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Feedback Controls for Continuous Priority Models in Supply Chain Management

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Abstract — We are interested in closed loop feedback control laws for supply chains. The mathematical modeling is based on Boltzmann–type equations. These equations allow to model a supply chain with priorities. The latter influence the processing time in a nonlinear way. For this class of models we derive a control law and we show numerical results.

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1. Modeling of a supply chain with priorities

Production and supply chain modeling is characterized by a variety of mathematical approaches at different scales. Here, we are interested in continuous models for high volume production systems. These models use partial differential equations to model production flow and have been discussed for example in [1, 2, 4, 3, 11, 15]. This approach leads to deterministic and fast but coarse grain models. However, the additional mathematical structure of the partial differential equations allows for further analysis and control approaches as for example employed in [17, 22, 10]. In the preceeding we focus on the following model introduced recently in [23]. The supply chain is characterized by an interval $x \in (-\infty, \infty)$ At stage $x = -\infty$ the products enter the supply chain and leave as finished products at $x = +\infty$. Their evolution along the chain is described by a kinetic equation including scheduling policies y. The need of such policies arises when not all products are treated in the same way but distinguished by certain features. Such features might be the due date, the time spent in the production line or the experation date. Therefore, the products in the chain might not be processed in a sequence and the production order might change according to some policy or service rule. The focus of the present paper is on the optimal choice of such a service rule.

Based on the previous motivations the general model considered in [23] is as follows. Let x denote the production stage of the product, $y \in \mathbb{R}^d$ the attribute or property of an individual product and t the time. Newton's equations for the evolution of an individual product are

$$\frac{dx}{dt} = v, \ \frac{dy}{dt} = E,$$

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where $E = E(x, t, y)$ is function describing a possible change in attributes due e.g. to perishability of products and $v = v(x, t, z)$ the production velocity. Depending on the model at hand $z \in \mathbb{R}$ might be given by a function depending on macroscopic properties as for example the product density. We introduce a prototype in equation (1.2) below. For convenience y is treated as continuous variable. Hence, if $f(t, x, y)$ denotes the product density with attribute y at stage x and time t, then f satisfies the transport equation

$$\partial_t f + \partial_x (vf) + \nabla_y \cdot Ef = 0. \tag{1.1}$$

In the model [23] this velocity is obtained using certain scheduling policies or priority functions p. The latter describes the order of processing the parts and the basic concept is as follows: production of parts of high priority is enforced compared with low priority parts. Given a set of sufficiently regular priority functions $p(y, \alpha) : \mathbb{R}^d \times \mathbb{R}^m \to \mathbb{R}$, where α is an external control, the production velocity is defined by

$$v = v(x, t, \phi_f(p(y, \alpha), \alpha)) \text{ and } \phi_f(q, \alpha) = \int H(p(\eta, \alpha) - q) f(\eta) d\eta. \tag{1.2}$$

The function $H(x)$ denotes the Heavi-side function. Note that α might be time–dependent in order to allow for dynamic priority changes. In the following we assume that function $z \to v(x, t, z), z, \alpha \to p(z, \alpha)$ are arbitrary but sufficiently smooth. The function $\phi_f(q, \alpha)$ measures the number of parts with priority larger than q at the given control value α. Hence, only parts with priority higher than q will be considered in th the computation of ϕ_f and therefore in turn the production velocity. The function v is such that a higher priority guarantees a larger processing speed, i.e., v is monotone in z. Typical examples for v are [2, 1, 4, 3, 11, 15, 16]

$$v(x, t, \phi) = v_0(x) H(c(x) - v_0(x)\phi) \tag{1.3}$$

or

$$v(x, t, \phi) = v_0(x) \exp(-v_0(x)\phi / c(x)), \tag{1.4}$$

where $v_0(x)$ and $c(x)$ are the known velocities and capacities of the machines.

The parameter α was not present in the original model [23, 11]. We offer the folllowing interpretation: the function $p(y, \alpha)$ is a measure of the priority associated with a property value y and the parameter α acting as control. E.g., in case of perishable goods one might want to temporarily change the priority in order to guarantee on–time delivery.

We are interested in control problems related to the supply chain model (1.1). For simplicity we consider the Cauchy problem on real axis x. Ultimately, we want to derive a feedback law to chose an optimal policy $\alpha = \alpha(t)$ in order to obtain a product density $f(t, x, y)$ close to a some given production demand $\bar{f}(t, x, y)$. This question leads to controllability and optimal control problems studied for example in [16, 12, 13, 14]. In the existing work there has always been the assumption that the demand $\bar{f}$ is known a priori at any time $t > 0$. However, this is typically **not** the case in supply chain problems. The reasons are unpredictable changes in demand or stochastic fluctuations.

Therefore, we devise a strategy chosing $\alpha(t)$ even when $\bar{f}$ is unknown a priori. To be more precise we derive an closed-loop control concept for the priority supply chain model (1.1). This will allow to explicitly chose $\alpha(t)$ at every time t based on predictions of the underlying model (1.1) and depending *only* on the current desired demand $(x, y) \to \bar{f}(t, x, y)$. The closed loop control concept is similar to model predictive control [5, 6] and receding horizon control [24], developed in engineering context. In applied mathematics closed loop control concepts have been studied to stabilize fluid flow in [9, 25, 7, 21, 20, 19, 18, 8]. Here, we proceed *different* as in the previous references. We introduce a linear relation between multiplier and state which additionally is space and velocity dependent. The linear factor itself satisfies a newly derived equation. After further linearizations we observe that this factor in fact is determined by an ordinary differential equation. In order to compute the value of α this ordinary differential equation has to be solved. The current state and demand $\bar{f}(t,x,y)$ as well as the derivatives of ϕ with respect to f enter in this computations. The precise derivation will be given in Section 2 together with some numerical results.

2. The control law

We derive the closed loop control law in this section. We assume at first that the demand $\bar{f}(t,x,y)$ is known over a time interval $[0,T]$. In this case the best possible choice for the parameter α is obtained as solution to an optimal control problem. Consider the mean squared deviation in f on the full space $\mathbb{R}\times[0,T]\times\mathbb{R}^n$ and minimize this distance with respect to α. To be more precise, consider the problem

$$\min \int \frac{1}{2}\left(f-\bar{f}\right)^2 dxdydt \tag{2.1}$$

$$\text{subject to} \tag{2.2}$$

$$\partial_t f + \partial_x(vf) + \nabla_y\cdot Ef = 0, f(x,0,y) = f_0 \tag{2.3}$$

Here, f_0 is the given initial supply of parts. Formally [26], the optimal priority α satisfies the first–order optimality system

$$\partial_t f + \partial_x(vf) + \nabla_y\cdot Ef = 0, \quad f(x,0,y) = f_0 \tag{2.4}$$

$$-\partial_t\lambda - v\partial_x\lambda + v^* + \nabla_y\lambda\cdot E = f-\bar{f}, \quad \lambda(x,T,y) = 0 \tag{2.5}$$

$$\int(-\partial_x\lambda) f v_{\delta\alpha} dxdy = 0. \tag{2.6}$$

The functions v^* and $v_{\delta\alpha}$ are defined as follows:

$$v^* = \int v_z(x,t,\eta)\partial_x\left(\lambda(t,x,\eta)\right) f(t,x,\eta) H(p(\eta,\alpha)-p(y,\alpha))d\eta \tag{2.7}$$

$$v_{\delta\alpha} = \int v_z(x,t,\eta)\left(\Delta_\epsilon H(p(\eta,\alpha)-p(y,\alpha))\right)\left(\partial_\alpha p(\eta,\alpha)-\partial_\alpha p(y,\alpha)\right) f(\eta)d\eta. \tag{2.8}$$

For the detailed computations we refer to the appendix A. Since H is not differentiable we introduce a smoothed version as

$$\Delta_\epsilon H(x) = \frac{1}{2\epsilon}\left(H(x+\epsilon) - H(x-\epsilon)\right),$$

and where

$$v_z(x,t,y) := \partial_{\phi_f} v(x,t,\phi_f(p(y,\alpha),\alpha)).$$

The previous equations can be easily deduced. The only interesting part in the previous derivation of the optimality system is the derivative of $v = v(x,t,\phi_f(p(y,\alpha),\alpha))$. Since ϕ_f is depending on the non–differentiable Heavi–side function H. We have the following lemma.

Lemma 2.1. *Let $v = v(x,t,z)$ and $p = p(y,\alpha)$ be smooth functions in their arguments. Let $\phi_f(q,\alpha) = \int H(q - p(\eta,\alpha)f(\eta)d\eta$. Let $\epsilon > 0$ be given. Then, an approximation of order ϵ to the derivative in direction $\delta\alpha$ is*

$$\partial_\alpha v(x,t,\phi_f(p(y,\alpha),\alpha)) := v_z(x,t,\phi_f(p(y,\alpha),\alpha)) \cdot \int \Delta_\epsilon H(p(\eta,\alpha) - p(y,\alpha))(\partial_\alpha p(\eta,\alpha) - \partial_\alpha p(y,\alpha))f(\eta)d\eta$$

and the derivative in direction δf is

$$\partial_f v(x,t,\phi_f(p(y,\alpha),\alpha)\delta f = \int v_z(x,t,\phi_f(p(y,\alpha),\alpha)H(p(\eta,\alpha) - p(y,\alpha)\delta f(x,t,\eta)d\eta$$

respectively.

Proof. The second derivative is obvious. We compute the derivative with respect to α by replacing $H(x)$ with a finite–difference approximation of $\partial_x(xH(x))$:

$$H(x) = \frac{1}{2\epsilon}\Big((x+\epsilon)H(x+\epsilon) - (x-\epsilon)H(x-\epsilon)\Big) + O(\epsilon^2).$$

Hence, we obtain an approximation on $\partial_x H(x)$ as $\partial_x H(x) = \Delta_\epsilon H(x)$ which yields the previous formula.

The major drawback of the optimality system (2.4) is the explicit dependence of its solution (f,λ,α) on the full time evolution $\bar{f}$. Thinking of possibly random demands over time this knowledge is not easily available. Therefore, we look for a formulation allowing the controls α depending only on the current demand $(x,y) \to \bar{f}(t,x,y)$ at time t and the state of the system $f(t,x,y)$. Obviously, any choice depending on this local information yields a suboptimal control to the system. However, we show later on in the numerical results that this choice is at least better than any a priori choice of $\alpha(t)$.

A reasonable control strategy is devised as follows. Starting from the optimality system (2.4) and we approximate the full adjoint state λ by a linear function G. More precisely, assume from now on that (2.9) holds true. □

Assumption 2.1.

$$\lambda(x,t,y) = G(x,y)\left(f(x,t,y) - \bar{f}(x,t,y)\right). \tag{2.9}$$

Remark 2.1. The idea to replace the adjoint variable by an unknown linear operator $G(x,y)$ is in analogy to LQ–controllers. Those are common practice in the design of a feedback controller. There, typically the dynamics is $x'(t) = Ax(t) + Bu(t)$ with a cost $\int_0^\infty x'(t)Qx(t) + u'(t)Ru(t)dt$. The ansatz $u(t) = Kx(t)$ leads to the so–called Riccati equation. We try to mimic this procedure in the nonlinear and partial differential equations case and obtain in equation (2.13) the corresponding equation for G.

We furthermore assume that the given demand $\bar{f}$ is a solution to the kinetic equation, i.e., $\bar{f}$ is a reachable state. Additionally, we assume that the difference between $\bar{f}$ and f is sufficiently small at some given time t.

Assumption 2.2.

$$\bar{f}(x,t,y) \textit{ solves } (1.1), \|f(t,\cdot,\cdot) - \bar{f}(t,\cdot,\cdot)\|_{W^{1,\infty}} \leqslant \delta. \tag{2.10}$$

Now, we insert the ansatz for λ in (2.5), multiply (2.4) by G and add both equations.

$$-vG_x(f-\bar{f}) - vG(f-\bar{f})_x + G(vf)_x - G(\bar{v}\bar{f})_x + v^* \tag{2.11}$$
$$+\nabla_y \cdot \big(EG(f-\bar{f})\big) + GE\cdot\nabla_y(f-\bar{f}) = f-\bar{f} \tag{2.12}$$

Here, $\bar{v}$ denotes the velocity corresponding to $\bar{f}$, i.e., $\bar{v} = v(x,t,\phi_{\bar{f}})$. Note that ϕ_f is linear in f and v is supposed to be a smooth function in x, t and ϕ_f. Hence, $\|f(t,\cdot) - \bar{f}(t,\cdot)\| \leqslant \delta$ implies that $\bar{v} = v + O(\delta)$. Furthermore, we have

$$v^* = \int v_z(x,t,\eta)\partial_x\left(G(f-\bar{f})\right)(t,x,\eta)f(t,x,\eta)H(p(\eta,\alpha)-p(y,\alpha))d\eta.$$

Due to assumption (2.10), we have $v^* = O(\delta)$. We further simplify the equation by considering only terms linear in $f-\bar{f}$. We neglect all terms having derivatives on f and $\bar{f}$. Hence, the simplified equation for G up to order $O(\delta)$ reads

$$\Big(-vG_x + Gv_x - 1 + \nabla_y\cdot(EG)\Big)(f-\bar{f}) = 0. \tag{2.13}$$

Note that the term -1 appears to the right-hand side in equation (2.12). The term v^* is of order $O(\delta)$ and has been dropped, simiarly, the term $\nabla_y(f-\bar{f})$ has been dropped. The terms $G(vf)_x - G(\bar{v}\bar{f})_x = G(vf) - G(v\bar{f})_x + o(\delta) = G(v(f-\bar{f}))_x + o(\delta) \approx Gv_x(f-\bar{f})$. Note that this equation can be solved for G independent of $f-\bar{f}$, similar to the Riccati equation for a linear feedback controller.

Last, we use the representation of λ in terms of G in order to obtain the final control law. We apply a single step of a Gauss–Seidel method to (2.4)–(2.6). Note that this corresponds to a single step of a steepest descent method applied to the probem (2.1). Starting with $\alpha \equiv 0$ we obtain

$$\alpha^{new}(t) = \int \partial_x\lambda f v_{\delta\alpha} dxdy \ a.e.\ t \tag{2.14}$$

We use this result and the previous computations to define the closed loop feedback control

$$\alpha^{ctrl}(t) := \int \partial_x(G(x,y)\cdot(f(t,\cdot)-\bar{f}(t,\cdot)))f(t,\cdot)v_{\delta\alpha}dxdy \tag{2.15}$$

Here, G is the solution to (2.13) at time t. The concept of using a one–step steepest descent method as control law has also been applied in the case of the Navier–Stokes equation, see e.g. [19]. Therein, it has shown that this control leads to a suboptimal control strategy. However, here we have an additional linearization procedure employed in order to obtain a suitable formulation for $G(x,y)$ not depending on the full adjoint state.

Note that α^{ctrl} is explicit and depends only on the knowledge of the current state of the system and the current data $\bar{f}$. There is no need to solve an optimal control problem.

Clearly, due to the previous assumptions α^{ctrl} is only an approximation to the first iteration of a steepest descent method for the full control problem (2.1). However, the focus is not on a time–optimal control but rather on a feedback control with validation given through the equations.

We test the control law in a numerical simulation of the kinetic equation with $x, y_i \in [0,1]^2$. We chose $E \equiv 0$ and $p(y,\alpha) = \alpha y_1 + y_2$. The smoothing parameter for the Heavi–side function is $\epsilon = 10^{-2}$ and we use N_y and N_x discretization points in space y and x, respectively. We use a maximal time horizon $T = 2$ and set $f(t=0,x,y_1,y_2) = H(\frac{1}{4}-x)$ The time–discretization is chosen such that the CFL condition is satisfied, i.e., $\frac{\Delta t}{\Delta x} \max_{x,y,t} \|v\| \leqslant 1$. Since $v \geqslant 0$ we apply a first–order Upwind discretization in x and an explicit Euler–discretization in t to equation (1.1). We compute the controlled case and the case of an a priori fixed control $\alpha \equiv 0$. In the controlled case we compute α according to equation (2.15). This amounts in solving (2.13) at every time step. The ordinary differential equation is solved using the implicit Euler method on the same grid as f. As example we prescribe as desired state $\bar{f}(t, x, y_1, y_2) = 1 - H(y_2 - \frac{4}{5})$. We show the evolution of the costs over time as well as the chosen control α^{ctrl}. The evolution is compared with a fixed choice of α. The results are depicted in Figure 2.1. In the second example we consider two states $\bar{f}_0$ and $\bar{f}_1$. The initial data is $f^0 := f(t = 0, x, y_1, y_2) = y_1 H(\frac{1}{4} - x)$. The states $\bar{f}_0$ and $\bar{f}_1$ are the solution to (1.1) with initial data f^0 and for $\alpha \equiv 0$ and $\alpha \equiv 1$, respectively. The desired state is $\bar{f} = \bar{f}_0 H(2 - t) + \bar{f}_1 H(t - 2) H(7.5 - t) + \bar{f}_0 H(7.5 - t)$. Hence, there is a need to modify the control in the time–interval $t \in [2, 7.5]$. The costs over time for fixed controls and the control law are depicted in Figure 2.2.

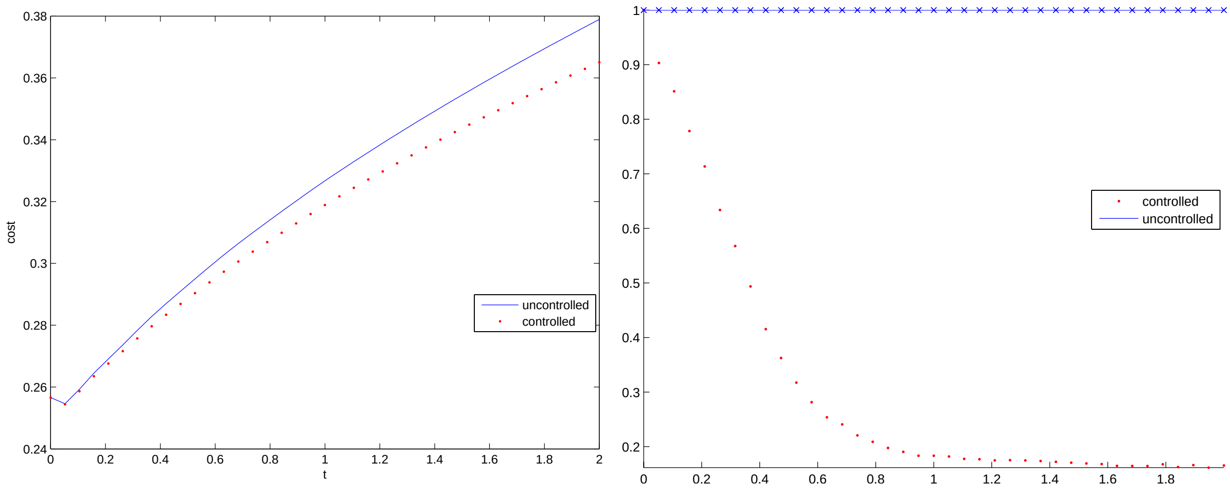

Figure 2.1. Fully discretized kinetic equation with $N_x = N_y = 20$. Left part is the cost functional over time in the controlled (dotted) and uncontrolled (solid line) case in the right part depicts the control over time for both cases

3. Summary

Using the first–order optimality system for a priority supply chain model we derive a feedback control law for priority scheduling. Numerical results for the kinetic equation are given and show the expected behavior.

A. Formal derivation of the optimality system to (2.1)

The first–order optimality system to (2.1) can formally be derived using the following calculus. For a given terminal time $T > 0$ we introduce the Lagrange function to the system (2.1) as

$$L(f, \lambda, \alpha) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^T \frac{1}{2} \left(f - \bar{f}\right)^2 - \lambda \partial_t f - \lambda \partial_x (vf) - \lambda \nabla_y \cdot E f dx dy dt.$$

The dependence of L on α is through the function v defined in equation (1.2). Using integration by parts and assuming that f decays to zero for $x, y \to \pm\infty$ we obtain a reformulation

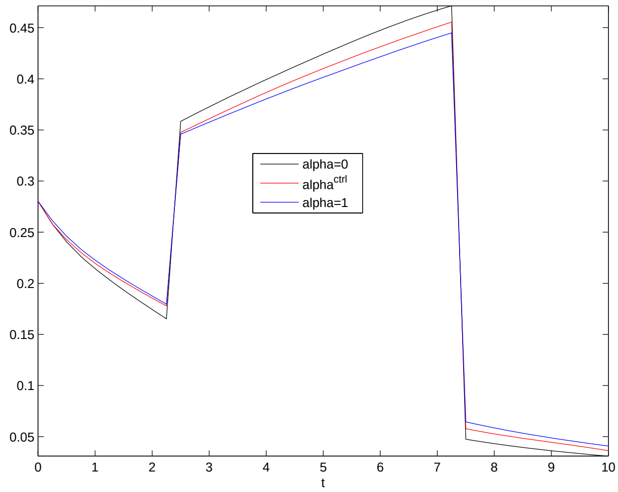

Figure 2.2. Fully discretized kinetic equation with $N_x = N_y = 20$. Evolution of the cost functional over time for various choices of α

of L as

$$L(f,\lambda,\alpha) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^T \frac{1}{2}\left(f-\bar{f}\right)^2 + \left(\partial_t\lambda + \partial_x\lambda(v) + \nabla_y\lambda\cdot E\right) f dxdydt$$
$$- \int_{\mathbb{R}}\int_{\mathbb{R}} \lambda(x,T,y)f(x,T,y) - \lambda(x,0,y)f_0(x,y)dxdy.$$

Now, the formal first–order optimality system is obtained by computing the derivative with respect to f, λ and α, respectively. The derivative with respect to λ yields equation (2.4). The derivative with respect to f yields equation (2.5) and the derivative with respect to α yields (2.6). We discuss the derivative with respect to $\alpha = \alpha(t)$ in more detail: we consider a smooth variation of $\delta\alpha(t)$. Since only v depends on α we are lead to consider the difference

$$v_{\alpha(t)+\delta\alpha(t)} - v_{\alpha(t)} = \int_{\mathbb{R}}\int_{\mathbb{R}} (-\partial_x\lambda) f v_{\delta\alpha}\left(\delta\alpha(t)\right) dtdxdy + o(\|\delta\alpha\|).$$

Equation (2.4) is the pointwise in time formulation of the previous equation.

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