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A Substructuring Domain Decomposition Scheme for Unsteady Problems

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Abstract — Domain decomposition methods are used for the approximate solution of boundary-value problems for partial differential equations on parallel computing systems. Specific features of unsteady problems are fully taken into account in iteration-free domain decomposition schemes. Regionally-additive schemes are based on various classes of splitting schemes. In this paper we highlight a class of domain decomposition schemes which are based on the partition of the initial domain into subdomains with common boundary nodes. Using a partition of unity we construct and analyze unconditionally stable schemes for domain decomposition based on a two-component splitting: the problem within each subdomain and the problem at their boundaries. As an example we consider a Cauchy problem of first or second order in time with a non-negative self-adjoint second order operator in space. The theoretical discussion is supplemented with the numerical solution of a model problem for a two-dimensional parabolic equation.

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1. Introduction

Theory and practice of domain decomposition (DD) methods for stationary boundary value problems is comprehensively presented in the books [16, 18, 28, 31]. Different versions of the domain decomposition method, with and without overlapping subdomains, are used. The approximate solution of unsteady problems can be derived from the standard implicit approximations in time by solving the corresponding discrete equations at each new time level by one of the variants of the DD method for stationary problems. Taking into account the transient character of unsteady problems (see, e.g., the implementation based on the Schwartz method [3, 4]), we can construct optimal iterative DD methods, where the number of iterations is independent of the mesh size in time and space.

This specific feature of unsteady problems is fully taken into account in iteration-free DD schemes. In some cases it is possible [12, 13] to use only a single iteration step of the Schwarz alternating method for a second order parabolic equation without loosing accuracy in the approximate solution. Iteration-free DD schemes are associated with special variants of the additive (splitting) schemes: the regionally- additive schemes [21].

Domain decomposition schemes for unsteady problems can be distinguished by (i) the method of domain decomposition used, (ii) the choice of decomposition operators (exchange

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of boundary conditions), and (iii) the splitting scheme. For differential problems it is natural to choose domain decomposition methods

$$\overline{\Omega} = \bigcup_{\alpha=1}^{p} \overline{\Omega}_\alpha, \quad \overline{\Omega}_\alpha = \Omega_\alpha \cup \partial\Omega_\alpha, \quad \alpha = 1, 2, ..., p \tag{1.1}$$

with overlapping of subdomains ($\Omega_{\alpha\beta} \equiv \Omega_\alpha \cap \Omega_\beta \neq \varnothing$) or without overlapping ($\Omega_{\alpha\beta} = \varnothing$) [18, 31]. Methods without overlapping of the subdomains are associated with an explicit formulation of the boundary conditions at the interface boundaries. These methods are in common use for problems where in each particular subdomain its own specific computational grid (triangulation) is introduced. In general also DD schemes with overlapping subdomains are used. Domain decomposition methods with a minimal overlap, where the width of overlap is equal to the grid size ($\Omega_{\alpha\beta} = \mathcal{O}(h)$), can often be interpreted as methods without overlap supplemented with appropriate boundary conditions at the interfaces.

The domain decomposition (1.1) is associated with a corresponding additive representation of the problem operator

$$\mathcal{A} = \sum_{\alpha=1}^{p} \mathcal{A}_\alpha. \tag{1.2}$$

In this case, the operator $\mathcal{A}_\alpha$ is associated with the solution of some problem in the subdomains Ω_α, $\alpha = 1, 2, ..., p$. The most common approach to construct the decomposition operators is based on a partition of unity over for the computational domain. For the decomposition (1.1) we can associate with each separate subdomain Ω_α the function $\eta_\alpha(\mathbf{x})$, $\alpha = 1, 2, ..., p$ such that

$$\eta_\alpha(\mathbf{x}) = \begin{cases} > 0, & \mathbf{x} \in \Omega_\alpha, \\ 0, & \mathbf{x} \notin \Omega_\alpha, \end{cases} \quad \alpha = 1, 2, ..., p, \tag{1.3}$$

and also

$$\sum_{\alpha=1}^{p} \eta_\alpha(\mathbf{x}) = 1, \quad \mathbf{x} \in \Omega. \tag{1.4}$$

Suppose, for example, that the operator A is the diffusion operator

$$\mathcal{A} = -\operatorname{div} k(\mathbf{x}) \operatorname{grad}, \quad \mathbf{x} \in \Omega. \tag{1.5}$$

Then we can define the operators of decomposition by one of the following three basic forms:

$$\mathcal{A}_\alpha = \eta_\alpha \mathcal{A}, \tag{1.6}$$

$$\mathcal{A}_\alpha = -\operatorname{div} k(\mathbf{x})\eta_\alpha(\mathbf{x}) \operatorname{grad}, \tag{1.7}$$

$$\mathcal{A}_\alpha = \mathcal{A} \eta_\alpha, \quad \alpha = 1, 2, ..., p. \tag{1.8}$$

This technique was introduced in [14] (decomposition (1.7)), and in [32] (decomposition (1.6)–(1.8)). More recent developments are summarized in the books [21, 26]. Various versions of the decomposition operators differ by the different boundary conditions at the boundary interfaces. They ensure the convergence of the approximate solution in different spaces of grid functions. The construction of decomposition operators for unsteady problems with non-selfadjoint operators requires special attention [24, 27, 34].

For unsteady problems with splitting (1.2) different splitting schemes are used. In the theory of additive operator-difference schemes [26, 19, 15, 39] we first can distinguish the case of the simplest two-component splitting. In this case, we construct unconditionally stable factorized splitting schemes, such as the classical scheme of alternating directions, or the predictor-corrector scheme. Two-component regionally-additive schemes are constructed and studied in [14, 32, 33] as well as in the above mentioned papers [24, 27, 34] for convection-diffusion problems.

Also the splitting of the problem operator into the sum of three or more non-commutative operators ($p > 2$ in (1.2)) is of great interest. Classic schemes [19, 15, 39] of multi-component splitting are based on the concept of summarized approximation. Additively-averaged schemes of summarized approximation [26, 8] are more explicitly oriented to parallel computations. Regionally-additive schemes of component-wise splitting are investigated in [37]. A variant of two-component splitting with the Crank-Nicolson scheme for the individual subproblems with a minimal overlap and decomposition (1.7) is considered in the article [6].

Nowadays, the schemes of full approximation are in common use for general multi-component splitting. In this regard, we note the regularized additive schemes [25], where the stability condition is achieved by perturbation of the operators of the difference scheme. In the vector additive schemes [1, 35] instead of one equation we solve a system of similar equations. Such schemes are also constructed for the evolutionary equations of second order [22, 2]. Vector regionally-additive schemes are investigated in [23, 38]. In [36] more general regularized schemes of domain decomposition are proposed, with different structures both for the splitting operators and for the operators of the discrete problem at the new time-level.

Among the other DD methods for solving boundary value problems for parabolic equations it is necessary to mention the explicit-implicit methods considered in many papers (see, for example, [41, 40, 30, 9, 10, 11]). In this case the domain decomposition is made without overlap and the transition to a new time-level is organized as follows. First, the approximate solution at the common boundaries of the subdomains is predicted using the explicit scheme. Next, these boundary conditions are used to compute the approximate solution in the individual subdomains. And finally, a correction of the interface boundary conditions is carried out by implicit schemes. It will be shown below that such schemes of domain decomposition do completely fit in the above general scheme of DD methods with the choice of operators according to decomposition (1.6).

In this paper, we construct DD schemes for parabolic and hyperbolic equations with self-adjoint elliptic operators of second order. Unconditionally stable, factorized regionally-additive schemes are constructed using the decomposition (1.6) and the two-component or the general multi-component splitting. Domain decomposition schemes with a self-adjoint operator for the discrete problem at the new time-level are derived. For these schemes we can construct special iteration methods of conjugate gradient type for solving stationary problems.

This paper is organized as follows. In Section 2 we formulate our model Cauchy problem for parabolic and hyperbolic equations in a rectangle. Next, Section 3 provides the stability conditions for the standard two- and tree-level implicit schemes for model problems in a rectangle. The domain decomposition and the construction of the operators is discussed in Section 4. The possibilities of the standard factorized schemes for domain decomposition are considered in Section 5. The stability condition, appropriate a priori estimates for the approximate solution and the convergence rate are derived for factorized regionally-additive schemes. In Section 6 multi-component splitting schemes are constructed. Hyperbolic equations of second order are considered in Section 7. The theoretical results are illustrated by numerical results in Section 8.

2. Model boundary problems

Let us consider a model boundary value problem for the parabolic equation of second order. In a bounded domain Ω the unknown function $u(\mathbf{x}, t)$ satisfies the following equation

$$\frac{\partial u}{\partial t} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\mathbf{x}) \frac{\partial u}{\partial x_\alpha} \right) = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad 0 < t \leqslant T, \tag{2.1}$$

where $k(\mathbf{x}) \geqslant \kappa > 0, \ \mathbf{x} \in \Omega$. Equation (2.1) is supplemented with the homogeneous Dirichlet boundary conditions

$$u(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \quad 0 < t < T. \tag{2.2}$$

In addition, the initial condition is prescribed

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.3}$$

So, the unsteady diffusion problem (2.1)–(2.3) is considered on the set of functions $u(\mathbf{x}, t)$ satisfying boundary conditions (2.2). Instead of (2.1), (2.2) we also use the differential operator notation

$$\frac{du}{dt} + \mathcal{A}u = f(t), \quad 0 < t < T, \tag{2.4}$$

with

$$\mathcal{A}u = - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha} \left(k(\mathbf{x}) \frac{\partial u}{\partial x_\alpha} \right).$$

The Cauchy problem is considered for the evolutionary equation (2.4) with initial condition

$$u(0) = u^0. \tag{2.5}$$

On the set of functions (2.2) let us define the Hilbert space $\mathcal{H} = \mathcal{L}_2(\Omega)$ with the scalar product and norm

$$(u, v) = \int_\Omega u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \quad \|u\| = (u, u)^{1/2}.$$

In $\mathcal{H}$ the operator of the diffusive transport $\mathcal{A}$ is self-adjoint and positive definite

$$\mathcal{A} = \mathcal{A}^* \geqslant \kappa \delta \mathcal{E}, \quad \delta = \delta(\Omega) > 0, \tag{2.6}$$

where $\mathcal{E}$ is the identity operator in $\mathcal{H}$.

We present now the simplest a priori estimate for the solution of problem (2.4)–(2.6) which will be for us the check point for the considering grid problems. The self-adjoint positive definite operator $\mathcal{D}$ can be associated with the Hilbert space $\mathcal{H}_\mathcal{D}$ having the inner product and norm

$$(u, v)_\mathcal{D} = (\mathcal{D}u, v), \quad \|u\|_\mathcal{D} = (u, u)_\mathcal{D}^{1/2}$$

respectively. In $\mathcal{H}$ multiply scalarly equation (2.4) by $\mathcal{A}u$. In view of (2.6) we obtain inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|_\mathcal{A}^2 + \|\mathcal{A}u\|^2 = (f, \mathcal{A}u). \tag{2.7}$$

Taking into account

$$(f, \mathcal{A}u) \leqslant \|\mathcal{A}u\|^2 + \frac{1}{4}\|f\|^2,$$

from (2.7) we have

$$\frac{d}{dt}\|u\|^2_{\mathcal{A}} \leqslant \frac{1}{4}\|f\|^2.$$

In view of the Gronwall lemma we obtain the desired estimate

$$\|u\|^2_{\mathcal{A}} \leqslant \|u^0\|^2_{\mathcal{A}} + \int_0^t \|f(\theta)\|^2 d\theta, \tag{2.8}$$

which expresses the stability of the solution of problem (2.4)–(2.6) with respect to the initial data and right-hand side.

In addition to the parabolic equation (2.1), we consider the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \sum_{\alpha=1}^{m} \frac{\partial}{\partial x_\alpha}\left(k(\mathbf{x})\frac{\partial u}{\partial x_\alpha}\right) = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, \quad 0 < t \leqslant T \tag{2.9}$$

with boundary conditions (2.2). Equation (2.9) is supplemented with two initial conditions

$$u(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \frac{\partial u}{\partial t}(\mathbf{x}, 0) = v^0(\mathbf{x}), \quad \mathbf{x} \in \Omega. \tag{2.10}$$

Problem (2.2), (2.9), (2.10) is associated with the following Cauchy problem for the evolutionary equation of second order:

$$\frac{d^2 u}{dt^2} + \mathcal{A}u = f(t), \quad 0 < t < T, \tag{2.11}$$

$$u(0) = u^0, \quad \frac{du}{dt}(0) = v^0. \tag{2.12}$$

Scalar multiplication of equation (2.11) with $\mathcal{A}du/dt$ yields

$$\frac{1}{2}\frac{d}{dt}\left(\left\|\frac{du}{dt}\right\|^2_{\mathcal{A}} + \|\mathcal{A}u\|^2\right) = \left(f, \mathcal{A}\frac{du}{dt}\right).$$

For the right-hand side we use the estimate

$$\left(f, \mathcal{A}\frac{du}{dt}\right) \leqslant \frac{1}{2}\left\|\frac{du}{dt}\right\|^2_{\mathcal{A}} + \frac{1}{2}\|f\|^2_{\mathcal{A}}.$$

The result is

$$\frac{d}{dt}\|u\|^2_* \leqslant \|u\|^2_* + \|f\|^2_{\mathcal{A}},$$

where

$$\|u\|^2_* = \left\|\frac{du}{dt}\right\|^2_{\mathcal{A}} + \|\mathcal{A}u\|^2.$$

The desired a priori estimate

$$\|u(t)\|^2_* \leqslant \exp(t)\left(\|\mathcal{A}u^0\|^2 + \|v^0\|^2_{\mathcal{A}} + \int_0^t \exp(-\theta)\|f(\theta)\|^2_{\mathcal{A}} d\theta\right) \tag{2.13}$$

expresses the stability with respect to the initial data and right-hand side of the Cauchy problem for operator-differential equation (2.11).

3. Standard difference approximations

We will conduct a detailed study of approximations in space and time using as an example the boundary problems in a rectangle

$$\Omega = \{ \ \mathbf{x} \mid \mathbf{x} = (x_1, x_2), \ 0 < x_\alpha < l_\alpha, \ \alpha = 1, 2\}.$$

The approximate solution is given at the nodes in a uniform rectangular grid Ω

$$\bar{\omega} = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2), \quad x_\alpha = i_\alpha h_\alpha, \quad i_\alpha = 0, 1, ..., N_\alpha, \quad N_\alpha h_\alpha = l_\alpha\}$$

and by ω we denote the set of internal nodes ($\bar{\omega} = \omega \cup \partial\omega$). For the grid functions $y(\mathbf{x}) = 0, \ \mathbf{x} \in \partial\omega$ we define the Hilbert space $H = L_2(\omega)$ with the scalar product and norm

$$(y, w) = \sum_{\mathbf{x}\in\omega} y(\mathbf{x})w(\mathbf{x})h_1h_2, \quad \|y\| = (y, y)^{1/2}.$$

Assuming that the coefficient $k(\mathbf{x})$ in Ω is sufficiently smooth, we use the discrete diffusion operator

$$\begin{aligned} Ay = &- \frac{1}{h_1^2}k(x_1 + 0.5h_1, x_2)(y(x_1 + h_1, x_2) - y(x_1, x_2)) \\ &+ \frac{1}{h_1^2}k(x_1 - 0.5h_1, x_2)(y(x_1, x_2) - y(x_1 - h_1, x_2)) \\ &- \frac{1}{h_2^2}k(x_1, x_2 + 0.5h_2)(y(x_1, x_2 + h_2) - y(x_1, x_2)) \\ &+ \frac{1}{h_2^2}k(x_1, x_2 - 0.5h_2)(y(x_1, x_2) - y(x_1, x_2 - h_2)). \end{aligned} \tag{3.1}$$

In H the operator A is self-adjoint and positive definite

$$A = A^* \geqslant \kappa(\delta_1 + \delta_2)E, \quad \delta_\alpha = \frac{4}{h_\alpha^2}\sin^2\frac{\pi h_\alpha}{2l_\alpha}, \quad \alpha = 1, 2. \tag{3.2}$$

After the approximation in space we go from (2.1), (2.2) to the differential-difference equation

$$\frac{dy}{dt} + Ay = f(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \quad 0 < t < T. \tag{3.3}$$

Taking into account (2.3), let us supplement equation (3.3) with the initial condition

$$y(\mathbf{x}, 0) = u^0(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{3.4}$$

For the solution of the differential-difference Cauchy problem (3.3), (3.4) the following a priori estimate holds (see (2.8))

$$\|y\|_A^2 \leqslant \|u^0\|_A^2 + \int_0^t \|f(\theta)\|^2 d\theta. \tag{3.5}$$

Similarly, the approximation in space leads us from (2.2), (2.9), (2.10) to the problem

$$\frac{d^2y}{dt^2} + Ay = f(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \quad 0 < t < T, \tag{3.6}$$

$$y(\mathbf{x},0) = u^0(\mathbf{x}), \quad \frac{dy}{dt}(\mathbf{x},0) = v^0(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{3.7}$$

The discrete analog of (2.13) is the estimate

$$\|y(t)\|_*^2 \leqslant \exp(t) \left(\|Au^0\|^2 + \|v^0\|_A^2 + \int_0^t \exp(-\theta) \|f(\theta)\|_A^2 d\theta \right), \tag{3.8}$$

where the left-hand-side is defined by

$$\|y\|_*^2 = \left\| \frac{dy}{dt} \right\|_A^2 + \|Ay\|^2.$$

The emphasis now is on the approximation in time. For the construction of a domain decomposition scheme for problem (3.3), (3.4), we take the usual two-level scheme as the starting point. Let τ be a uniform time-step and let $y^n = y(t^n)$, $t^n = n\tau$, $n = 0, 1, ..., N$, $N\tau = T$. Equation (3.3) is approximated by a two-level scheme with weights

$$\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1-\sigma) y^n) = \varphi^n, \quad n = 0, 1, ..., N-1, \tag{3.9}$$

where, for example, $\varphi^n = f(\sigma t^{n+1} + (1-\sigma) t^n)$. It is supplemented by the initial condition

$$y^0 = u^0. \tag{3.10}$$

Difference scheme (3.9), (3.10) has the approximation error $\mathcal{O}(\tau^2 + (\sigma - 1/2)\tau + h^2)$, where $h^2 = (h_1^2 + h_2^2)/2$.

Theorem 1. *The difference scheme (3.9), (3.10) is unconditionally stable for $\sigma \geqslant 1/2$, and for the numerical solution the estimate*

$$\|y^{n+1}\|_D^2 \leqslant \|y^n\|_D^2 + \frac{\tau}{2} \|\varphi^n\|^2, \quad n = 0, 1, ..., N-1, \tag{3.11}$$

holds, where

$$D = A + \left(\sigma - \frac{1}{2}\right) \tau A^2.$$

Proof. Let us write difference scheme (3.9) as

$$\left(E + \left(\sigma - \frac{1}{2}\right) \tau A \right) \frac{y^{n+1} - y^n}{\tau} + A \frac{y^{n+1} + y^n}{2} = \varphi^n,$$

and apply scalar multiplication with $\tau A(y^{n+1} + y^n)$. Using the fact that $\sigma \geqslant 1/2$, and hence $D \geqslant A$, we have

$$\|y^{n+1}\|_D^2 - \|y^n\|_D^2 + \frac{\tau}{2} \|A(y^{n+1} + y^n)\|^2 = \tau(\varphi^n, A(y^{n+1} + y^n)).$$

Taking into account

$$(\varphi^n, A(y^{n+1} + y^n)) \leqslant \frac{1}{2} \|A(y^{n+1} + y^n)\|^2 + \frac{1}{2} \|\varphi^n\|^2,$$

we obtain the required estimate (3.11). □

The a priori estimate (3.11) for the solution of problem (3.9), (3.10) is the discrete analog of the a priori estimate (3.5) for the solution of differential-difference problem (3.3), (3.4) ($D = A + \mathcal{O}(\tau)$).

To solve problem (3.6), (3.7) numerically, it is natural to use three-level schemes of second order accuracy in time. Let

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + A(\sigma y^{n+1} + (1-2\sigma)y^n + \sigma y^{n-1}) = \varphi^n,$$
$$n = 1, 2, ..., N-1, \tag{3.12}$$

where, for example, $\varphi^n = f(t^n)$. In view of (3.7) we can for the solution of equation (3.6) approximate the initial condition by:

$$y^0 = u^0, \quad \frac{y^1 - y^0}{\tau} = v^0 + \frac{\tau}{2}(\varphi^0 - Au^0). \tag{3.13}$$

The error of the difference scheme (3.12), (3.13) is $\mathcal{O}(\tau^2 + h^2)$.

Theorem 2. *The difference scheme (3.12), (3.13) is unconditionally stable for $\sigma \geqslant 1/4$, and for the numerical solution the estimate*

$$S^{n+1} \leqslant \exp(\tau) S^n + \frac{\tau^2}{2} \frac{\exp(\tau)}{\exp(0.5\tau) - 1} \|\varphi^n\|_A^2, \quad n = 0, 1, ..., N-1, \tag{3.14}$$

holds, where

$$S^n = \left\| \frac{y^n - y^{n-1}}{\tau} \right\|_D^2 + \left\| A \frac{y^n + y^{n-1}}{2} \right\|^2,$$
$$D = A + \left(\sigma - \frac{1}{4}\right) \tau^2 A^2.$$

Proof. We introduce the notation

$$\zeta^n = \frac{y^n + y^{n-1}}{2}, \quad \eta^n = \frac{y^n - y^{n-1}}{\tau}.$$

Taking into account the identity

$$y^n = \frac{1}{4}(y^{n+1} + 2y^n + y^{n-1}) - \frac{1}{4}(y^{n+1} - 2y^n + y^{n-1}),$$
$$\sigma y^{n+1} + (1-2\sigma)y^n + \sigma y^{n-1} = y^n + \sigma(y^{n+1} - 2y^n + y^{n-1})$$

we rewrite (3.12) as

$$\left(E + \left(\sigma - \frac{1}{4}\right)\tau^2 A\right) \frac{\eta^{n+1} - \eta^n}{\tau} + A \frac{\zeta^{n+1} + \zeta^n}{2} = \varphi^n. \tag{3.15}$$

Apply scalar multiplication of (3.15) in H with

$$2A(\zeta^{n+1} - \zeta^n) = \tau A(\eta^{n+1} + \eta^n).$$

With this notation for $\sigma \geqslant 1/4$ we obtain

$$S^{n+1} - S^n = \tau(\varphi^n, A(\eta^{n+1} + \eta^n)). \tag{3.16}$$

Using the estimates for the right-hand side

$$\tau A(\varphi^n, (\eta^{n+1} + \eta^n)) \leqslant \frac{\tau}{2\varepsilon}\|\eta^{n+1} + \eta^n\|_A^2 + \frac{\tau}{2}\varepsilon\|\varphi^n\|_A^2,$$

$$\|\eta^{n+1} + \eta^n\|_A^2 \leqslant 2(\|\eta^{n+1}\|_A^2 + \|\eta^n\|_A^2),$$

with $\varepsilon > 0$, from (3.16) we obtain

$$\Big(1 - \frac{\tau}{\varepsilon}\Big) S^{n+1} \leqslant \Big(1 + \frac{\tau}{\varepsilon}\Big) S^n + \frac{\tau}{2}\varepsilon\|\varphi^n\|_A^2. \tag{3.17}$$

We choose ε so that

$$1 - \frac{\tau}{\varepsilon} = \exp(-0.5\tau),$$

and therefore

$$1 + \frac{\tau}{\varepsilon} < \exp(0.5\tau).$$

With this in mind from (3.17) we obtain the level-wise stability estimate (3.14). □

Estimate (3.14) can be treated as the discrete analog of the a priori estimate (3.8). For the difference schemes (3.9), (3.10) and (3.12), (3.13) we can obtain many other a priori estimates of stability with respect to the initial data and right-hand side [21, 19]. We have restricted ourselves to only those estimates that we can associate with the corresponding estimates for the domain decomposition schemes considered below.

4. Substructuring domain decomposition

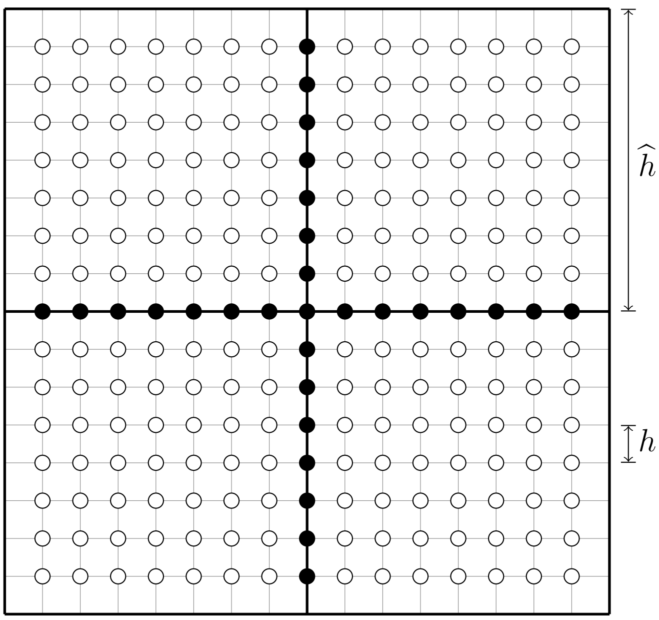

Figure 4.1. Grid decomposition

Let us consider a special class of domain decomposition methods. At the discrete level we define in the domain a set of subdomains and interface nodes and then we solve the subproblems separately. At the continuous level, this decomposition is associated with subdomains

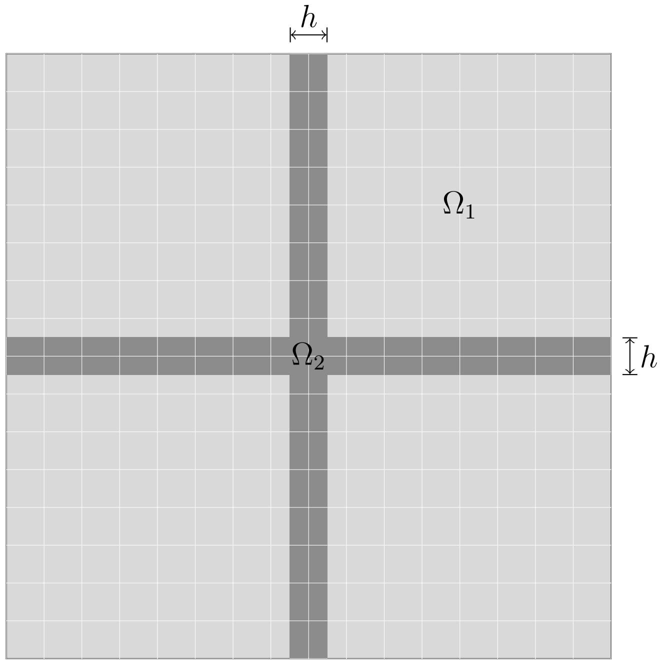

Figure 4.2. Domain decomposition

of which the width is equal to the corresponding discretization step in space. We illustrate this on the model grid problems in a rectangle.

The computational grid ω is partitioned into rectangular subdomains with a (coarse grid) size $\widehat{h}$. The boundaries of the subdomains (interface lines) consist of the nodes of the fine computational grid. Denote this set of interior boundary nodes as $\widehat{\omega}$. A fragment of the grid is shown in Fig. 4.1. Such a decomposition of the fine computational grid corresponds to the domain decomposition depicted in Fig. 4.2: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$, $\Omega_{12} = \varnothing$. Subdomain Ω_2 is a wireframe, and the width of the individual edges of the lattice is h. Domain Ω_1 consists of the disconnected individual subdomains.

With the partition of unity (1.3), (1.4) we associate a corresponding additive representation of the identity operator E in the space of grid functions H, defined on the set of internal nodes of ω. Let

$$\sum_{\alpha=1}^{p} \chi_\alpha = E, \quad \chi_\alpha \geqslant 0, \quad \alpha = 1, 2, ..., p. \tag{4.1}$$

Similar to (1.6), the operators of decomposition can be given in the form

$$A_\alpha = \chi_\alpha A, \quad \alpha = 1, 2, ..., p. \tag{4.2}$$

In view of (4.1), in this splitting we have for the problem operator the following additive representation

$$A = \sum_{\alpha=1}^{p} A_\alpha. \tag{4.3}$$

Splitting (4.3) allows us to go from equation (3.3) to the equation

$$\frac{dy}{dt} + \sum_{\alpha=1}^{p} A_\alpha y = f(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \quad 0 < t < T. \tag{4.4}$$

Direct construction of various splitting schemes for problem (3.4), (4.4) is complicated by the fact that individual operator terms $A_\alpha,\ \alpha = 1, 2, ..., p$ do not inherit the basic properties of the operator A, i.e., the self-adjointness and the non-negativity. However, using decomposition operators (4.2), equation (4.4) can be easily transformed in symmetric form. Multiplying equation (4.4) by the self-adjoint operator A, we obtain the equation

$$\tilde{B}\frac{dy}{dt} + \sum_{\alpha=1}^{p} \tilde{A}_\alpha y = Af(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \quad 0 < t < T, \tag{4.5}$$

where the operators

$$\tilde{B} = A, \quad \tilde{A}_\alpha = A\chi_\alpha A, \quad \alpha = 1, 2, ..., p$$

are self-adjoint and non-negative. Moreover, we can introduce new variables $v = A^{1/2}y$ and instead of (4.5) we can consider the equation

$$\frac{dv}{dt} + \sum_{\alpha=1}^{p} \tilde{A}_\alpha v = A^{1/2} f(\mathbf{x}, t), \quad \mathbf{x} \in \omega, \quad 0 < t < T, \tag{4.6}$$

with self-adjoint and non-negative operators

$$\tilde{A}_\alpha = A^{1/2}\chi_\alpha A^{1/2}, \quad \alpha = 1, 2, ..., p.$$

Standard estimates for the solution of equation (4.6) in the norm of H (for $\|v\|$) correspond to using estimates in H_A (for $\|y\|_A$). This explains our in some sense unusual choice of the the priori estimate (3.5) for problem (3.3), (3.4) and estimate (3.8) for problem (3.6), (3.7).

The particular specification of the decomposition operators of type (4.1), (4.2) is provided via the selection of terms $\chi_\alpha,\ \alpha = 1, 2, ..., p$. Some advanced features are discussed below, but we start from the simplest version. If we use the substructuring domain decomposition (see Fig. 4.1), it is natural to put

$$\chi_2(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \widehat{\omega}, \\ 0, & \mathbf{x} \notin \widehat{\omega}, \end{cases} \quad \chi_1(\mathbf{x}) = 1 - \chi_2(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{4.7}$$

The operator A_2 is associated with interface nodes $\widehat{\omega}$, and A_1 with the internal nodes of the subdomains.

5. Factorized schemes of domain decomposition

After selecting the operators in decomposition (4.3) the construction of the domain decomposition schemes is carried out using one or another additive scheme. For (4.2), (4.7) we can consider the simplest two-component ($p = 2$) splitting scheme. In this situation, we can try to use the operator analogues of the classical schemes of alternating directions[17, 5].

We begin with the scheme of stabilizing correction [5], where the transition to a new time level in problem (3.4), (4.4) with $p = 2$ is performed as follows:

$$\frac{y^{n+1/2} - y^n}{\tau} + A_1 y^{n+1/2} + A_2 y^n = \varphi^n, \tag{5.1}$$

$$\frac{y^{n+1} - y^n}{\tau} + A_1 y^{n+1/2} + A_2 y^{n+1} = \varphi^n, \tag{5.2}$$

where, for example, $\varphi^n = f(t^{n+1})$, $n = 0, 1, ..., N-1$. Equations (5.1), (5.2) are complemented by the initial condition (3.10).

If the decomposition (4.2), (4.7) is used, we have

$$\frac{y^{n+1/2} - y^n}{\tau} + \chi_1 A y^{n+1/2} + \chi_2 A y^n = \varphi^n, \tag{5.3}$$

$$\frac{y^{n+1} - y^n}{\tau} + \chi_1 A y^{n+1/2} + \chi_2 A y^{n+1} = \varphi^n. \tag{5.4}$$

The implementation of this scheme can be different.

Taking into account that

$$\varphi^n = \chi_1 \varphi^n + \chi_2 \varphi^n,$$

let us introduce the auxiliary function $\widetilde{y}^{n+1/2}$ and split equation (5.3) into the following two:

$$\frac{\widetilde{y}^{n+1/2} - y^n}{\tau} + \chi_2 A y^n = \chi_2 \varphi^n, \tag{5.5}$$

$$\frac{y^{n+1/2} - \widetilde{y}^{n+1/2}}{\tau} + \chi_1 A y^{n+1/2} = \chi_1 \varphi^n. \tag{5.6}$$

The function $\widetilde{y}^{n+1/2}$ is determined by the explicit scheme (5.5). Moreover, taking into account (4.7), the calculations are made only on the set of interface nodes.

Stage 1. Evaluation of the conditions at the boundaries of the subdomains by the explicit scheme

$$\frac{\widetilde{y}^{n+1/2} - y^n}{\tau} + A y^n = \varphi^n, \quad \mathbf{x} \in \widehat{\omega},$$

$$\widetilde{y}^{n+1/2} = y^n, \quad \mathbf{x} \notin \widehat{\omega}.$$

After such a predictor for the boundary conditions we solve the problems in the subdomains (5.6).

Stage 2. Evaluation of the solution in subdomains by the implicit scheme

$$\frac{y^{n+1/2} - \widetilde{y}^{n+1/2}}{\tau} + A y^{n+1/2} = \varphi^n, \quad \mathbf{x} \notin \widehat{\omega}.$$

$$y^{n+1/2} = \widetilde{y}^{n+1/2}, \quad \mathbf{x} \in \widehat{\omega}.$$

The last step is to correct the conditions at the boundaries, which provides, in particular, the stability of the approximate solution. For the subdomains it is convenient to replace equation (5.4) by the difference of (5.4), (5.3)

$$\frac{y^{n+1} - y^{n+1/2}}{\tau} + \chi_2 A (y^{n+1} - y^n) = 0.$$

Taking into account (4.7), we calculate the approximate solution at the new time level.

Stage 3. Correction of the conditions at the boundaries of subdomains by the implicit scheme

$$\frac{y^{n+1} - y^n}{\tau} + A y^{n+1} = \varphi^n, \quad \mathbf{x} \in \widehat{\omega},$$

$$y^{n+1} = y^{n+1/2}, \quad \mathbf{x} \notin \widehat{\omega}.$$

This numerical implementation (stages 1–3) of regionally-additive scheme (4.7), (5.3), (5.4) is nothing but the scheme of the domain decomposition [41, 40, 30, 9, 10, 11]) with the explicit-implicit procedure for calculating the boundary conditions at the boundaries of subdomains.

The regionally-additive scheme (4.7), (5.3), (5.4) gives a first order approximation in τ. It is also possible to use the second order schemes, where

$$\frac{y^{n+1/2} - y^n}{\tau/2} + \chi_1 A y^{n+1/2} + \chi_2 A y^n = \varphi^n, \tag{5.7}$$

$$\frac{y^{n+1} - y^{n+1/2}}{\tau/2} + \chi_1 A y^{n+1/2} + \chi_2 A y^{n+1} = \varphi^n. \tag{5.8}$$

with $\varphi^n = f(t^{n+1/2})$. We consider the schemes (5.3), (5.4) and (5.7), (5.8) as the operator analogs of the classical schemes of alternating directions. They are special cases of more general factorized schemes.

Consider the factorized scheme

$$B_1 B_2 \frac{y^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \tag{5.9}$$

where

$$B_\alpha = E + \sigma\tau\chi_\alpha A, \quad \alpha = 1, 2, \tag{5.10}$$

with the right-hand side specified in the form $\varphi^n = f(\sigma t^{n+1} + (1-\sigma)t^n)$. Direct substitution verifies that scheme (5.9), (5.10) coincides with scheme (5.3), (5.4) with $\sigma = 1$ and with scheme (5.7), (5.8) with $\sigma = 1/2$.

For the factorized scheme (5.9), (5.10) it is possible to use the three-stage computational implementation with explicit-implicit calculation of the interface boundary conditions. We introduce, for example, the new grid function $\widetilde{y}^{n+1}$ and instead of (5.9) in view of (5.10) we solve two differential equations:

$$(E + \sigma\tau\chi_1 A)\frac{\widetilde{y}^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \tag{5.11}$$

$$(E + \sigma\tau\chi_2 A)\frac{y^{n+1} - y^n}{\tau} = \frac{\widetilde{y}^{n+1} - y^n}{\tau}. \tag{5.12}$$

Taking into account (4.7), we obtain from (5.11) that for nodes at common boundaries (*Stage 1* — the explicit scheme for boundary nodes)

$$\frac{\widetilde{y}^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \quad \mathbf{x} \in \widehat{\omega}. \tag{5.13}$$

For subdomains we have

$$(E + \sigma\tau A)\frac{y^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \quad \mathbf{x} \notin \widehat{\omega}.$$

This corresponds to (*Stage 2* — the implicit scheme in the subdomains) the use the implicit scheme with weight σ for the difference solution in the subdomains. The implementation of (5.12) (*Stage 3* — the implicit scheme for the boundary nodes) in view of (4.7) is

$$(E + \sigma\tau A)\frac{y^{n+1} - y^n}{\tau} = \frac{\widetilde{y}^{n+1} - y^n}{\tau}, \quad \mathbf{x} \in \widehat{\omega}, \tag{5.14}$$

$$y^{n+1} = \widetilde{y}^{n+1}, \quad \mathbf{x} \notin \widehat{\omega}.$$

Taking into account (5.13), equation (5.14) can be written as

$$(E + \sigma\tau A)\frac{y^{n+1} - y^n}{\tau} + Ay^n = \varphi^n, \quad \mathbf{x} \in \widehat{\omega}.$$

At this stage all computational work is associated only with the correction of the internal boundary conditions by this weighted implicit scheme.

Theorem 3. *The factorized regionally-additive difference scheme (4.1), (5.9), (5.10) is unconditionally stable for $\sigma \geqslant 1/2$, and for the discrete solution the following estimate holds*

$$\|B_2 y^{n+1}\|_A \leqslant \|B_2 y^n\|_A + \tau\|B_1^{-1}\varphi^n\|_A, \quad n = 0, 1, ..., N-1. \tag{5.15}$$

Proof. It is convenient first to symmetrize the factorized scheme (5.9), (5.10). Let $v^n = A^{1/2}y^n$ and

$$\widetilde{B}_\alpha = E + \sigma\tau\widetilde{A}_\alpha, \quad \widetilde{A}_\alpha = A^{1/2}\chi_\alpha A^{1/2}, \quad \alpha = 1, 2.$$

Then equation (5.9) can be rewritten as

$$\widetilde{B}_1\widetilde{B}_2\frac{v^{n+1} - v^n}{\tau} + Av^n = A^{1/2}\varphi^n. \tag{5.16}$$

Assuming that $\widetilde{B}_2 v^n = w^n$, from (5.16) we obtain

$$w^{n+1} = Sw^n + \tau\widetilde{B}_1^{-1}A^{1/2}\varphi^n, \tag{5.17}$$

where S is the operator of the transition to the new time level

$$S = E - \tau\widetilde{B}_1^{-1}A\widetilde{B}_2^{-1}. \tag{5.18}$$

Taking into account the above notation, from (5.18) we obtain

$$S = \frac{2\sigma - 1}{2\sigma}E + \frac{1}{2\sigma}\widetilde{B}_1^{-1}(\widetilde{B}_1\widetilde{B}_2 - 2\sigma(\widetilde{A}_1 + \widetilde{A}_2))\widetilde{B}_1^{-2}$$

$$= \frac{2\sigma - 1}{2\sigma}E + S_1S_2,$$

where

$$S_\alpha = (E + \sigma\tau\widetilde{A}_\alpha)^{-1}(E - \sigma\tau\widetilde{A}_\alpha), \quad \alpha = 1, 2.$$

If $\sigma \geqslant 0$, taking into account the non-negativity of the operators $\widetilde{A}_\alpha,\ \alpha = 1, 2$, we have

$$\|S_\alpha\| \leqslant 1, \quad \alpha = 1, 2.$$

With the stronger restriction $\sigma \geqslant 1/2$ we find that $\|S\| \leqslant 1$. From (5.7) we obtain the estimate

$$\|w^{n+1}\| = \|w^n\| + \tau\|\widetilde{B}_1^{-1}A^{1/2}\varphi^n\|.$$

This is the required estimate (5.15). □

The fundamental issue in the construction of domain decomposition schemes for unsteady problems is to estimate the convergence rate for the approximate solution. The accuracy depends on the computational grid (the width of the overlapping) and therefore regionally-additive schemes are conditionally convergent. The situation can be illustrated by the example of the above factorized decomposition schemes (5.9), (5.10).

The accuracy is analyzed in a standard way by considering the corresponding problem for the error

$$z^n(\mathbf{x}) = y^n(\mathbf{x}) - u^n(\mathbf{x}), \quad \mathbf{x} \in \omega,$$

where $u^n(\mathbf{x}) = u(\mathbf{x}, t^n)$ is the exact solution of the differential problem (2.1)–(2.3). From (4.1), (5.9), (5.10) we obtain the problem for the error

$$B_1 B_2 \frac{z^{n+1} - z^n}{\tau} + A z^n = \psi^n, \tag{5.19}$$

$$z^0 = 0. \tag{5.20}$$

In view of (5.15) for problem (5.19), (5.20) we have

$$\|B_2 z^{n+1}\|_A \leqslant \sum_{k=0}^{n} \tau \|B_1^{-1} \psi^k\|_A, \quad n = 0, 1, ..., N-1. \tag{5.21}$$

For the approximation error we have

$$\psi^n = \varphi^n - B_1 B_2 \frac{u^{n+1} - u^n}{\tau} - A u^n. \tag{5.22}$$

Taking into account (5.10), from (5.22) we obtain

$$\psi^n = \psi_1^n + \psi_2^n,$$

$$\psi_1^n = \varphi^n - \left(E + \left(\sigma - \frac{1}{2}\right) \tau A\right) \frac{u^{n+1} - u^n}{\tau} - A \frac{u^{n+1} + u^n}{2},$$

$$\psi_2^n = -\sigma^2 \tau^2 \chi_1 A \chi_2 A \frac{u^{n+1} - u^n}{\tau}.$$

The first term of the error is standard for the schemes with weights, whereas the second term results from the splitting in subdomains. For sufficiently smooth solutions of problem (2.1)–(2.3) we have

$$\psi_1^n = \mathcal{O}(h^2 + \tau^2 + \left(\sigma - \frac{1}{2}\right) \tau).$$

Let us consider the term ψ_2^n in more detail.

Taking into account (5.21) and using the notation from the proof of Theorem 3, we have

$$\begin{aligned}
\|B_1^{-1} \psi_2^n\|_A &= \|\widetilde{B}_1^{-1} A^{1/2} \psi^n\| \\
&= \sigma^2 \tau^2 \left\| \widetilde{B}_1^{-1} A^{1/2} \chi_1 A \chi_2 A \frac{u^{n+1} - u^n}{\tau} \right\| \\
&= \sigma \tau \left\| Q A^{1/2} \chi_2 A \frac{u^{n+1} - u^n}{\tau} \right\| \leqslant \sigma \tau \left\| A^{1/2} \chi_2 A \frac{u^{n+1} - u^n}{\tau} \right\|,
\end{aligned}$$

where

$$Q = (E + \sigma\tau \widetilde{A}_1)^{-1} \sigma\tau \widetilde{A}_1.$$

Thus

$$\|B_1^{-1} \psi_2^n\|_A = \mathcal{O}(\sigma\tau \|\chi_2\|_A).$$

These arguments allow us to formulate the following statement.

Theorem 4. *For the error of the factorized regionally-additive difference scheme (4.1), (5.9), (5.10) with $\sigma \geqslant 1/2$ we have for problem (2.1)–(2.3) the following estimate*

$$\|B_2 z^{n+1}\|_A \leqslant M \left(h^2 + \tau^2 + \left(\sigma - \frac{1}{2} \right) \tau + \sigma\tau \|\chi_2\|_A \right). \quad (5.23)$$

For the substructuring domain decomposition schemes considered here, with the discrete elliptic operators of second order (3.1) and splitting (4.1), (4.7) the estimate (5.23) gives

$$\|B_2 z^{n+1}\|_A \leqslant M \left(h^2 + \tau^2 + \left(\sigma - \frac{1}{2} \right) \tau + \sigma\tau \widehat{h}^{-1/2} h^{-1/2} \right). \quad (5.24)$$

Note also that $\sigma = 1/2$ does not increase the order of accuracy. But in this case the main error term is two times lower compared to $\sigma = 1$.

A slightly different algorithm can be implemented. Instead of (5.9) we apply the factorized scheme

$$B_2 B_1 \frac{y^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \quad (5.25)$$

i.e., we commute the operators B_1 and B_2.

The implementation of scheme (5.25) will differ slightly from the implementation of scheme (5.9). Similar to (5.11), (5.12) we have

$$(E + \sigma\tau \chi_2 A) \frac{\widetilde{y}^{n+1} - y^n}{\tau} + A y^n = \varphi^n, \quad (5.26)$$

$$(E + \sigma\tau \chi_1 A) \frac{y^{n+1} - y^n}{\tau} = \frac{\widetilde{y}^{n+1} - y^n}{\tau}. \quad (5.27)$$

At stage (5.26) we use the implicit scheme for the nodes at the boundaries of the subdomains and explicit scheme in the subdomains. Note that for the explicit scheme it is enough to evaluate only the boundary nodes. At stage (5.27) the solution in the subdomains is calculated using the implicit scheme. Thus, the computational cost in case of the factorized scheme (5.25) remains practically the same as for scheme (5.9).

6. Schemes of multi-component splitting

The factorized schemes of the two-component splitting constructed above can be generalized in various directions. The most fundamental issue is to construct such schemes in the case of general multi-component splitting.

The need for such an extension results from, in particular, calculations of conditions at the boundaries of subdomains, i.e., the solution of problems on graphs for two-dimensional problems. In the two-dimensional problems in a rectangle and rectangular grids, the implementation of, for example, (5.4) does not face significant problems. However, for more

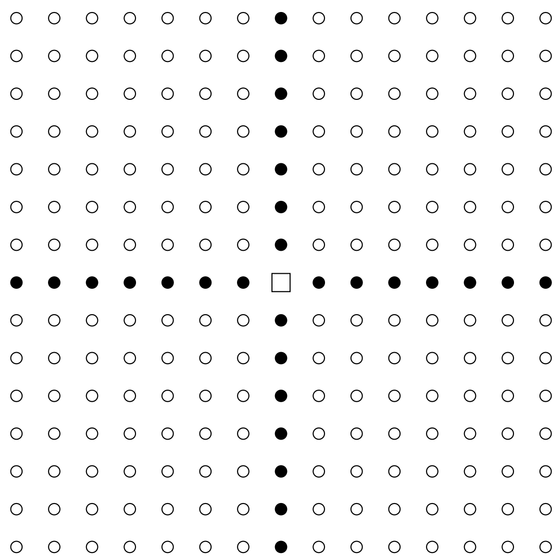

Figure 6.1. Three-component decomposition without the overlapping of subdomains

general situations, for example, for three-dimensional boundary value problems, the solution of these grid problems can be difficult. Such considerations lead to the construction of decomposition procedures for the set of boundary nodes of subdomains. A characteristic example is shown in Fig. 6.1. The set of boundary nodes is divided into two parts: $\widehat{\omega} = \widehat{\omega}_s \cup \widehat{\omega}_m$. Here the set of nodes at the boundary of two subdomains is denoted by $\widehat{\omega}_s$ (in Fig. 6.1 depicted as $\bullet$). The set of nodes that lie at the boundaries of a greater number of subdomains is designated as $\widehat{\omega}_m$ (in Fig. 6.1 represented by $\square$).

Instead of the two-component splitting (4.1), (4.7), we now use the three-component splitting (4.1) with $p = 3$ and

$$\chi_2(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \widehat{\omega}_s, \\ 0, & \mathbf{x} \notin \widehat{\omega}_s, \end{cases} \qquad \chi_3(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \widehat{\omega}_m, \\ 0, & \mathbf{x} \notin \widehat{\omega}_m, \end{cases}$$

$$\chi_1(\mathbf{x}) = 1 - \chi_2(\mathbf{x}) - \chi_3(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{6.1}$$

With such a decomposition, the calculations can be executed independent of each other in the different parts of the subdomain boundaries (on the set $\widehat{\omega}_s$) because of the known conditions at the nodes of crossing (on the set $\widehat{\omega}_m$).

Local computation of the solution at the boundary crossings introduces additional errors. To improve the accuracy of the approximate solution at the boundaries of subdomains, it is possible to apply algorithms with overlapping subdomains. Such a situation is shown in Fig. 6.2. There the set of boundary nodes $\widehat{\omega}_m$, which lie near the boundary crossing and $\widehat{\omega}_s \cap \widehat{\omega}_m \neq \varnothing$ is indicated. With this in mind, instead of (6.1) we set

$$\chi_2(\mathbf{x}) = \begin{cases} > 0, & \mathbf{x} \in \widehat{\omega}_s, \\ 0, & \mathbf{x} \notin \widehat{\omega}_s, \end{cases} \qquad \chi_3(\mathbf{x}) = \begin{cases} > 0, & \mathbf{x} \in \widehat{\omega}_m, \\ 0, & \mathbf{x} \notin \widehat{\omega}_m, \end{cases}$$

$$\chi_2(\mathbf{x}) + \chi_3(\mathbf{x}) = 1, \quad \mathbf{x} \in \widehat{\omega}, \quad \chi_1(\mathbf{x}) = 1 - \chi_2(\mathbf{x}) - \chi_3(\mathbf{x}), \quad \mathbf{x} \in \omega. \tag{6.2}$$

For the general multi-component ($p > 2$) decomposition it is possible to construct regularized additive schemes [26, 25] in a simpler way. For solving problem (3.4), (4.2), (4.4) we

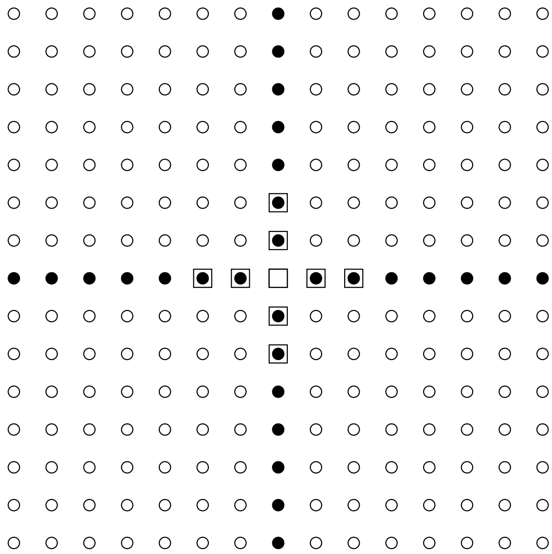

Figure 6.2. Three-component decomposition with the overlapping of subdomains

can use the additive scheme of full approximation

$$\frac{y^{n+1} - y^n}{\tau} + \widetilde{A} y^n = \varphi^n, \quad (6.3)$$

where

$$\widetilde{A} = \sum_{\alpha=1}^{p} \widetilde{A}_\alpha, \quad \widetilde{A}_\alpha = (E + \sigma\tau\chi_\alpha A)^{-1}\chi_\alpha A, \quad \alpha = 1, 2, ..., p. \quad (6.4)$$

This scheme is characterized by the fact that each operator term $\chi_\alpha A, \ \alpha = 1, 2, ..., p$ is perturbed with an error $\mathcal{O}(\tau)$.

Theorem 5. *The regularized difference scheme (6.3), (6.4) is unconditionally stable for $\sigma \geqslant p/2$, and for the discrete solution we have the estimate*

$$\|y^{n+1}\|_A \leqslant \|y^n\|_A + \tau\|\varphi^n\|_A, \quad n = 0, 1, ..., N-1. \quad (6.5)$$

Proof. The operator $\widetilde{A}$ can be written in the form

$$\widetilde{A} = \sum_{\alpha=1}^{p} A^{-1/2} R_\alpha A^{1/2}, \quad (6.6)$$

where

$$R_\alpha = (E + \sigma\tau C_\alpha)^{-1} C_\alpha, \quad C_\alpha = A^{1/2}\chi_\alpha A^{1/2}, \quad \alpha = 1, 2, ..., p.$$

Thus

$$C_\alpha = C_\alpha^* \geqslant 0, \quad R_\alpha = R_\alpha^* \geqslant 0, \quad \alpha = 1, 2, ..., p.$$

With this in mind the difference scheme (6.3), (6.4) is written as

$$\frac{v^{n+1} - v^n}{\tau} + \sum_{\alpha=1}^{p} R_\alpha v^n = A^{1/2}\varphi^n, \quad (6.7)$$

where, as before, $v^n = A^{1/2}y^n$. From (6.7) we have

$$v^{n+1} = Sv^n + \tau A^{1/2}\varphi^n \tag{6.8}$$

with the transition operator

$$S = E - \tau \sum_{\alpha=1}^{p} R_\alpha.$$

Using this representation, we set

$$S = \frac{1}{p}\sum_{\alpha=1}^{p} S_\alpha, \quad S_\alpha = E - p\tau R_\alpha, \quad \alpha = 1, 2, ..., p. \tag{6.9}$$

For the individual terms we obtain

$$S_\alpha = (E + \sigma\tau C_\alpha)^{-1}(E + (\sigma - p)\tau C_\alpha).$$

Under the constraint $\sigma \geqslant p/2$ we have $\|S_\alpha\| \leqslant 1$, which allows us to obtain from (6.8) the following estimate

$$\|v^{n+1}\| \leqslant \|v^n\| + \tau\|A^{1/2}\varphi^n\|,$$

which is nothing but (6.5). □

Standard finite-difference schemes for component-wise splitting [19, 15, 39] can be easily constructed using the transition operator. With regard to our problem, we shall again start with notation (6.8), but instead the additive structure (see (6.9)) we now use a multiplicative structure

$$S = \prod_{\alpha=p}^{1} S_\alpha, \quad S_\alpha = E - \tau R_\alpha, \quad \alpha = 1, 2, ..., p. \tag{6.10}$$

In this case we have $\|S_\alpha\| \leqslant 1$ for $\sigma \geqslant 1/2$.

The implementation of the component-wise splitting scheme is performed as a sequence of intermediate difference problems similar to (6.8):

$$v^{n+\alpha/p} = S_\alpha v^{n+(\alpha-1)/p} + \tau A^{1/2}\varphi_\alpha^n, \quad \alpha = 1, 2, ..., p. \tag{6.11}$$

Comparing this with (6.8), (6.9), we obtain

$$\varphi^n = \sum_{\alpha=p}^{1}\prod_{\beta=p}^{\alpha-1} S_\beta \varphi_\alpha^n.$$

Without loss of accuracy, we can consider only the simplest choice for φ_α^n, $\alpha = 1, 2, ..., p$, where

$$\varphi^n = \sum_{\alpha=p}^{1} \varphi_\alpha^n. \tag{6.12}$$

With this notation the difference equations (6.11) can be written as

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + (E + \sigma\tau\chi_\alpha A)^{-1}\chi_\alpha A y^{n+(\alpha-1)/p} = \varphi_\alpha^n,$$

$$\alpha = 1, 2, ..., p. \tag{6.13}$$

We can formulate now the following statement.

Theorem 6. *The additive component-wise splitting scheme (6.12), (6.13) is unconditionally stable for $\sigma \geqslant 1/2$, and for the discrete solution estimate (6.5) holds.*

The scheme is first-order accurate in time. However, in the case of two-component splitting with $\sigma = 1/2$ the approximation error is $\mathcal{O}(\tau^2)$ (see, e.g., [20]). This variant is used for the construction of DD schemes in [6]. For the general multi-component splitting the additive schemes of second order in time are based on the symmetrization of transition operator [29, 7]. In this case, instead of (6.10) we can use, for example,

$$S = \prod_{\beta=1}^{p} S_\beta \prod_{\alpha=p}^{1} S_\alpha, \quad S_\alpha = \Big(E + \frac{\tau}{4} C_\alpha\Big)^{-1} \Big(E - \frac{\tau}{4} C_\alpha\Big), \quad \alpha = 1, 2, ..., p.$$

Thus we make two half-steps in time for $\sigma \geqslant 1/2$, sequentially solving problems for operators $\chi_\alpha A, \ \alpha = 1, 2, ..., p$, and then for the operators $\chi_\beta A, \ \beta = p, p-1, ..., 1$.

The regularized scheme (6.3), (6.4) can be written in the form similar to (6.13)

$$\frac{y^{n+\alpha/p} - y^{n+(\alpha-1)/p}}{\tau} + (E + \sigma\tau\chi_\alpha A)^{-1} \chi_\alpha A y^n = \varphi_\alpha^n,$$
$$\alpha = 1, 2, ..., p. \qquad (6.14)$$

In contrast to (6.13) here $y^{n+(\alpha-1)/p}$ is only partially used for solving the problem for $y^{n+\alpha/p}$. This increasing explicitness results in a stronger stability condition (instead of $\sigma \geqslant 1/2$ we have $\sigma \geqslant p/2$).

The numerical implementation of scheme (6.12), (6.13) is shown below for decomposition (4.7). In accordance with (6.12) we set

$$\varphi_\alpha^n = \chi_\alpha \varphi^n, \quad \alpha = 1, 2, ..., p.$$

From (6.13) we obtain

$$(E + \sigma\tau\chi_1 A) \frac{y^{n+1/2} - y^n}{\tau} + \chi_1 A y^n = (E + \sigma\tau\chi_1 A) \chi_\alpha \varphi^n, \qquad (6.15)$$

$$(E + \sigma\tau\chi_2 A) \frac{y^{n+1} - y^{n+1/2}}{\tau} + \chi_2 A y^{n+1/2} = (E + \sigma\tau\chi_2 A) \chi_2 \varphi^n. \qquad (6.16)$$

In finding $y^{n+1/2}$ from (6.15) (*Stage 1*) we solve the boundary value problems in the subdomains by the implicit scheme. The boundary conditions are taken from the previous time level, i.e.,

$$(E + \sigma\tau A) \frac{y^{n+1/2} - y^n}{\tau} + A y^n = (E + \sigma\tau A) \varphi^n, \quad \mathbf{x} \notin \widehat{\omega}.$$
$$y^{n+1/2} = y^n, \quad \mathbf{x} \in \widehat{\omega}.$$

Conditions at the common boundaries are corrected during evaluation y^{n+1} from (6.16) (*Stage 2*)

$$(E + \sigma\tau A) \frac{y^{n+1} - y^{n+1/2}}{\tau} + A y^{n+1/2} = (E + \sigma\tau A) \varphi^n, \quad \mathbf{x} \in \widehat{\omega},$$
$$y^{n+1} = y^{n+1/2}, \quad \mathbf{x} \notin \widehat{\omega}.$$

The numerical implementation of the component-wise splitting scheme (6.12), (6.13) is slightly less compared with the factorized domain decomposition scheme (5.3), (5.4) (there is no explicit calculation of the interface boundary conditions). Similarly, two-stage implementation can be constructed for the regularized scheme (6.12), (6.14).

7. Hyperbolic equations of second order

The possibilities to construct domain decomposition schemes for the hyperbolic boundary value problems (2.2), (2.9), (2.10) are more restricted. Here we only consider the regularized schemes similar to (6.3), (6.4) for the parabolic problem (2.1)–(2.3),

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + \widetilde{A} y^n = \varphi^n, \tag{7.1}$$

where

$$\widetilde{A} = \sum_{\alpha=1}^{p} \widetilde{A}_\alpha, \quad \widetilde{A}_\alpha = (E + \sigma\tau^2 \chi_\alpha A)^{-1} \chi_\alpha A, \quad \alpha = 1, 2, ..., p. \tag{7.2}$$

This scheme is second order accurate in time and we have the following theorem.

Theorem 7. *The regularized difference scheme (3.13), (7.1), (7.2) is unconditionally stable for $\sigma \geqslant p/4$, and for the discrete solution the following estimate holds*

$$S^{n+1} \leqslant \exp(\tau) S^n + \frac{\tau^2}{2} \frac{\exp(\tau)}{\exp(0.5\tau) - 1} \|\varphi^n\|^2_{D^{-1}}, \quad n = 1, 2, ..., N-1, \tag{7.3}$$

where

$$S^n = \left\| \frac{y^n - y^{n-1}}{\tau} \right\|^2_D + \left\| \frac{y^n + y^{n-1}}{2} \right\|^2_{A\widetilde{A}},$$

$$D = D^* = A \left(E - \frac{\tau^2}{4} \widetilde{A} \right).$$

Proof. The proof is similar to that for Theorem 2. Similar to (6.6), for the operator $\widetilde{A}$, taking into account (7.2), we have representation

$$\widetilde{A} = \sum_{\alpha=1}^{p} A^{-1/2} R_\alpha A^{1/2}, \tag{7.4}$$

where now

$$R_\alpha = (E + \sigma\tau^2 C_\alpha)^{-1} C_\alpha, \quad C_\alpha = A^{1/2} \chi_\alpha A^{1/2}, \quad \alpha = 1, 2, ..., p,$$

with self-adjoint and non-negative operators C_α and R_α, $\alpha = 1, 2, ..., p$.

Difference scheme (7.1), (7.2) can be written as

$$\frac{v^{n+1} - 2v^n - v^{n-1}}{\tau} + \sum_{\alpha=1}^{p} R_\alpha v^n = A^{1/2} \varphi^n \tag{7.5}$$

for $v^n = A^{1/2} y^n$. Using the notation

$$\zeta^n = \frac{v^n + v^{n-1}}{2}, \quad \eta^n = \frac{v^n - v^{n-1}}{\tau},$$

write (7.4) as

$$\left(E - \frac{\tau^2}{4} R \right) \frac{\eta^{n+1} - \eta^n}{\tau} + R \frac{\zeta^{n+1} + \zeta^n}{2} = A^{1/2} \varphi^n, \tag{7.6}$$

where

$$R = R^* = \sum_{\alpha=1}^{p} R_\alpha.$$

Multiply (7.6) in H by the scalar

$$2(\zeta^{n+1} - \zeta^n) = \tau(\eta^{n+1} + \eta^n)$$

and obtain the equality

$$S^{n+1} - S^n = \tau(A^{1/2}\varphi^n, (\eta^{n+1} + \eta^n)), \tag{7.7}$$

where

$$S^n = \|\eta^n\|^2_{\widetilde{D}} + \|\zeta^n\|^2_R = \left\| \frac{v^n - v^{n-1}}{\tau} \right\|^2_{\widetilde{D}} + \left\| \frac{v^n + v^{n-1}}{2} \right\|^2_R,$$

$$\widetilde{D} = E - \frac{\tau^2}{4} R.$$

and $\widetilde{D} > 0$. For the first term we have

$$\left\| \frac{v^n - v^{n-1}}{\tau} \right\|^2_{\widetilde{D}} = \left\| \frac{y^n - y^{n-1}}{\tau} \right\|^2_D$$

that results from

$$A^{1/2}\widetilde{D}A^{1/2} = A - \frac{\tau^2}{4} A^{1/2} R A^{1/2} = A\left(E - \frac{\tau^2}{4}\widetilde{A}\right) = D.$$

To prove the inequality $\widetilde{D} > 0$ for $\sigma \geqslant p/4$, we set

$$\widetilde{D} = \frac{1}{p}\sum_{\alpha=1}^{p} \widetilde{D}_\alpha, \quad \widetilde{D}_\alpha = E - \frac{\tau^2}{4} p R_\alpha, \quad \alpha = 1, 2, ..., p.$$

For each individual term we have $\widetilde{D}_\alpha,\ \alpha = 1, 2, ..., p$, if

$$E - \frac{\tau^2}{4} p (E + \sigma\tau^2 C_\alpha)^{-1} C_\alpha > 0.$$

We have

$$E + \sigma\tau^2 C_\alpha - \frac{\tau^2}{4} p C_\alpha > E.$$

for $\sigma \geqslant p/4$ for each $\alpha = 1, 2, ..., p$.

For the right-hand side of (7.7) with $\widetilde{\varphi}^n = A^{1/2}\varphi^n$, we use the estimates

$$\tau(\widetilde{\varphi}, (\eta^{n+1} + \eta^n)) \leqslant \frac{\tau}{2\varepsilon}\|\eta^{n+1} + \eta^n\|^2_{\widetilde{D}} + \frac{\tau}{2}\varepsilon\|\widetilde{\varphi}^n\|^2_{\widetilde{D}^{-1}},$$

$$\|\eta^{n+1} + \eta^n\|^2_{\widetilde{D}} \leqslant 2(\|\eta^{n+1}\|^2_{\widetilde{D}} + \|\eta^n\|^2_{\widetilde{D}}).$$

We obtain the inequality

$$\left(1 - \frac{\tau}{\varepsilon}\right) S^{n+1} \leqslant \left(1 + \frac{\tau}{\varepsilon}\right) S^n + \frac{\tau}{2}\varepsilon\|\widetilde{\varphi}^n\|^2_{\widetilde{D}^{-1}}. \tag{7.8}$$

We assume that

$$1 - \frac{\tau}{\varepsilon} = \exp(-0.5\tau),$$

and therefore

$$1 + \frac{\tau}{\varepsilon} < \exp(0.5\tau).$$

Therefore the required stability estimate (7.3) is easily obtained from (7.8). □

This stability estimate is characterized by more complex norms in comparison with the case of standard schemes with weights (compare (3.14) and (7.3)). The numerical implementation of the regularized scheme (7.1), (7.2) is similar to that of scheme (6.3), (6.4).

8. Numerical results for model problems

Numerical experiments are made for the parabolic equation (2.1), where

$$k(\mathbf{x}) = 1, \quad f(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Omega, \quad 0 < t \leqslant T. \tag{8.1}$$

Problem (2.1)–(2.3), (7.8) is considered in the unit square $l_1 = l_2 = 1$, and the initial condition has the form

$$u^0(\mathbf{x}) = \sin(n_1 \pi x_1)\sin(n_2 \pi x_2), \quad \mathbf{x} \in \Omega, \tag{8.2}$$

for natural n_1 and n_2. The solution of problem (2.1)–(2.3), (8.1), (8.2) is written as

$$u(\mathbf{x}, t) = \exp(-\pi^2(n_1^2 + n_2^2)t)\sin(n_1 \pi x_1)\sin(n_2 \pi x_2). \tag{8.3}$$

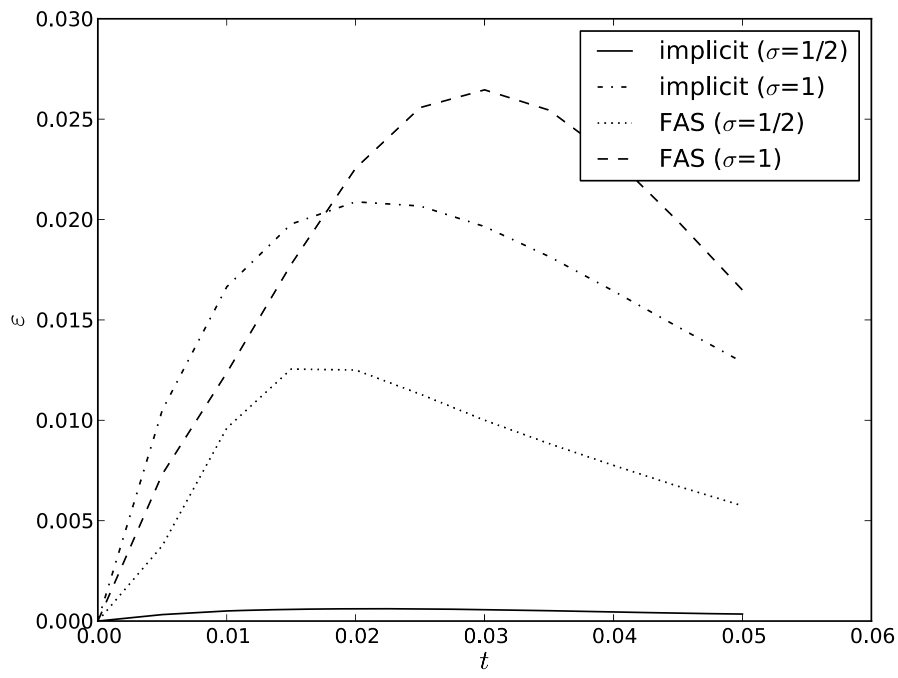

Figure 8.1. The error of the factorized regionally-additive scheme

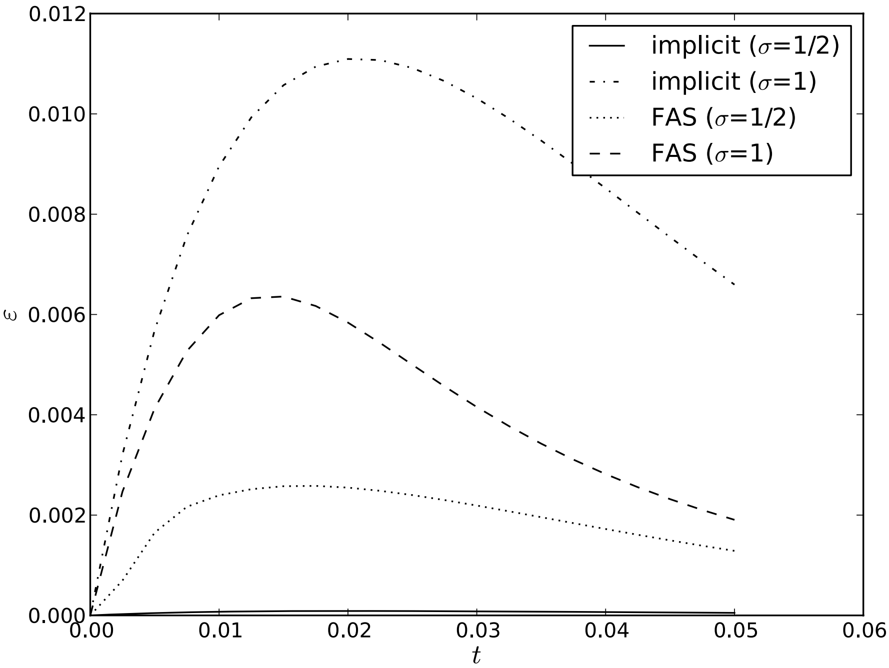

Figure 8.2. Reducing of the time step ($\tau = 0.005$)

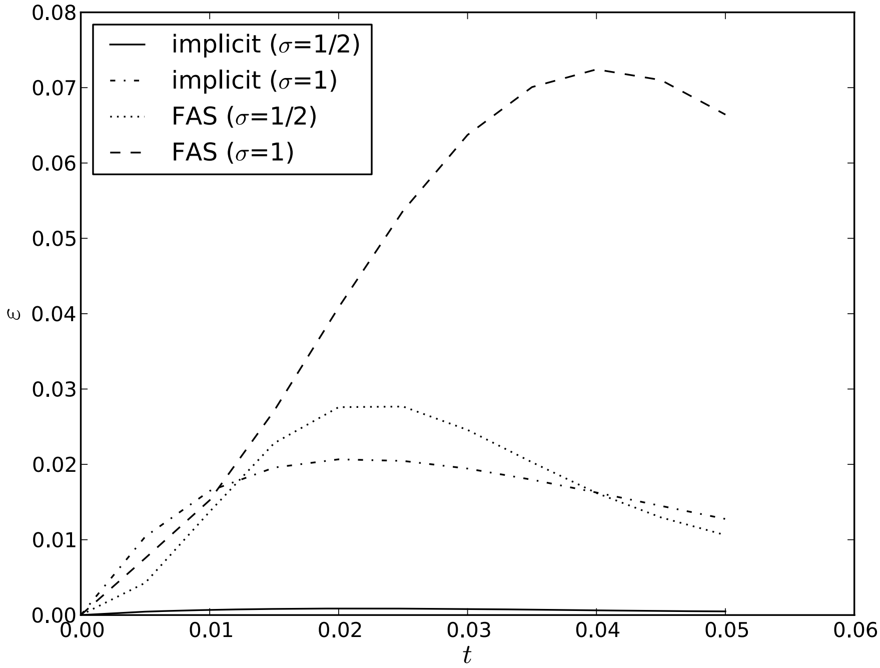

Figure 8.3. Reducing of the spatial step ($h = 1/80$)

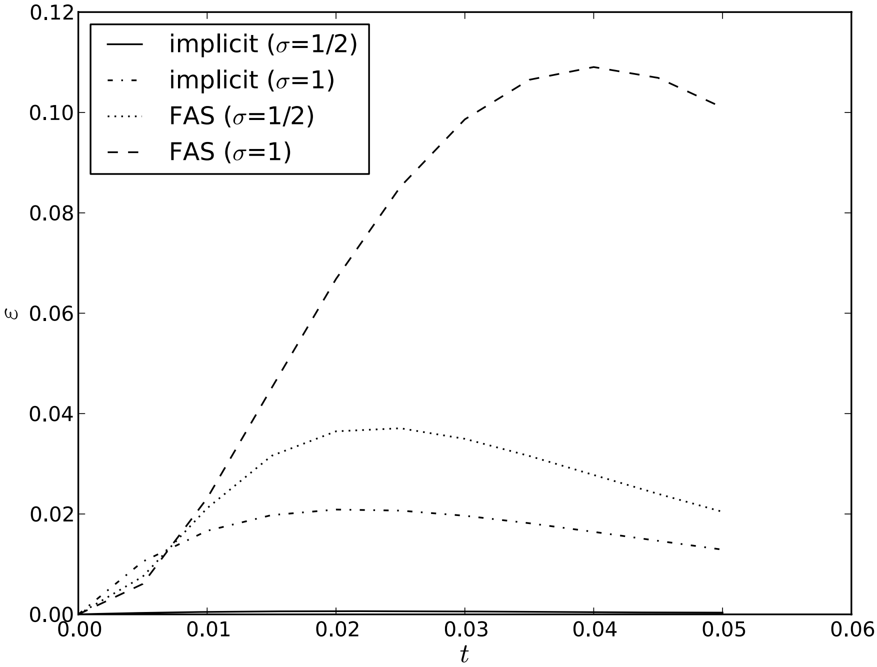

Figure 8.4. Increasing of the number of subdomains ($\widehat{h} = 0.25$)

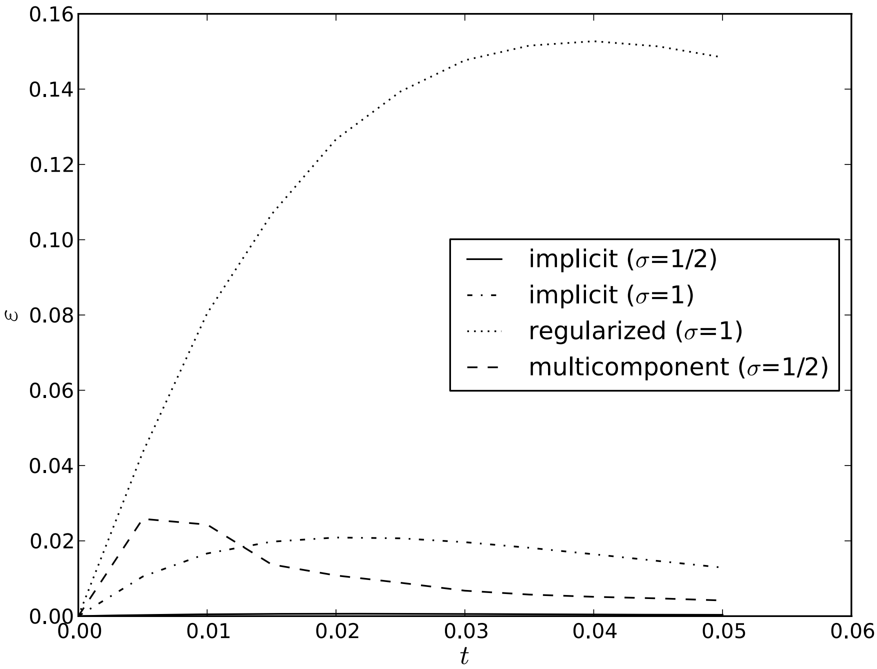

Figure 8.5. The error of the regularized and the regionally-additive component-wise splitting schemes

The numerical results computed with the regionally-additive schemes are compared with the difference solution obtained by means of the implicit scheme (3.9), (3.10) for $\sigma = 1/2$ and $\sigma = 1$. The error of the approximate solution was computed as $\varepsilon(t^n) = \|y^n(\mathbf{x}) - u(\mathbf{x}, t^n)\|$ (norm in $H = L_2(\omega)$) at each particular time level.

In the basic case we used $n_1 = 2, n_2 = 1$, $N_1 = N_2$, $h_1 = h_2 = h = 1/40$, $T = 0.05, N = 10, \tau = 0.01$. The decomposition is carried out by cutting the Ω into four squares ($\widehat{h} = 0.5$).

Results of solving the test problem using the standard implicit schemes with weights (3.9), (3.10) of second ($\sigma = 1/2$) and first ($\sigma = 1$) orders of accuracy with respect to τ and factorized regionally-additive scheme (4.1), (5.9), (5.10) (FAS) for the same values of the weight parameter σ are presented in Fig. 8.1.

With the selected parameters the domain decomposition scheme, constructed on the basis of classical factorized schemes, yields the approximate solution with a slightly larger error than the standard two-level scheme with weights. The effect of the time step is shown in Fig. 8.2, where a smaller time step ($\tau = 0.005$) was used. A more interesting effect is connected with the discretization in space (Fig. 8.3). The effect of conditional convergence becomes more evident for the regionally-additive scheme at $\sigma = 1/2$.

In the study of the decomposition schemes particular attention should be paid to the dependence of the accuracy on the number of subdomains. The error for an increasing number of subdomains (four times, $\widehat{h} = 0.25$) is shown in Fig. 8.4. The decrease of the accuracy (compare Fig. 8.1 and Fig. 8.4) is more significant for the factorized regionally-additive schemes with $\sigma = 1/2$.

We made computations with the above multicomponent splitting schemes for the model problem (2.1)–(2.3), (8.1), (8.2) and decomposition (4.2), (4.7). In Fig. 8.5 the error of the approximate solution for the regularized regionally-additive scheme (6.3), (6.4) with $\sigma = 1$ is compared with the results of the regionally-additive component-wise splitting scheme (6.12), (6.13) with $\sigma = 1/2$. Clearly the accuracy of the regularized scheme is much lower, whereas the accuracy of the component-wise splitting scheme is practically the same as the accuracy of the factorized regionally-additive scheme (see Fig. 8.1 and Fig. 8.5).

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