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Layer-adapted Methods for a Singularly Perturbed Singular Problem

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Abstract — In the present paper we analyze linear finite elements on a layer-adapted mesh for a boundary value problem characterized by the overlapping of a boundary layer with a singularity. Moreover, we compare this approach numerically with the use of adapted basis functions, in our case modified Bessel functions. It turns out that as well adapted meshes as adapted basis functions are suitable where for our one-dimensional problem adapted bases work slightly better.

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1. Introduction

Let us consider the following singularly perturbed singular problem:

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{du}{dx}) + c(x)u = f(x), \qquad x \in (0,1) \quad \text{with} \qquad u(0) = u(1) = 0. \tag{1.1}$$

Here, $\alpha \in (0,1)$ is a fixed parameter. For $\alpha \geqslant 1$ the character of the singularity at zero changes such that only one boundary condition at $x = 1$ is correct. Moreover, the perturbation parameter ε is small ($0 < \varepsilon \ll 1$). We assume c and f to be sufficiently smooth and assume $c(x) > c_0^2 > 0$ for $x \in [0,1]$.

As we will show in Section 2, problem (1.1) admits a unique weak solution which additionally belongs to $C[0,1]$. In the vicinity of $x = 0$, however, we have to take into account the boundary layer and the singularity of u in the sense that only $x^\alpha u' \in C[0,1]$, for instance.

Even for more general singular problems S. Meyer thoroughly discussed in [8] the analytical behaviour of the solutions of (1.1) (cf. [9] for a short version). Concerning numerical methods for solving (1.1) almost nothing is known.

In [4] we proposed to solve the problem on a uniform mesh with finite elements but operator adapted basis functions. However, these basis functions turn out to be the relatively complicated modified Bessel functions and we were not able to realize the method in that time. By now the available software allows the implementation of the idea given in [4]. In Section 3.2, we shall describe the basic features of the method and present numerical results. Moreover, we shall compare the results with a currently very popular technique: the use of standard finite elements on a layer-adapted mesh.

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So far the analysis of finite element methods on layer-adapted meshes is mostly restricted to problems with "standard" boundary layers, see [12] and [7]. It is the main aim of our paper to analyze a more complicated layer structure, characterized by the overlapping with a singularity. Because our problem (1.1) is relatively simple, the basic principles of the mesh construction are easy to understand and can then hopefully be extended to more complicated problems in several space dimensions in the future.

In Section 2, we shall present some basic facts of the analytical structure of the solution and the related mesh construction in the case $\varepsilon = 1$. In Section 3, we present for the singularly perturbed case as well results for linear splines on layer-adapted meshes as for adapted splines on an uniform mesh. Here we repeat some basic facts from the not widely known paper [4]. Finally, in Section 4 we present a numerical comparison of both methods.

Throughout the text we use standard notation for Sobolev spaces, i.e., $|\cdot|_{k,\Omega}$ and $\|\cdot\|_{k,\Omega}$ are the usual Sobolev seminorm and norm on $H^k(\Omega) = W^{k,2}(\Omega)$. Furthermore, if not otherwise remarked, the occuring numbers C are generic constants that are independent of N and ε.

2. The non-singularly perturbed case $\varepsilon = 1$

In this Section, we consider the boundary value problem

$$-\frac{d}{dx}(x^\alpha \frac{du}{dx}) + c(x)u = f(x), \qquad x \in (0,1), \quad u(0) = u(1) = 0 \tag{2.1}$$

and its discretization with linear finite elements. Introducing

$$H^{1,\alpha}(0,1) := \{v : \int_0^1 x^\alpha (v')^2 + \int_0^1 v^2 < \infty\} \tag{2.2}$$

and

$$H_0^{1,\alpha}(0,1) := \{v \in H^{1,\alpha}(0,1) : v(0) = v(1) = 0\} \tag{2.3}$$

it is easy to show that (2.1) admits a unique solution in $H_0^{1,\alpha}(0,1)$. Moreover, from

$$v(x) = \int_0^x t^{-\alpha/2}(t^{\alpha/2}v'(t))dt$$

it follows

$$| v(x) | \leqslant (\int_0^1 t^{-\alpha} dt)^{1/2} (\int_0^1 t^\alpha v'^2(t) dt)^{1/2}. \tag{2.4}$$

Thus, the continuous embedding $H_0^{1,\alpha}(0,1) \hookrightarrow C[0,1]$ holds and we have a Friedrichs-type inequality

$$\int_0^1 v^2 \leqslant C \int_0^1 t^\alpha v'^2(t) dt \quad \text{for all} \quad v \in H_0^{1,\alpha}(0,1). \tag{2.5}$$

Consequently, we can use the norm

$$\|v\|_{1,\alpha} := \|v\|_{H_0^{1,\alpha}} := (\int_0^1 x^\alpha v'^2)^{1/2} \qquad \text{for} \qquad v \in H_0^{1,\alpha}(0,1). \tag{2.6}$$

Next let be given some grid $\{x_i\}_{i=0}^N$, i.e.,

$$0 = x_0 < x_1 < \ldots. \quad < x_{N-1} < x_N = 1$$

with mesh sizes $h_i = x_i - x_{i-1}$. Based on the bilinear form

$$a(v,w) := \int_0^1 x^\alpha v'w' + \int_0^1 c\,vw$$

we discretize (2.1) with linear finite elements. Let $V_h \cap H_0^{1,\alpha}(0,1)$ be the corresponding finite element space. Then $u_h \in V_h$ satisfies

$$a(u_h, v_h) = (f, v_h) \quad \text{for all} \quad v_h \in V_h$$

and the Cea Lemma yields the error estimate

$$\|u - u_h\|_{1,\alpha} \leqslant C\|u - u^I\|_{1,\alpha}, \tag{2.7}$$

if u^I denotes the nodal interpolant of the given continuous function u.

The classical theory of weakly singular differential equations (cf. [10]) tells us that the solution u of (2.1) admits a decomposition

$$u = \tilde{u} + \phi \tag{2.8}$$

into a smooth part $\tilde{u} \in H^2(0,1)$ and a singular part ϕ. The singular part satisfies

$$|\phi^{(k)}(x)| \leqslant Cx^{-\alpha-(k-1)} \quad \text{for} \quad k = 1, 2. \tag{2.9}$$

Next we introduce a graded mesh related to

$$x_i = (ih)^{1/\mu} \quad \text{with some constant} \quad \mu \leqslant 1, \tag{2.10}$$

here $i = 0, 1, ..., N$ and $h \cdot N = 1$. Then we have the following estimates for the local mesh size $h_i = x_i - x_{i-1}$:

$$\frac{2^{1-\frac{1}{\mu}}}{\mu}(ih)^{\frac{1}{\mu}-1} \leqslant \frac{h_i}{h} \leqslant \frac{1}{\mu}(ih)^{\frac{1}{\mu}-1}, \tag{2.11}$$

The interpolation error of the smooth part of the solution satisfies, as usual,

$$|\tilde{u} - \tilde{u}^I|_1 \leqslant Ch|\tilde{u}|_2, \quad \|\tilde{u} - \tilde{u}^I\|_0 \leqslant Ch^2|\tilde{u}|_2, \tag{2.12}$$

Hence,

$$\|\tilde{u} - \tilde{u}^I\|_{1,\alpha} \leqslant |\tilde{u} - \tilde{u}^I|_1 \leqslant \frac{C}{\mu} h|u|_2$$

because $h_i \leqslant \frac{1}{\mu}h$.

Next we estimate $\|\phi - \phi^I\|_{1,\alpha}$, first on $(x_k, 1)$ and then on the interval $(0, x_k)$, for some $k < N$ still to be chosen

$$\begin{aligned} \textstyle\int_{x_k}^1 x^\alpha(\phi - \phi^I)'^2 \quad &= \quad \sum_{i=k+1}^{N} \textstyle\int_{x_{i-1}}^{x_i} x^\alpha(\phi - \phi^I)'^2 \\ &\overset{(2.9)}{\leqslant} \quad C \sum_{i=k+1}^{N} x_i^\alpha h_i (h_i x_{i-1}^{-\alpha-1})^2 \overset{(2.10),(2.11)}{\leqslant} C\tfrac{1}{\mu^3}(1+\tfrac{1}{k})^{\frac{2+2\alpha}{\mu}} h^{\frac{1-\alpha}{\mu}} \sum_{i=k+1}^{N} \frac{1}{i^{3-\frac{1-\alpha}{\mu}}}. \end{aligned}$$

The choice $\mu = (1-\alpha)/2$ leads to the order $\mathcal{O}(h^2 \ln \frac{1}{h})$ but with $\mu < (1-\alpha)/2$ the term of the right hand side is of optimal order $\mathcal{O}(h^2)$ for some fixed $\mu \geqslant \mu_0 > 0$. Since

$$\sum_{i=k+1}^{N} \frac{1}{i^{3-\frac{1-\alpha}{\mu}}}. \leqslant C \int_{k+1}^{N} s^{\frac{1-\alpha}{\mu}-3} ds \leqslant C \frac{\mu}{1-\alpha-2\mu} h^{2-\frac{1-\alpha}{2}}, \tag{2.13}$$

we then have

$$\int_{x_k}^{1} x^\alpha (\phi - \phi^I)'^2 \leqslant \underbrace{C \frac{(1+\frac{1}{k})^{\frac{2+2\alpha}{\mu}}}{\mu^2 (1-\alpha-2\mu)}}_{=:C_1(k)} \; h^2 \tag{2.14}$$

On the first k subintervals using (2.9) one directly gets

$$\int_0^{x_k} x^\alpha (\phi - \phi^I)'^2 \leqslant C(\int_0^{x_k} x^\alpha \phi'^2 + \int_0^{x_k} x^\alpha (\phi^I)'^2) \tag{2.15}$$

$$\leqslant C \int_0^{x_k} x^{-\alpha} \tag{2.16}$$

$$\leqslant \underbrace{C \frac{k^{\frac{1-\alpha}{\mu}}}{1-\alpha}}_{=:C_2(k)} \; h^{\frac{1-\alpha}{\mu}} \leqslant C_2 \, h^2 \tag{2.17}$$

As C_1 gets smaller and C_2 gets bigger for $k \to N$, there must be an optimal choice for our degree of freedom x_k. Furthermore some value for $\mu \in (0, \frac{1-\alpha}{2})$ has to be fixed. In practise it turns out that a value of about $0.8 \ldots 0.95$ times $\frac{1-\alpha}{2}$ yields the best results.

Lemma 2.1. *On the graded mesh* (2.10) *with* $0 < \mu_0 \leqslant \mu < (1-\alpha)/2$ *the error of the linear finite element approximation of the boundary value problem* (2.1) *satisfies*

$$\|u - u_h\|_{1,\alpha} \leqslant Ch. \tag{2.18}$$

Note that on a uniform mesh ($\mu = 1$) we get an error of order $\mathcal{O}(h^{(1-\alpha)/2})$.

3. The singularly perturbed case

3.1. Linear elements on a layer-adapted mesh

From [8] and [9] it is known that the solution u of problem (1.1) admits a decomposition

$$u = S + E_0 + E_1, \tag{3.1}$$

where S is smooth and its derivatives are uniformly bounded with respect to ε. With $\eta = (1-x)/\varepsilon$ the layer function $\hat{E}_1$ at $x = 1$, defined by $\hat{E}_1(\eta) = E_1(x)$, satisfies

$$-\frac{d^2 \hat{E}_1}{d\eta^2} + c(1)\, \hat{E}_1 = 0. \tag{3.2}$$

Consequently, for large η we can use the estimate $|\hat{E}_1(\eta)| = |\exp(-\sqrt{c(1)}\eta)| \leqslant \exp(-c_0\eta)$. On the other side, with $\xi = x/(\varepsilon^{\frac{2}{2-\alpha}})$ the layer function $\hat{E}_0$ at $x = 0$ given by $\hat{E}_0(\xi) = E_0(x)$ is the exponentially decreasing solution of

$$-\frac{d}{d\xi}(\xi^\alpha \frac{d\hat{E}_0}{d\xi}) + c(0)\,\hat{E}_0 = 0. \tag{3.3}$$

Similar as in Section 2 the singularity of $\hat{E}_0$ at $\xi = 0$ is characterized by

$$|\frac{d^k\hat{E}_0(\xi)}{d\xi^k}| \leqslant C\xi^{-\alpha-(k-1)} \quad \text{for} \quad k = 1, 2. \tag{3.4}$$

To get an asymptotic estimate for $\xi \to \infty$ we substitute in (3.3), (cf. also [6], 2·162 (Ia))

$$\hat{E}_0(\xi) = \xi^{\frac{1-\alpha}{2}} W(\frac{2\sqrt{c(0)}}{(2-\alpha)}\xi^{\frac{2-\alpha}{2}}) \tag{3.5}$$

$$\text{and} \quad z = \frac{2\sqrt{c(0)}}{(2-\alpha)}\xi^{\frac{2-\alpha}{2}} \tag{3.6}$$

and obtain the following second order differential equation

$$z^2\frac{d^2}{dz^2}W + z\frac{d}{dz}W - (z^2 + (\frac{1-\alpha}{2-\alpha})^2)W = 0. \tag{3.7}$$

Its solutions are the well known modified Bessel functions I_ν, K_ν of order $\nu = \frac{1-\alpha}{2-\alpha}$. Finally, resubstituting yields the fundamental solutions to (3.3). Since we are only interested in the exponentially decreasing solutions we know the structure of our layer function $\hat{E}_0$:

$$\hat{E}_0(\xi) = C\xi^{\frac{1-\alpha}{2}} K_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{c(0)}}{(2-\alpha)}\xi^{\frac{2-\alpha}{2}}\right). \tag{3.8}$$

Thereafter the asymptotic behaviour of the modified Bessel functions for large arguments (cf. [1], 9.7.2) tells us for large ξ

$$|\hat{E}_0(\xi)| \leqslant C\exp\left(-c_0\frac{\xi^{\frac{2-\alpha}{2}}}{\frac{2-\alpha}{2}}\right). \tag{3.9}$$

Remark 3.1. Similarly, with the knowledge that $K'_{\frac{1-a}{2-a}}$ has a similar asymptotic behavior as $K_{\frac{1-a}{2-a}}$ (cf. [1], 9.7.4) one can show that as well it holds

$$|\frac{d}{d\xi}\hat{E}_0(\xi)| \leqslant C\exp\left(-c_0\frac{\xi^{\frac{2-\alpha}{2}}}{\frac{2-\alpha}{2}}\right). \tag{3.10}$$

Based on that information we define the transition points τ_0 and τ_1 of a modified Shishkin-type mesh [7, 12, 13]. While τ_1 is as usual defined by $\exp(-c_0(1-\tau_1)/\varepsilon) = N^{-2}$, thus

$$\tau_1 = 1 - 2\frac{\varepsilon}{c_0}\ln N,$$

we define the transition point τ_0 similarly by

$$\exp(-\frac{2c_0}{2-\alpha}\frac{\tau_0^{\frac{2-\alpha}{2}}}{\varepsilon}) = N^{-2}.$$

Therefore we get

$$\tau_0 = ((2-\alpha)\frac{\varepsilon}{c_0}\ln N)^{\frac{2}{2-\alpha}}. \tag{3.11}$$

Assume, for simplicity, that N is divisible by 3 and set

$$\tau_0 = x_{N/3} \quad , \quad \tau_1 = x_{2N/3}.$$

We decompose $[0,\tau_0],[\tau_0,\tau_1],[\tau_1,1]$ in $\frac{N}{3}$ subintervals, on $[\tau_0,\tau_1]$ and $[\tau_1,1]$ we choose an equidistant subdivision. In $[0,\tau_0]$, however, as in Section 2 we define with $h=(3\tau_0)/N$

$$x_i = \tau_0(\frac{i\,h}{\tau_0})^{\frac{1}{\mu}} \quad \text{for} \quad i=0,1,...,N/3. \tag{3.12}$$

Introducing

$$\|v\|^2_{\varepsilon,\alpha} := \varepsilon^2\|v\|^2_{1,\alpha} + \|v\|^2_0 \tag{3.13}$$

the finite element approximation satisfies

$$\|u-u^N\|_{\varepsilon,\alpha} \leqslant C\|u-u^I\|_{\varepsilon,\alpha}$$

and we have to estimate the interpolation error for the linear interpolant on our modified Shishkin mesh.

Remark that for the non-singular problem ($\alpha=0$) it is well known (cf. [12], p. 405, (3.149)) that the interpolation error satisfies

$$\|u-u^I\|_\varepsilon \leqslant C(\varepsilon^{1/2}N^{-1}\ln N + N^{-2}),$$

here the L_2 interpolation error leads to $\mathcal{O}(N^{-2})$ and the weighted H^1 semi-norm generates the term of order $\mathcal{O}(\varepsilon^{1/2}N^{-1}\ln N)$.

On our mesh for the interpolation error of S and E_1 we can simply take the known results but we carefully have to study the interpolation error of E_0. To do that we transform the corresponding integrals based on the substitution $x=\xi\varepsilon^{\frac{2}{2-\alpha}}$

$$\varepsilon^2\int_0^{\tau_0} x^\alpha(E_0-E_0^I)'^2dx = \varepsilon^{\frac{2}{2-\alpha}}\int_0^{\xi_0}\xi^\alpha(\frac{d}{d\xi}(\hat{E}_0-\hat{E}_0{}^I))^2d\xi$$

with $\xi_0 = (\dfrac{2-\alpha}{c_0}\ln N)^{\frac{2}{2-\alpha}}$. Next we can apply the results of Section 2

$$\varepsilon^2\int_0^{\tau_0} x^\alpha(E_0-E_0^I)'^2dx \leqslant C\,\varepsilon^{\frac{2}{2-\alpha}}h^2 \leqslant C\varepsilon^{\frac{2}{2-\alpha}}(N^{-1}(\ln N)^{\frac{2}{2-\alpha}})^2 \tag{3.14}$$

For $x \geqslant \tau_0$ we first apply an inverse inequality

$$\begin{aligned}\varepsilon^2\int_{\tau_0}^1 x^\alpha(E_0-E_0^I)'^2dx \leqslant& C\varepsilon^2(\int_{\tau_0}^1 x^\alpha E_0'^2dx + \int_{\tau_0}^1 x^\alpha(E_0^I)'^2dx) && (3.15)\\ \leqslant& C\varepsilon^2(\int_{\tau_0}^1 x^\alpha E_0'^2dx + N^2\int_{\tau_0}^1 x^\alpha(E_0^I)^2dx) && (3.16)\end{aligned}$$

and use, as usual on Shishkin meshes, the smallness of E_0 and its derivative (compare (3.10))

$$\varepsilon^2 \int_{\tau_0}^{1} x^\alpha (E_0 - E_0^I)'^2 dx \leqslant C\varepsilon^2 (N^{-4} + N^{-2}). \tag{3.17}$$

To estimate the L_2 interpolation error of E_0 we use for $x \geqslant \tau_0$ the smallness of E_0 to obtain

$$\|E_0 - E_0^I\|_{0,(\tau_0,1)} \leqslant CN^{-2},$$

and transform again on $(0, \tau_0)$:

$$\int_0^{\tau_0} (E_0 - E_0^I)^2 dx = \varepsilon^{\frac{2}{2-\alpha}} \int_0^{\xi_0} (\hat{E}_0 - \hat{E}_0{}^I)^2 d\xi.$$

Next we apply (2.5) to replace the L_2 error by the error with respect to $\|\cdot\|_{1,\alpha}$ and obtain similarly to (3.14)

$$\|E_0 - E_0^I\|_{0,(0,\tau_0)} \leqslant C\varepsilon^{\frac{1}{2-\alpha}} N^{-1} (\ln N)^{\frac{2}{2-\alpha}}.$$

Summarizing, we proved

Theorem 3.1. *On the singularity-adapted Shishkin-type mesh with* $0 < \mu_0 \leqslant \mu < (1 - \alpha)/2$ *the error of the finite element approximation of problem* (1.1) *satisfies*

$$\|u - u^N\|_{\varepsilon,\alpha} \leqslant C(\varepsilon^{\frac{1}{2-\alpha}} N^{-1} (\ln N)^{\frac{2}{2-\alpha}} + N^{-2} + \varepsilon^{1/2} N^{-1} (\ln N)).$$

3.2. Adapted elements on an uniform mesh

Exponentially fitted basis functions have been introduced by Hemker [5], later its analysis was substantially simplified in [11]. The recent state of the art is to find in [2] or [12]. The splines used in most papers are solutions of local boundary value problems for equations with *constant* coefficients or (in 2D) tensor products of such functions. Some exception is [3], where for a turning point problem parabolic cylinder functions are used to represent the basic splines.

For simplicity we use an equidistant mesh and denote by h the step size. To define our adapted splines, we first replace the functions c, f that occur in the given boundary value problem

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{du}{dx}) + c(x)u = f(x), \qquad x \in (0,1) \quad \text{with} \qquad u(0) = u(1) = 0 \tag{3.18}$$

by piecewise constant approximations c_h, f_h, i.e.,

$$c_h(x) = c_i, \qquad f_h(x) = f_i \qquad \forall x \in (x_{i-1}, x_i), \quad i = 1, \ldots, N,$$

with appropriate c_i, $f_i \in \mathbb{R}$. A natural choice is

$$c_i = c(x_{i-1} + h/2), \quad f_i = f(x_{i-1} + h/2). \tag{3.19}$$

The approximation u_h of u is now defined as the solution of

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{du_h}{dx}) + c_h(x)u_h(x) = f_h(x), \quad x \in (0,1) \quad \text{with} \qquad u_h(0) = u_h(1) = 0. \tag{3.20}$$

As supposed, we have $c(x) \geqslant c_0^2$ for $x \in [0,1]$. This leads to $c_h > 0$ and as a consequence problem (3.20) possesses a unique weak solution u_h which is continuously differentiable in $(0,1)$. Using the weak maximum principle one can prove, cf. [4], for every first order approximation c_h, f_h of c, f

$$\|u - u_h\|_\infty \leqslant C\,h. \tag{3.21}$$

In [4] the c_h, f_h were chosen in such a way that u_h represents an upper (or lower) approximations of u. Then the possible order is indeed one. For the choice (3.19) we conjecture that the method even achieves the order 2.

In the non-singularly perturbed case convergence of order 2 can be proved based on Lemma 5.1 in [11] and the properties of the related Green's function. A similar result can be found in [14]. In the singularly perturbed case worse properties of the related Green's function make an analogue analysis more delicate.

The structure of u_h over each subinterval $\Omega_i := (x_{i-1}, x_i)$ is known. If $u_i := u_h(x_i)$, then u_h locally forms the solution of

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{du_h}{dx}) + c_i u_h(x) = f_i, \quad x \in \Omega_i \quad \text{with} \qquad u_h(x_{i-1}) = u_{i-1},\ u_h(x_i) = u_i. \tag{3.22}$$

The bounday value problems (3.22) are linear and each related homogeneous differential equation

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{du_h}{dx}) + c_i u_h(x) = 0, \quad x \in \Omega_i \tag{3.23}$$

can be transformed into the defining differential equation for the modified Bessel functions (compare (3.7)). Thus, one obtains the following two linearly independent solutions on every subinterval.

$$v_{i1}(x) := x^{\frac{1-\alpha}{2}} I_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{c_i}}{\varepsilon\,(2-\alpha)} x^{\frac{2-\alpha}{2}}\right), \qquad v_{i2}(x) := x^{\frac{1-\alpha}{2}} K_{\frac{1-\alpha}{2-\alpha}}\left(\frac{2\sqrt{c_i}}{\varepsilon\,(2-\alpha)} x^{\frac{2-\alpha}{2}}\right),$$

where I_ν and K_ν again denote the modified Bessel functions of order ν. With the aid of v_{i1} and v_{i2} we define basis functions $\phi_j \in C(0,1)$, $j = 1, \dots, N-1$ and $\psi_j \in C(0,1)$, $j = 1, \dots, N$ such that they locally form the solutions of the boundary value problems

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{d\phi_j}{dx}) + c_i \phi_j(x) = 0, \quad x \in \Omega_i \quad \text{with} \qquad \phi_j(x_i) = \delta_{ij} \tag{3.24}$$

and

$$-\varepsilon^2 \frac{d}{dx}(x^\alpha \frac{d\psi_j}{dx}) + c_i \psi_j(x) = \delta_{ij}, \quad x \in \Omega_i \quad \text{with} \qquad \psi_j(x_{i-1}) = 0,\ \psi_j(x_i) = 0, \tag{3.25}$$

respectively.

Thus, the basis functions ϕ_i and ψ_i can be represented by

$$\phi_i(x) = \begin{cases} k_{i1}\, v_{i1}(x) + k_{i2}\, v_{i2}(x), & \text{for} \quad x \in \overline{\Omega}_i \\ d_{i+1,1}\, v_{i+1,1}(x) + d_{i+1,2}\, v_{i+1,2}(x), & \text{for} \quad x \in \Omega_{i+1}\,. \\ 0, & \text{otherwise} \end{cases} \tag{3.26}$$

with appropriate constants k_{i1}, k_{i2}, d_{i1}, $d_{i2} \in \mathbb{R}$ and by

$$\psi_i(x) = \begin{cases} \dfrac{1}{c_i}\,(1-\phi_{i-1}(x)-\phi_i(x)), & \text{for} \quad x \in \Omega_i \\ 0, & \text{otherwise} \end{cases}, \tag{3.27}$$

respectively. The constants k_{i1}, k_{i2}, d_{i1}, d_{i2} in the definition (3.26) of the functions ϕ_i are defined by the condition $\phi_i(x_j) = \delta_{ij}$.

Now, we have to find the parameters u_i of the representation

$$u_h(x) = \sum_{i=1}^{N-1} u_i\,\phi_i(x) \;+\; \sum_{i=1}^{N} f_i\,\psi_i(x), \quad x \in [0,1]. \tag{3.28}$$

The known differentiability of u_h in $(0,1)$ yields conditions at the inner grid points. With the structure (3.26), (3.27) we obtain the tridiagonal system

$$a_i\,u_{i-1} + b_i\,u_i + c_i\,u_{i+1} = r_i \quad i = 1,\dots,N-1 \tag{3.29}$$

with $u_0 := 0$, $u_N := 0$ and the coefficients

$$a_i = \phi'_{i-1}(x_i-0), \quad b_i = \phi'_i(x_i-0) - \phi'_i(x_i+0), \quad c_i = -\phi'_{i+1}(x_i+0) \tag{3.30}$$

and the right hand side

$$r_i = -f_i\,\psi'_i(x_i-0) + f_{i+1}\,\psi'_{i+1}(x_i+0). \tag{3.31}$$

The monotonicity properties of the basis functions ϕ_i imply that the coefficients of the linear system satisfy $a_i < 0, b_i > 0$ and $c_i < 0$. Together with the regularity of system (3.29) which is a consequence of the coercivity of the operator related to (3.22) it proves that the matrix of the discrete system is a M-matrix. The M-matrix property allows us to determine the unknown coefficients $\{u_i\}_{i=1}^{N-1}$ of (3.28) in a stable way.

However, the magnitude of the occurring modified Bessel functions requires a more elaborate way to assemble the linear system. For ε smaller than 10^{-3} the respective function values $v_{i1}(x_i)$ and $v'_{i1}(x_i)$ are beyond 10^{350} whereas $v_{i2}(x_i)$ and $v'_{i2}(x_i)$ are numbers smaller than 10^{-350}. In this way it is impossible for a computer with regular precision to represent these numbers. The coefficients $k_{i1}, k_{i2}, d_{i1}, d_{i2}$ have to compensate in magnitude for the extreme values to combine the correct basis functions. A closer look leads to the idea not to compute these very numbers, e.g. $v_{i1}(x_i)$ or k_{i1}, but their logarithms. Since only products of very huge and very tiny numbers will be incorporated in the linear system, these logarithms can be added and the corresponding base can be taken to that sum yielding the correct entry in the linear system. Logarithms of the modified bessel functions can be accessed e.g. via an asymptotic expansion (cf. [1], 9.7.1 - 9.7.4).

Remark 3.2. A different way to assemble a discrete system for the modified Bessel function approach can be described by defining the functions v_{i1} and v_{i2} as discontinuous basis functions on Ω_i. The discrete solution then has the following structure:

$$u_h(x) = c_{i1}v_{i1}(x) + c_{i2}v_{i2}(x) + \frac{f_i}{c_i}, \quad x \in \Omega_i,\; i = 1,\dots,N.$$

The $2N$ unknowns $\bigcup\limits_{i=1}^{N}\{c_{i1}, c_{i2}\}$ and $2N$ equations (continuity of u_h and its derivative at the inner grid points and the two boundary conditions) together constitute the linear system that theoretically yields the same results as the former implementation. However, the coefficient matrix contains function values of the fundamental solutions v_{i1}, v_{i2} and their derivatives, respectively, that are of cosmic magnitude in the singular perturbed case. Thus, even for moderate perturbation parameters ε the condition number of the coefficient matrix virtually explodes. As previously discussed, for even smaller numbers ε the computer cannot even represent all coefficients correctly.

4. Numerical experiments

To illustrate the theoretical results in Section 2 and 3 we numerically investigate two test problems and compare the results. At first we consider a singular equation like (1.1) but with $\varepsilon = 1$

$$\begin{aligned} -\frac{d}{dx}(x^{\frac{1}{2}}\frac{du}{dx}) + (1 + x^{\frac{1}{2}})u &= f, \qquad x \in (0,1) \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{4.1}$$

The right hand side f is constructed in such a way that the exact solution has the typical singular behavior at zero: $u(x) = x^{\frac{1}{2}}(1 - \sin(\frac{\pi}{2}x))$. Moreover, on the one hand we use linear FEM on a graded mesh with the choice $\mu = 0.2$ for the grading parameter, on the other hand the adapted basis functions on an equidistant grid with the same number of unknowns. Additionally both methods are compared with the convergence of linear FEM on the same equidistant grid.

Table 4.1 compares the errors of the two approaches in some discrete approximation of the maximum norm, $\|\cdot\|_{\infty,*}$, together with its experimental order of convergence (EOC). The discrete norm $\|\cdot\|_{\infty,*}$ serves as a tight lower bound for the continuous maximum norm $\|\cdot\|_\infty$ and is defined by the maximal absolute function value of all grid points *and* of one hundred additional sample points on every interval. Figure 4.1 displays the decay of the errors for the three approaches on a double logarithmic scale. Additionally, we present the error decay of the finite element solution in the $H^{1,\alpha}$-norm.

One can observe that the linear FEM solution on the graded mesh converges with second order in the $\|\cdot\|_{\infty,*}$-norm. Also the solution using adapted basis functions converges with almost second order in the same norm. As expected, linear FEM on a uniform grid has some difficulties with the singular behaviour of the exact solution performing an approximate $\|\cdot\|_{\infty,*}$-convergence of order $\frac{1}{2}$ (see Fig. 4.1).

In the second example we want to throw some light on the singular perturbed case. Our model equation is of the following form:

$$\begin{aligned} -\varepsilon^2\frac{d}{dx}(x^{\frac{1}{2}}\frac{du}{dx}) + (1 + x^2)u &= 2 + \sin(2\pi x), \quad x \in (0,1) \\ u(0) &= 0, \\ u(1) &= 0. \end{aligned} \tag{4.2}$$

For this boundary value problem we do not know the exact solution. Instead we compute a reference solution using linear FEM on a very fine graded S-mesh of about one million unknowns.

k ($3 \cdot 2^k$ unknowns)	Bessel in $\|\cdot\|_{\infty,*}$	EOC	FEM in $\|\cdot\|_{\infty,*}$	EOC	FEM in $\|\cdot\|_{1,\alpha}$	EOC
4 (48)	4.522e-05	1.927	5.155e-03	1.894	5.420e-02	0.904
5 (96)	1.191e-05	1.924	1.344e-03	1.939	2.829e-02	0.938
6 (192)	3.141e-06	1.923	3.448e-04	1.963	1.456e-02	0.958
7 (384)	8.280e-07	1.924	8.775e-05	1.974	7.425e-03	0.972
8 (768)	2.181e-07	1.925	2.221e-05	1.982	3.763e-03	0.981
9 (1536)	5.740e-08	1.926	5.601e-06	1.988	1.899e-03	0.987
10 (3072)	1.509e-08	1.927	1.409e-06	1.991	9.556e-04	0.991
11 (6144)	3.971e-09	1.927	3.537e-07	1.994	4.800e-04	0.994

Table 4.1. Errors in example 1

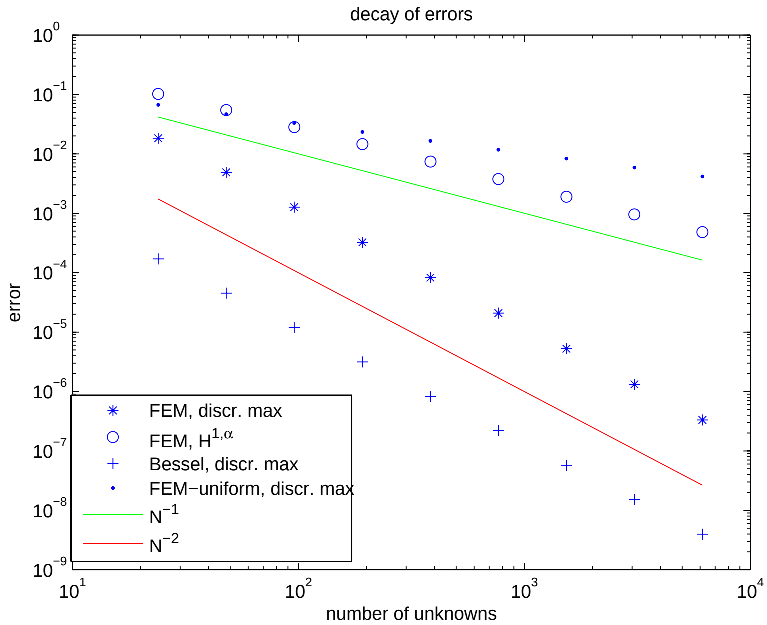

Figure 4.1. Decay of errors in example 1

For four values of ε we compare, similar to the previous example, the error decay and rate of convergence of the linear FEM on a graded Shishkin mesh and the modified ansatz function approach on an equidistant grid having the same number of unknowns. Again, we also compute the error decay of the FEM solution in the ε-weighted $H^{1,\alpha}$ seminorm. Remark that in our calculations the number of unknowns are chosen in such a way that they are log-equidistributed numbers divisible by three.

The results are presented in Table 4.2 to Table 4.5. The Bessel function approach shows a behavior dependent on the perturbation parameter ε: In the most interesting case $\varepsilon < N^{-1}$ we have first order convergence, in accordance with (3.21). In the case $\varepsilon > N^{-1}$ the performance improves and the method converges even faster than second order. As expected the finite element solution converges almost second order in the $\|\cdot\|_{\infty,*}$-norm and almost first order in the weighted $H^{1,\alpha}$ seminorm.

Remark that it seems possible to extend the adapted mesh approach to two-dimensional problems while the generation of adapted bases in 2D is extremely complicated.

number of unknowns	Bessel in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\varepsilon\|\cdot\|_{1,\alpha}$	EOC
30	6.747e-02	1.248	9.858e-02	0.928	2.861e-02	0.655
45	3.833e-02	1.395	6.487e-02	1.032	2.147e-02	0.708
63	2.270e-02	1.557	4.398e-02	1.156	1.667e-02	0.752
90	1.230e-02	1.717	2.808e-02	1.257	1.259e-02	0.787
⋮	⋮	⋮	⋮	⋮	⋮	⋮
23169	3.530e-08	2.465	3.918e-06	1.792	9.572e-05	0.909
32766	1.503e-08	2.463	2.098e-06	1.803	6.980e-05	0.911

Table 4.2. Errors in example 2, $\varepsilon^2 = 10^{-4}$

number of unknowns	Bessel in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\varepsilon\|\cdot\|_{1,\alpha}$	EOC
30	8.441e-02	1.038	9.862e-02	0.929	8.203e-03	0.622
45	5.548e-02	1.035	6.492e-02	1.031	6.217e-03	0.683
63	3.967e-02	0.997	4.402e-02	1.155	4.858e-03	0.733
90	2.776e-02	1.001	2.811e-02	1.257	3.688e-03	0.772
126	1.977e-02	1.008	1.820e-02	1.292	2.817e-03	0.801
180	1.373e-02	1.023	1.164e-02	1.253	2.100e-03	0.824
255	9.507e-03	1.055	7.273e-03	1.351	1.567e-03	0.841
360	6.482e-03	1.111	4.441e-03	1.431	1.168e-03	0.852
510	4.278e-03	1.193	2.632e-03	1.502	8.654e-04	0.861
723	2.716e-03	1.303	1.527e-03	1.560	6.394e-04	0.867
⋮	⋮	⋮	⋮	⋮	⋮	⋮
23169	2.829e-06	2.327	3.923e-06	1.791	2.925e-05	0.904
32766	1.249e-06	2.360	2.100e-06	1.803	2.136e-05	0.906

Table 4.3. Errors in example 2, $\varepsilon^2 = 10^{-6}$

number of unknowns	Bessel in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\varepsilon\|\cdot\|_{1,\alpha}$	EOC
30	8.566e-02	1.003	9.862e-02	0.929	2.474e-03	0.607
45	5.717e-02	0.997	6.493e-02	1.031	1.884e-03	0.672
63	4.086e-02	0.998	4.402e-02	1.155	1.476e-03	0.725
90	2.860e-02	1.000	2.812e-02	1.257	1.124e-03	0.765
⋮	⋮	⋮	⋮	⋮	⋮	⋮
5790	4.172e-04	1.063	4.600e-05	1.742	3.154e-05	0.888
8190	2.827e-04	1.122	2.501e-05	1.757	2.314e-05	0.893
11583	1.860e-04	1.208	1.354e-05	1.770	1.697e-05	0.896
16383	1.176e-04	1.321	7.301e-06	1.782	1.243e-05	0.898
23169	7.095e-05	1.458	3.924e-06	1.791	9.090e-06	0.902
32766	4.058e-05	1.612	2.101e-06	1.803	6.645e-06	0.904

Table 4.4. Errors in example 2, $\varepsilon^2 = 10^{-8}$

number of unknowns	Bessel in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\|\cdot\|_{\infty,*}$	EOC	FEM-Smesh in $\varepsilon\|\cdot\|_{1,\alpha}$	EOC
30	8.566e-02	1.003	9.862e-02	0.929	7.648e-04	0.599
45	5.717e-02	0.997	6.493e-02	1.031	5.837e-04	0.667
63	4.086e-02	0.998	4.402e-02	1.155	4.581e-04	0.720
90	2.860e-02	1.000	2.812e-02	1.257	3.490e-04	0.762
126	2.043e-02	1.000	1.820e-02	1.292	2.674e-04	0.792
180	1.430e-02	1.000	1.164e-02	1.253	1.998e-04	0.817
255	1.010e-02	1.000	7.274e-03	1.351	1.494e-04	0.834
⋮	⋮	⋮	⋮	⋮	⋮	⋮
23169	1.083e-04	1.002	3.924e-06	1.791	2.851e-06	0.901
32766	7.652e-05	1.003	2.101e-06	1.803	2.085e-06	0.903

Table 4.5. Errors in example 2, $\varepsilon^2 = 10^{-10}$

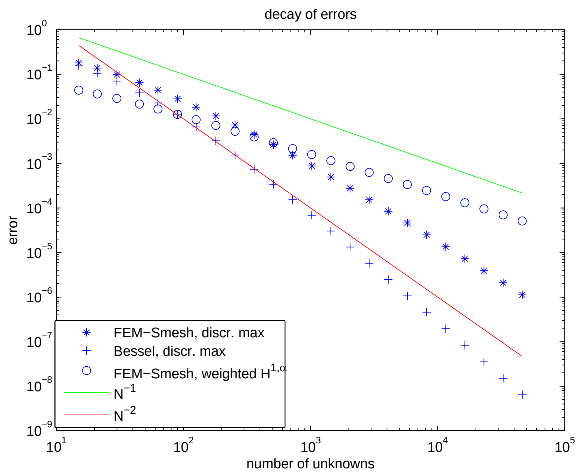

Figure 4.2. Exp. 2: Decay of errors, $\varepsilon^2 = 10^{-4}$

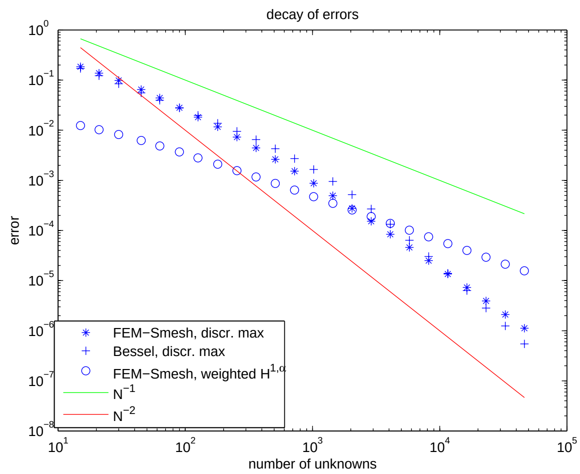

Figure 4.3. Exp. 2: Decay of errors, $\varepsilon^2 = 10^{-6}$

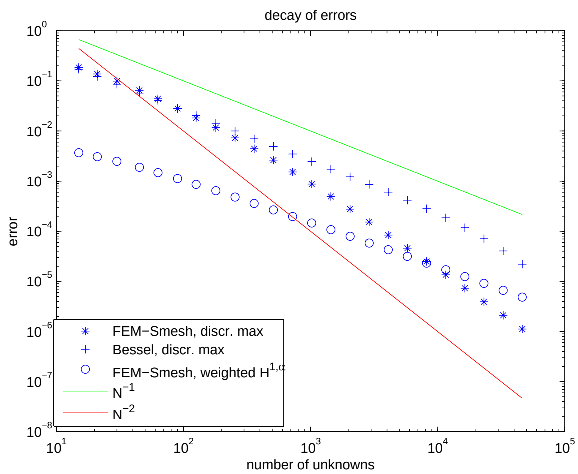

Figure 4.4. Exp. 2: Decay of errors, $\varepsilon^2 = 10^{-8}$

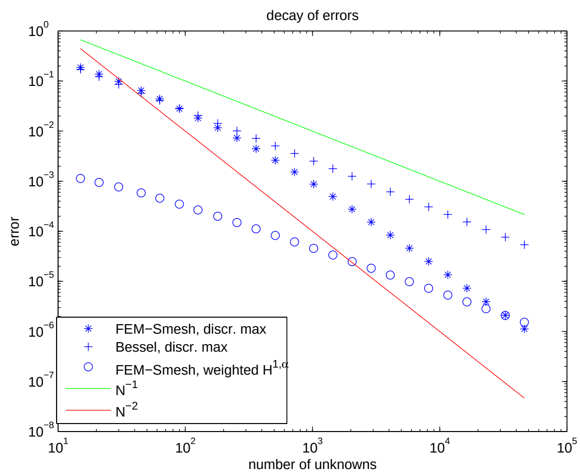

Figure 4.5. Exp. 2: Decay of errors, $\varepsilon^2 = 10^{-10}$

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