r}_1 = (\mathbf{0}, -\mathbf{x}_3, \mathbf{x}_2)^T.$$

We arrange these vectors in a matrix as follows:

$$\mathbf{N} = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{r}_1 \quad \mathbf{r}_2 \quad \mathbf{r}_3]. \quad (5.1)$$

The finite element functions corresponding to these coefficient vectors are \mathbf{w}_i^h , $i = 1, \dots, 6$.

We shall also need the symmetric and semi-definite stiffness matrix \mathbf{A} with element i, j

$$\mathbf{A}_{i,j} = \mathcal{A}(\phi_j^h; \phi_i^h), \quad i, j = 1, \dots, 3N. \quad (5.2)$$

and the discrete source term

$$\mathbf{b}_i = \mathcal{F}(\phi_i^h), \quad i = 1, \dots, 3N. \quad (5.3)$$

Our ordering convention induces a natural splitting of \mathbf{A} and \mathbf{b} into blocks of size $N \times N$ and $N \times 1$, respectively. For example the ij th element in the kl th block of \mathbf{A} is given by $\mathcal{A}(\mathbf{e}_l \phi_j^h; \mathbf{e}_k \phi_i^h)$. For a nodal basis

$$\ker(\mathbf{A}) = \text{span}\{\mathbf{c}_i, \mathbf{r}_i\}_{i=1}^3$$

and so $\mathbf{A}\mathbf{N} = \mathbf{0}$.

We will have occasion to use two discrete inner products. The first is the standard Euclidean inner product of \mathbf{x} and \mathbf{y} denoted by $\mathbf{x}^T \mathbf{y}$. The Euclidean norm of a vector will be denoted by $\|\cdot\|$. The second choice of inner product is the discrete $L^2(\Omega)$ inner product of \mathbf{x} and \mathbf{y} denoted by $\mathbf{x}^T \mathbf{M} \mathbf{y}$ where \mathbf{M} is the mass matrix containing the $L^2(\Omega)$ inner products of finite element basis functions. Recall that if \mathbf{x} and \mathbf{y} are the nodal coefficient associated with the functions $g = \sum \xi_i \phi_i^h$ and $h = \sum \eta_i \phi_i^h$, then $(g, h) = \mathbf{x}^T \mathbf{M} \mathbf{y}$ is the discrete $L^2(\Omega)$ inner product of g and h .

To complete our formulation of finite element methods for pure traction problems in linear elasticity, we need to discuss the computation of the finite element minimizers. This is the topic of the next few sections.

5.1. Factor space FEM

Consider first the minimization problem (4.1) and its restriction to a finite element subspace $\mathbf{X}_{\mathcal{N}}^h = \mathbf{X}^h / \mathcal{N}$ of $\mathbf{H}_{\mathcal{N}}^1(\Omega)$. This space is isomorphic to $\mathbf{R}^{3N} / \ker(\mathbf{A})$ and so

$$\min_{\hat{\mathbf{u}}^h \in \mathbf{X}_{\mathcal{N}}^h} \mathcal{J}(\hat{\mathbf{u}}^h; \mathbf{f}) \equiv \min_{\hat{\mathbf{u}} \in \mathbf{R}^{3N} / \ker(\mathbf{A})} \frac{1}{2} \hat{\mathbf{u}}^T \mathbf{A} \hat{\mathbf{u}} - \hat{\mathbf{u}}^T \mathbf{b}. \quad (5.4)$$

Therefore, the quadratic program on the right hand side of (5.4) represents the minimum potential energy principle (4.1) on the finite dimensional subspace $\mathbf{X}_{\mathcal{N}}^h$.

The first-order optimality condition for (5.4) is

$$\mathbf{A} \hat{\mathbf{u}} = \mathbf{b}. \quad (5.5)$$

A solution of (5.5) should be interpreted in the sense of computing *any* member of the minimizing class $\hat{\mathbf{u}}$. Since \mathbf{A} is singular, a direct method cannot be used for this purpose

(at least not without modification so that a zero pivot can be detected; see [4] for a direct solver that handles the hydrostatic pressure mode in the incompressible Stokes equations).

However, an iterative method can be used to solve the minimization problem (5.4) instead of computing a solution to the Euler-Lagrange equation (5.5). Indeed, as long as the \mathbf{b} is in the range of \mathbf{A} the quadratic functional in (5.4) has a finite lower bound. As a result, an iterative method such as the conjugate gradient method will generate a minimizing sequence that will converge modulo $\ker(\mathbf{A})$; see Theorem 13.11, [1, p. 583]. The rate of convergence of an iterative method depends on the ratio $\kappa_\epsilon(\mathbf{A}) = \lambda_{3N}(\mathbf{A})/\lambda_7(\mathbf{A})$ or the effective condition number. We remind that the eigenvalues of the stiffness matrix are ordered as $0 = \lambda_1 = \dots = \lambda_6 \leq \lambda_7, \dots, \leq \lambda_{3N}$. The paper [11] provides further information about the iterative solution of consistent singular systems arising in incompressible flows.

An important practical consideration for (5.5) is that the discrete source \mathbf{b} must be discretely orthogonal to the vectors \mathbf{c}_i and \mathbf{r}_i or equivalently $\mathbf{N}^T \mathbf{b} = \mathbf{0}$; this is a consequence of (2.4) because

$$\mathbf{c}_i^T \mathbf{b} = \int_{\Omega} \mathbf{f} \cdot \mathbf{e}_i \quad \text{and} \quad \mathbf{r}_i^T \mathbf{b} = \int_{\Omega} \mathbf{f} \cdot (\mathbf{e}_i \times \mathbf{r}). \quad (5.6)$$

In other words, solvability of the pure traction problem implies solvability of (5.5) provided integration is carried out *exactly*.

In practice, the source \mathbf{b} and the stiffness matrix \mathbf{A} are computed in floating point arithmetic via quadrature and may not satisfy (5.6). To restore consistency of the system we introduce the discrete analogue of the projection \mathcal{P} onto the rigid displacements space \mathcal{N} of Lemma 4.1 as

$$(\mathcal{P}^h \mathbf{f}, \mathbf{w}_i^h) = (\mathbf{f}, \mathbf{w}_i^h) \quad i = 1, \dots, 6.$$

The functions \mathbf{w}_i^h are the finite element functions whose coefficients are the six vectors in (5.1). The matrix $\mathbf{P} = \mathbf{N}(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{N}^T \mathbf{M}$ is the equivalent algebraic form of \mathcal{P}^h acting directly on the coefficients of finite element functions. A direct calculation reveals that the matrix form of the companion projection \mathbf{Q}^h onto the complement of \mathcal{N} in the finite element space \mathbf{X}^h is

$$\mathbf{Q} = \mathbf{I} - \mathbf{N}(\mathbf{N}^T \mathbf{M} \mathbf{N})^{-1} \mathbf{N}^T \mathbf{M} \quad (5.7)$$

Application of the projector to the linear system (5.5) results in

$$(\mathbf{Q} \mathbf{A} \mathbf{Q}^T) \mathbf{u} = \mathbf{Q} \mathbf{b}.$$

The computed solution $\mathbf{Q}^T \mathbf{u} (= \hat{\mathbf{u}})$ satisfies $\mathbf{w}^T \mathbf{Q}^T \mathbf{u} = 0$ and so the FEM solution is in \mathcal{N}^\perp .

We remark that the algebraic projection $\mathbf{Q}_a = \mathbf{I} - \mathbf{N}(\mathbf{N}^T \mathbf{N})^{-1} \mathbf{N}^T$ may also be used. When a structured triangulation is employed, then \mathbf{Q}_a is essentially an orthogonal discrete $L^2(\Omega)$ projector. If the triangulation is unstructured, then \mathbf{Q}_a is orthogonal with respect to the Euclidean inner product but not the discrete $L^2(\Omega)$ inner product. The use of a projector to maintain consistency of the linear system is not unfounded, especially for unstructured triangulations.

5.2. Constrained FEM

Consider the energy principle (4.2) posed on the complement space $\mathbf{H}_c^1(\Omega)$. We restrict (4.2) to a finite element subspace \mathbf{X}_C^h of $\mathbf{H}_c^1(\Omega)$. Since \mathbf{X}_C^h is isomorphic to \mathbf{R}^{3N-6} , the discrete problem is

$$\min_{\mathbf{u}^h \in \mathbf{X}_C^h} \mathcal{J}(\mathbf{u}^h; \mathbf{f}, \mathbf{t}) \equiv \min_{\mathbf{u} \in \mathbf{R}^{3N-6}} \frac{1}{2} \mathbf{u}^T \mathbf{A}_C \mathbf{u} - \mathbf{u}^T \mathbf{b}_C, \quad (5.8)$$

where \mathbf{A}_C , \mathbf{b}_C , and \mathbf{u} denote a stiffness matrix, right hand side and a coefficient vector relative to the nodal basis of \mathbf{X}_C^h . Again, the right hand side in (5.8) is a quadratic program that represents (4.2) on the finite dimensional subspace \mathbf{X}_C^h .

The first-order optimality condition is

$$\mathbf{A}_C \mathbf{u}_C = \mathbf{b}_C.$$

The caveat here is that in order to form the matrix \mathbf{A}_C and the discrete load \mathbf{b}_C , a conforming basis for \mathbf{X}_C^h is required. If the triangulation \mathcal{T}_h is non-uniform, imposing even the zero mean condition alone upon the basis may lead to significant increase in condition numbers; see [2].

An alternative quadratic program can be obtained from (4.8). The first-order necessary conditions for (4.9) are

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_C \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ \mathbf{0} \end{pmatrix} \quad (5.9)$$

The matrix in (5.9) is called the Karush-Kuhn-Tucker (KKT) matrix. The matrix \mathbf{B} is the FEM implementation of the operator $\mathbf{B}(\cdot)$ defined in (2.7).

There are several ways to solve the KKT linear system. The first approach would solve the indefinite linear system by either a sparse direct method or an iterative method. The second and third methods exploit the structure in the KKT matrix. The *range-space* method for the KKT system requires that \mathbf{A} is nonsingular and so it is not applicable to (5.9). The *null-space* method removes the requirement of nonsingularity of \mathbf{A} and can instead be used. However knowledge of the $(3N - 6)$ -dimensional null-space

$$Z = \{\mathbf{u} \in \mathbf{R}^{3N} \mid \mathbf{B}\mathbf{u} = \mathbf{0}\}$$

of \mathbf{B} is needed. Let $\mathbf{C} \in \mathbf{R}^{3N \times 3N-6}$ denote a matrix whose columns form a basis for Z . Then $\mathbf{u} = \mathbf{C}\mathbf{v}$ for $\mathbf{v} \in \mathbf{R}^{3N-6}$ and the constrained problem in terms of \mathbf{u} is equivalent to

$$\min_{\mathbf{v} \in \mathbf{R}^{3N-6}} \frac{1}{2} \mathbf{v}^T \mathbf{C}^T \mathbf{A} \mathbf{C} \mathbf{v} - \mathbf{v}^T \mathbf{C}^T \mathbf{b}. \quad (5.10)$$

The minimizer of (5.10) is computed by solving the symmetric positive-definite linear system

$$\mathbf{C}^T \mathbf{A} \mathbf{C} \mathbf{v} = \mathbf{C}^T \mathbf{b}. \quad (5.11)$$

The null-space method for (5.9) amounts to constructing the matrix \mathbf{C} and solving the linear system of equations (5.11).

5.3. Regularized FEM

The last quadratic program is derived from the regularized energy principle (4.3). After minimization in (4.3) is restricted to \mathbf{X}^h we obtain the following discrete problem:

$$\min_{\mathbf{u}^h \in \mathbf{X}^h} \mathcal{J}_R(\mathbf{u}^h; \mathbf{f}, \mathbf{t}) \equiv \min_{\mathbf{u} \in \mathbf{R}^{3N}} \frac{1}{2} \mathbf{u}^T \mathbf{A}_R \mathbf{u} - \mathbf{u}^T \mathbf{b}. \quad (5.12)$$

The elements of \mathbf{A}_R are given by $\mathcal{A}_R(\phi_i; \phi_j)$ of (4.12).

In this class of methods, the finite element solution of the pure traction problem is obtained by computing a minimizer of (5.12). The first-order necessary condition for (5.12) is

$$\mathbf{A}_R \mathbf{u} = \mathbf{b}. \quad (5.13)$$

The elements of the matrix \mathbf{A}_R are determined from (4.12) and result in

$$\mathbf{A}_R \equiv \mathbf{A} + \mathbf{B}\Upsilon\mathbf{B}^T$$

where \mathbf{A} is the singular stiffness matrix of order $3N$, \mathbf{B} is a rank-six matrix and Υ is a diagonal matrix of order six containing the regularization parameters. Depending on the choice of Θ_1 and Θ_2 of (2.7), \mathbf{A}_R may be dense even if \mathbf{A} is sparse because \mathbf{B} is a dense rank six matrix. This precludes the use of direct solver unless Θ_1 and Θ_2 are selected properly.

However, the possible disadvantage is only formal as we can solve the regularized system iteratively with very little additional cost added. For instance, to solve (5.13) by the pre-conditioned conjugate gradient algorithm, only the matrix vector product $\mathbf{A}_R\mathbf{u}$ is needed. The following two steps lead to an efficient implementation:

1. compute $\mathbf{w} = \mathbf{A}\mathbf{u}$;
2. compute the vector $\mathbf{y} = \Upsilon(\mathbf{B}^T\mathbf{u})$;
3. update $\mathbf{w} + \mathbf{B}\mathbf{y}$.

Step (1) is a standard part of any finite element solver, so the only additional work involved is in the rank-six update. Step (2) performs the six inner products needed followed by the rank size update in Step (3). The rank six matrix \mathbf{B} can be precomputed and stored rendering the computation of \mathbf{y} efficient.

The following theorem proves fundamental for understanding the structure of \mathbf{A}_R and how the rank-six update modifies the null-space of \mathbf{A} .

Theorem 5.1. *Suppose that $\mathbf{A} = \mathbf{Z}\Lambda\mathbf{Z}^T$ is an eigen-decomposition, where*

$$\mathbf{Z}_6 \equiv \mathbf{Z}\mathbf{E}_6 \equiv \mathbf{Z} \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_6 \end{bmatrix}$$

and $\mathbf{A}\mathbf{Z}_6 = \mathbf{0}$. If $\mathbf{B} = \mathbf{P}\mathbf{B} + \mathbf{Q}\mathbf{B}$ where \mathbf{Q} is defined in (5.7) and $\mathbf{P} = \mathbf{I} - \mathbf{Q}$ are discrete L^2 projectors, then

$$\|\mathbf{A}_R - \mathbf{Z}(\Lambda + \mathbf{E}_6\Upsilon\mathbf{E}_6^T)\mathbf{Z}^T\| \leq \|\Upsilon\| \|\mathbf{Q}\mathbf{B}\|(2 + \|\mathbf{Q}\mathbf{B}\|) \quad (5.14)$$

Proof. From the identity

$$\mathbf{A}_R = \mathbf{A} + \mathbf{B}\Upsilon\mathbf{B}^T = \mathbf{Z}(\Lambda + (\mathbf{Z}^T\mathbf{B})\Upsilon(\mathbf{Z}^T\mathbf{B})^T)\mathbf{Z}^T \quad (5.15)$$

the conclusion of the theorem will follow by examining $\mathbf{Z}^T\mathbf{B}$. By the hypothesis on \mathbf{Z}_6 , we have $\mathbf{Z}^T\mathbf{B} = \mathbf{Z}^T\mathbf{P}\mathbf{B} + \mathbf{Z}^T\mathbf{Q}\mathbf{B} = \mathbf{E}_6\mathbf{B} + \mathbf{Z}^T\mathbf{Q}\mathbf{B}$ and so

$$\mathbf{Z}^T\mathbf{B}\Upsilon(\mathbf{Z}^T\mathbf{B})^T = \mathbf{E}_6\Upsilon\mathbf{E}_6^T + \mathbf{E}_6\Upsilon(\mathbf{B}^T\mathbf{Q}\mathbf{Z})^T + \mathbf{B}^T\mathbf{Q}\mathbf{Z}\Upsilon\mathbf{E}_6^T + \mathbf{B}^T\mathbf{Q}\mathbf{Z}\Upsilon(\mathbf{B}^T\mathbf{Q}\mathbf{Z})^T. \quad (5.16)$$

Substituting (5.16) into (5.15) and using the identity that $\|\mathbf{Z}\| = \|\mathbf{Z}^T\| = 1$ gives the desired result. \square

The theorem demonstrates that the role of Υ is to shift the zero eigenvalues of \mathbf{A} while only perturbing the eigenvectors. Values that are at least as large as the smallest positive eigenvalue of \mathbf{A} but no larger than the largest eigenvalue will suffice.

6. Conclusions

We have developed a framework for the finite element solution of the pure traction linear elasticity based on direct discretization of a minimum potential energy principle. Our approach reveals important connections between finite element approximants and solutions of quadratic programs and prompts a novel regularized formulation of the pure traction problem. Among the principal advantages of this formulation are the possibility to obtain symmetric and positive definite algebraic systems using standard nodal finite element spaces and the elimination of the force equilibrium constraint. These systems can be solved efficiently by preconditioned conjugate gradients with the added cost of just few rank-six updates per iteration. The regularization approach is not a penalty approach and does not require large values of the weights in Υ . Theorem 5.1 reveals that weights just slightly larger than the smallest positive eigenvalue of \mathbf{A} are enough to shift its zero eigenvalues. In practice this means that we can always choose Υ so that conditioning of the regularized system does not exceed the effective condition number of the consistent singular system.

References

- [1] O. Axelsson, *Iterative Solution methods*, Cambridge University Press, Cambridge, 1994.
- [2] Pavel Bochev and R. B. Lehoucq. On the finite element solution of the pure Neumann problem. *SIAM Review*, **47** (2005), no. 1, pp. 50–66.
- [3] S. Brenner and R. Scott, *The mathematical theory of finite element methods*, Springer-Verlag, 1994.
- [4] M. S. Engelman, R. L. Sani, and P. M. Gresho, The implementation of normal and/or tangential boundary conditions in finite element codes for incompressible fluid flow, *International Journal for Numerical Methods in Fluids*, **2** (1982), no. 3, pp. 225–238.
- [5] C. Farhat and F.-X. Roux, An unconventional domain-decomposition method for an efficient parallel solution of large scale finite element systems, *SIAM J. Sci.Stat. Comp.*, **13** (1992), no. 1, pp. 379–396.
- [6] Charbel Farhat and Michel G eradin, On the general solution by a direct method of a large-scale singular system of linear equations: application to the analysis of floating structures, *International Journal on Numerical Methods in Engineering*, **41** (1998), pp. 675–696.
- [7] Morton Gurtin, *An introduction to Continuum Mechanics*, Academic Press, Boston, MA, 1981.
- [8] M. Papadrakakis and Y. Fragakis, An integrated geometric-algebraic method for solving semi-definite problems in structural mechanics, *Computer Methods in Applied Mechanics and Engineering*, **190** (2001), pp. 6513–6532.
- [9] P. Vanek, M. Brezina, and J. Mandel, Convergence of algebraic multigrid based on smoothed aggregation, *Numerische Mathematik*, **88** (2001), pp. 559–579.
- [10] P. Vanek, J. Mandel, and M. Brezina, Smoothed aggregation for second and fourth order elliptic problems, *Computing*, **65** (1996), pp. 179–196.
- [11] Andrew Yeckel and Jeffrey J. Derby, On setting a pressure datum when computing incompressible flows, *International Journal for Numerical Methods in Fluids*, **29** (1999), no. 1, pp. 19–34.