Research Article

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Finite sections of truncated Toeplitz operators

Abstract: We describe the C^* -algebra associated with the finite sections discretization of truncated Toeplitz operators on the model space K_u^2 where u is an infinite Blaschke product. As consequences, we get a stability criterion for the finite sections discretization and results on spectral and pseudospectral approximation.

Keywords: Model spaces; truncated Toeplitz operators; Widom's identity; stability of the finite sections discretization

DOI 10.2478/conop-2014-0002

Received January 24, 2014; accepted April 23, 2014

1 Truncated Toeplitz operators

Let H^2 denote the standard Hardy space on the unit disk $\mathbb{D}$, i.e. the Hilbert space of all holomorphic functions on $\mathbb{D}$ which have square-summable Taylor coefficients. As usual, we identify H^2 with its space of non-tangential boundary functions, which is a closed subspace of the Lebesgue space $L^2(\mathbb{T})$ with normalized Lebesgue measure m on the unit circle $\mathbb{T}$. The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 is denoted by P .

Every function $a \in L^\infty(\mathbb{T})$ defines an operator of multiplication on $L^2(\mathbb{T})$, which we denote by aI . The Toeplitz operator induced by a is the operator $T(a) := PaI|_{H^2}$, acting from H^2 to H^2 . The Toeplitz operator with generating function $a(z) = z$ is the operator S of forward shift, $(Sf)(z) = zf(z)$. Its adjoint S^* , the backward shift operator, is given by $(S^*f)(z) = z^{-1}(f(z) - f(0))$.

Let u be a non-constant inner function, i.e., u is holomorphic on $\mathbb{D}$ and $|u(t)| = 1$ for $t \in \mathbb{T}$ (the following becomes trivial when u is constant). The subspace $K_u^2 := H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of S^* . Conversely, every proper nontrivial invariant subspace of S^* is of this form by a celebrated theorem of Beurling. The spaces K_u^2 are also known as *model spaces*. We denote the orthogonal projection from $L^2(\mathbb{T})$ onto K_u^2 by P_u . If M_u and $M_{\bar{u}}$ denote the operators of multiplication by u and $\bar{u}$ on $L^2(\mathbb{T})$, then $P_u = P - M_u P M_{\bar{u}}$.

For $a \in L^\infty(\mathbb{T})$, the *truncated Toeplitz operator* (TTO for short) generated by a is the operator $T_u(a) := P_u a I|_{K_u^2}$ acting from K_u^2 to K_u^2 . Truncated Toeplitz operators share many of their properties with their relatives, the Toeplitz operators on H^2 , to which we here sometimes refer as *classical* Toeplitz operators, but there are also some striking differences. For example, the function a is in general not uniquely determined by the operator $T_u(a)$ it generates, and the truncated shift S_u , i.e., the TTO with generating function $a(z) = z$, is the sum of a unitary and a compact operator (hence a Fredholm operator with index 0), whereas its classical counterpart S is a proper isometry (and a Fredholm operator of index -1). Moreover, whereas the spectrum of the classical shift S is the closed unit disk $\mathbb{D}$, the spectrum of the truncated shift S_u coincides with the so-called *spectrum*

$$\sigma(u) := \{\lambda \in \mathbb{D} : \liminf_{z \rightarrow \lambda} |u(z)| = 0\}$$

of the inner function u [3][Lemma 2.5].

The following collection of positive results is taken from and proved in [8]. We denote by $T_u(C)$ the smallest closed C^* -subalgebra of $L(K_u^2)$ which contains the truncated shift S_u and the identity operator (the notation will be justified by assertion (d) in the theorem below). Further, we write $\text{Comm } \mathcal{A}$ for the commutator ideal of

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a C^* -algebra $\mathcal{A}$, i.e., for the smallest closed ideal of $\mathcal{A}$ which contains all commutators $ab - ba$ with $a, b \in \mathcal{A}$. The essential spectrum of an operator A is denoted by $\sigma_{\text{ess}}(A)$, and its essential norm by $\|A\|_{\text{ess}}$.

Theorem 1. *Let u be a non-constant inner function. Then*

- (a) *for $a, b \in C(\mathbb{T})$, $T_u(a)T_u(b) - T_u(ab)$ is compact.*
- (b) *$\text{Comm}(T_u(C)) = K(K_u^2)$.*
- (c) *$T_u(C)/K(K_u^2)$ is * -isomorphic to $C(\sigma(u) \cap \mathbb{T})$.*
- (d) *for $a \in C(\mathbb{T})$, the TTO $T_u(a)$ is compact if and only if $a(\sigma(u) \cap \mathbb{T}) = \{0\}$.*
- (e) *$T_u(C) = \{T_u(a) + K : a \in C(\mathbb{T}), K \in K(K_u^2)\}$.*
- (f) *for $a \in C(\mathbb{T})$, $\sigma_{\text{ess}}(T_u(a)) = a(\sigma_{\text{ess}}(S_u))$.*
- (g) *for $a \in C(\mathbb{T})$, $\|T_u(a)\|_{\text{ess}} = \sup\{|a(t)| : t \in \sigma(u) \cap \mathbb{T}\}$.*
- (h) *Every operator in $T_u(C)$ is the sum of a normal and a compact operator.*

Moreover,

$$\{0\} \longrightarrow K(K_u^2) \xrightarrow{\text{id}} T_u(C) \xrightarrow{\pi} C(\sigma(u) \cap \mathbb{T}) \longrightarrow \{0\}$$

is a short exact sequence, with the mapping π given by $T_u(a) + K \mapsto a|_{\sigma(u) \cap \mathbb{T}}$.

Here is an outline of the contents of the paper. In Section 2 we will single out a sequence (P_{u_n}) of finite rank projections which converge strongly to the identity operator on K_u^2 . The operator $P_{u_n}T_u(a)P_{u_n}$ is considered as a finite section of the truncated Toeplitz operator $T_u(a)$. In Section 4, Theorem 4, we describe the C^* -algebra $\mathcal{S}(T_u(C))$ generated by all sequences of the form $(P_{u_n}T_u(a)P_{u_n})_{n \geq 1}$ with a a continuous function. This description is based of a formula of Widom-type that we will derive in Section 3. As consequences of Theorem 4, we get a stability criterion and results on spectral approximation. The stability criterion (Theorem 5) says that a sequence (A_n) in $\mathcal{S}(T_u(C))$ is stable if and only if its strong limit A is invertible. In the case when the A_n are the finite sections of a truncated Toeplitz operator, this result is due to [4]. One advantage of Theorem 5 is that it implies (without any additional effort) results on spectral and pseudospectral approximation as well as on the asymptotic behavior of the small singular values of A_n ; see the end of Section 4.

2 A filtration and Widom's identity

Recall that a *filtration* on a Hilbert space H is a sequence $\mathcal{P} = (P_n)$ of orthogonal projections of finite rank on H which converges strongly to the identity operator on H . To define a filtration on the model space K_u^2 we specify u to be a Blaschke product, as follows. A single *Blaschke factor* is a function on the unit disk of the form

$$b_\lambda(z) := \begin{cases} z & \text{if } \lambda = 0, \\ \frac{\lambda - z}{1 - \bar{\lambda}z} \frac{|\lambda|}{\lambda} & \text{if } \lambda \in \mathbb{D} \setminus \{0\}. \end{cases} \quad (2.1)$$

A *Blaschke product* is then a function

$$u = \prod_{\lambda \in \mathbb{D} : k(\lambda) > 0} b_\lambda^{k(\lambda)} \quad (2.2)$$

which satisfies the *Blaschke condition*

$$\sum_{\lambda \in \mathbb{D}} k(\lambda)(1 - |\lambda|) < \infty. \quad (2.3)$$

If u is a finite Blaschke product, i.e., if u is of the form (2.2) with $k(\lambda) = 0$ for all but finitely many $\lambda \in \mathbb{D}$, then (2.3) is satisfied. Conversely, if (2.3) holds, then every disk $\{z \in \mathbb{D} : |z| \leq r\}$ with $0 < r < 1$ contains only finitely many λ with $k(\lambda) \neq 0$. Thus, if u in (2.2) is an (infinite) Blaschke product, the number of its non-one factors is countable. We order the λ with $k(\lambda) \neq 0$ in a sequence $(\lambda_k)_{k \geq 1}$ in such a way that $|\lambda_k| \leq |\lambda_{k+1}|$ for all k . Then (2.2) and (2.3) can be written as

$$u = \prod_{k=1}^{\infty} b_{\lambda_k} \quad \text{with} \quad \sum_{k=1}^{\infty} (1 - |\lambda_k|) < \infty. \quad (2.4)$$

Blaschke products are inner functions. If u is an infinite Blaschke product, we use its product representation (2.4) to define a filtration on the associated model space K_u^2 . For $n \geq 1$, set $u_n := \prod_{k=1}^n b_{\lambda_k}$ and let P_{u_n} denote the orthogonal projection from $L^2(\mathbb{T})$ onto $K_{u_n}^2$. The projections P_{u_n} own the following properties.

Proposition 1. (a) *The projections P_{u_n} have a finite spatial rank.*

(b) *$P_{u_n} \rightarrow P_u$ on $L^2(\mathbb{T})$ strongly as $n \rightarrow \infty$.*

(c) *$P_{u_m} P_{u_n} = P_{u_n} P_{u_m} = P_{u_{\min\{m, n\}}}$ for $m, n \geq 1$.*

(d) *$P_{u_n} P_u = P_u P_{u_n} = P_{u_n}$ and $P_{u_n} P = P P_{u_n} = P_{u_n}$ for $n \geq 1$.*

Hints to the proof. Assertion (a) is the "Lemma on Finite Dimensional Subspaces" on page 33 in [7]. For assertion (b) observe that the u_n converge to u uniformly on compact subsets of $\mathbb{D}$ by the "Lemma on Blaschke Products" on page 280 in [7]. Thus, the model spaces $K_{u_n}^2$ converge to K_u^2 by the "Theorem on Lower Limits" on page 34 in [7] (note that the limit of the u_n exists; so the set of its limit points contains only one element, which clearly coincides with the greatest common divisor in the formulation of that theorem). Thus, $P_{u_n} \rightarrow P_u$ on $L^2(\mathbb{T})$ strongly by the definition on page 34 in [7].

Finally, u_n is a divisor of u and of u_m for $m \geq n$. Corollary 8 on page 19 in [7] then implies that $K_{u_n}^2 \subseteq K_{u_m}^2 \subseteq K_u^2 \subseteq H^2$, whence assertions (c) and (d). $\square$

Thus, the restrictions of the projections P_{u_n} to the model space K_u^2 form a filtration on K_u^2 by the preceding proposition. We denote this filtration by $\mathcal{P}_u$ and write $\mathcal{P}_u = (P_{u_n})_{n \geq 1}$, not distinguishing between a projection P_{u_n} on $L^2(\mathbb{T})$ and its restriction to K_u^2 .

The study of the finite sections discretization (FSD for short) for (classical) Toeplitz operators is dominated by Widom's identity

$$P_n T(ab) P_n = P_n T(a) P_n T(b) P_n + P_n H(a) H(\tilde{b}) P_n + R_n H(\tilde{a}) H(b) R_n. \quad (2.5)$$

To explain this identity, we need some notation. Let $J : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ denote the operator $(Jf)(t) = t^{-1}f(t^{-1})$. One easily checks that, for every function $a \in L^\infty(\mathbb{T})$, JaJ is the operator of multiplication by the function $\tilde{a}(t) := a(t^{-1})$. Then $H(a) := PaJ|_{H^2}$ is the (classical) Hankel operator, $R_n := H(t^n)$, and $P_n := R_n^2$. Note that P_n is just the orthogonal projection onto the linear span of the functions t^n with $n \in \{0, 1, \dots, n-1\}$ and that $R_n = R_n^*$.

Our goal is to achieve a comparable identity for the finite sections of TTO, where the role of the R_n in Widom's identity is played by the operators

$$R_{u_n} := P_{u_n} M_{u_n} J \quad \text{and} \quad R_{u_n}^* = J M_{\overline{u_n}} P_{u_n}. \quad (2.6)$$

Theorem 2 (Widom's identity for TTO). *Let $a, b \in L^\infty(\mathbb{T})$. Then*

$$P_{u_n} T_u(ab) P_{u_n} = P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} + P_{u_n} H(a) H(\tilde{b}) P_{u_n} + R_{u_n} H(\tilde{a}) H(b) R_{u_n}^*.$$

Proof. By Proposition 1 (d) and since $P_{u_n} = P - M_{u_n} P M_{\bar{u}_n}$, we find

$$\begin{aligned} P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} &= P_{u_n} P a P_{u_n} b P P_{u_n} \\ &= P_{u_n} P a P b P P_{u_n} - P_{u_n} a M_{u_n} P M_{\bar{u}_n} b P_{u_n} \\ &= P_{u_n} P a b P P_{u_n} - P_{u_n} P a Q b P P_{u_n} - P_{u_n} M_{u_n} a P b M_{\bar{u}_n} P_{u_n} \quad \text{with } Q := I - P \\ &= P_{u_n} a b P_{u_n} - P_{u_n} P a Q J^2 Q b P P_{u_n} - P_{u_n} M_{u_n} P a P b M_{\bar{u}_n} P_{u_n} - P_{u_n} M_{u_n} Q a P b M_{\bar{u}_n} P_{u_n} \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} Q a P b M_{\bar{u}_n} P_{u_n}. \end{aligned}$$

In the last line we used that $P_{u_n} M_{u_n} P f \in P_{u_n} (u_n H^2) = 0$ for $f \in L^2(\mathbb{T})$, whence $P_{u_n} M_{u_n} P = 0$. Then $P M_{\bar{u}_n} P_{u_n} = (P_{u_n} M_{u_n} P)^* = 0$, too, and we can proceed with

$$\begin{aligned} P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} Q a P b P M_{\bar{u}_n} P_{u_n} - P_{u_n} M_{u_n} Q a P b Q M_{\bar{u}_n} P_{u_n} \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} Q a P b Q M_{\bar{u}_n} P_{u_n} \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} J^2 Q a P b Q J^2 M_{\bar{u}_n} P_{u_n} \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - R_{u_n} J Q a P b Q J R_{u_n}^*. \end{aligned}$$

Since $J Q a P b Q J = H(\tilde{a}) H(b)$, this is the assertion. $\square$

3 Hankel operators by Blaschke products

The operators R_n in Widom's identity (2.5) can be identified with the (classical) Hankel operators $H(t^n)$. Similarly,

$$\begin{aligned} R_{u_n} &= P_{u_n} M_{u_n} J \\ &= (P - M_{u_n} P M_{\bar{u}_n}) M_{u_n} J \\ &= (P M_{u_n} - M_{u_n} P) J \\ &= (P M_{u_n} - P M_{u_n} P) J \quad (\text{since } M_{u_n} P f \in H^2 \text{ for } f \in H^2) \\ &= P M_{u_n} Q J, \end{aligned}$$

which can be identified with the (classical) Hankel operator $H(M_{u_n})$ on H^2 . Analogously, $R_{u_n}^*$ can be identified with $H(M_{u_n})^* = H(\overline{M_{u_n}})$. We will see that the operators R_{u_n} in Widom's identity for TTO play a quite different role compared with the R_n in (2.5). We start with some general properties of Hankel operators generated by inner functions.

Proposition 2. *Let $u \in H^\infty$ and $|u(t)| = 1$ for $t \in \mathbb{T}$. Then the Hankel operator $H(u) = P u Q J$ is a partial isometry, the range and initial projection of which are given by $H(u) H(u)^* = P - u P \bar{u} I$ and $H(u)^* H(u) = P - \bar{u} P u I$.*

Proof. Since $u \bar{u} = 1$ and $P u P = u P$,

$$H(u) H(u)^* = P u Q J J Q \bar{u} P = P u Q \bar{u} P = P u \bar{u} P - P u P \bar{u} P = P - u P \bar{u} I. \quad (3.1)$$

Using this identity and $P \bar{u} Q = 0$ we obtain

$$H(u) H(u)^* H(u) = P u Q J - u P \bar{u} P u Q J = P u Q J - u P \bar{u} u Q J + u P \bar{u} Q P u Q J = P u Q J = H(u),$$

i.e., $H(u)$ is an isometry. Replacing u in (3.1) by $\bar{u}$ (which is also in H^∞) and taking into account that $H(\bar{u}) = H(u)^*$, the identity for $H(u)^* H(u)$ follows. $\square$

Corollary 1. $R_{u_n} R_{u_n}^* = P_{u_n}$, $R_{u_n}^* R_{u_n} = P_{\bar{u}_n}$, $P_{u_n} R_{u_n} = R_{u_n}$, $R_{u_n}^* P_{u_n} = R_{u_n}^*$.

The following convergence result for the R_{u_n} is in sharp contrast with the R_n in Widom's identity (2.5), which converge weakly to zero.

Theorem 3. $R_{u_n} = H(M_{u_n}) \rightarrow H(M_u)^*$ -strongly.

In the proof of this result, we will make use of the following well known assertion.

Lemma 1. Let A_n, A be bounded linear operators on a Hilbert space H . If $A_n \rightarrow A$ weakly and $\|A_n x\| \rightarrow \|Ax\|$ for all $x \in H$, then $A_n \rightarrow A$ strongly.

Proof. It is clearly sufficient to prove the following fact for elements x_n, x of H : if $x_n \rightarrow x$ weakly and $\|x_n\| \rightarrow \|x\|$, then $\|x_n - x\| \rightarrow 0$. This follows from

$$\begin{aligned} \|x_n - x\|^2 &= \langle x_n - x, x_n - x \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\ &= \|x_n\|^2 + \|x\|^2 - \langle x_n, x \rangle - \langle x, x_n \rangle \end{aligned}$$

which goes to 0 by hypothesis. $\square$

Proof of Theorem 3. We first show that $R_{u_n} = H(M_{u_n}) \rightarrow H(M_u)$ weakly. Indeed, the uniform convergence of u_n to u on compact subsets of $\mathbb{D}$ implies the convergence of the k th Taylor coefficient of u_n to the k th Taylor coefficient of u for every $k \in \mathbb{Z}^+$. Together with the uniform boundedness of the operators H_{u_n} , this fact implies the weak convergence of $H(M_{u_n})$ to $H(M_u)$.

Next we show that $\|H(M_{u_n})x\| \rightarrow \|H(M_u)x\|$ for every $x \in H^2$. Once this is done, the strong convergence of $H(M_{u_n})$ to $H(M_u)$ follows from Lemma 1.

We start with showing that

$$P_{\bar{u}_n} \rightarrow P_{\bar{u}} \quad \text{strongly as } n \rightarrow \infty. \quad (3.2)$$

Indeed, let b_λ be a single Blaschke factor as in (2.1). For $t \in \mathbb{T}$ and $\lambda \neq 0$ we then have

$$\overline{b_\lambda}(t) = \frac{\overline{\lambda - t^{-1}}}{1 - \bar{\lambda}t^{-1}} \frac{|\lambda|}{\bar{\lambda}} = \frac{\bar{\lambda} - t}{1 - \bar{\lambda}t} \frac{|\lambda|}{\bar{\lambda}} = b_{\bar{\lambda}}(t).$$

For $\lambda = 0$, the equality $\overline{b_\lambda} = b_{\bar{\lambda}}$ on $\mathbb{T}$ is evident. Moreover, if (λ_n) is a sequence in $\mathbb{D}$ satisfying the Blaschke condition, then the sequence $(\bar{\lambda}_n)$ also satisfies this condition. So we can apply the assertion of Proposition 1 (b) to the functions

$\prod_{k=1}^\infty \overline{b_{\lambda_k}}$ and $\prod_{k=1}^n \overline{b_{\lambda_k}}$ in place of u and u_n to get the assertion (3.2). From (3.2) and Corollary 1 we then conclude that

$$H(M_{u_n})^* H(M_{u_n}) \rightarrow H(M_u)^* H(M_u) \quad \text{strongly,}$$

from which we obtain

$$\langle H(M_{u_n})^* H(M_{u_n})x, x \rangle - \langle H(M_u)^* H(M_u)x, x \rangle = \|H(M_{u_n})x\|^2 - \|H(M_u)x\|^2 \rightarrow 0$$

for every $x \in H^2$. This proves the strong convergence of $R_{u_n} = H(M_{u_n})$ to $H(M_u)$. The strong convergence of the adjoint operators follows as above, by working with the Blaschke product $\bar{u}$ in place of u . $\square$

Corollary 2. Let L be a compact operator on H^2 . Then

$$\|R_{u_n} L R_{u_n}^* - P_{u_n} H(M_u) L H(M_u)^* P_{u_n}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Indeed, since L and $H(M_u) L H(M_u)^*$ are compact, we derive from Proposition 1 (b) and Theorem 3 that both sequences $(R_{u_n} L R_{u_n}^*)$ and $(P_{u_n} H(M_u) L H(M_u)^* P_{u_n})$ converge to $H(M_u) L H(M_u)^*$ in the norm. $\square$

4 The algebra of the FSD for TTO

With every filtration $\mathcal{P} = (P_n)$ on a Hilbert space H , there are naturally associated some algebraic objects. By $\mathcal{F}^{\mathcal{P}}$ we denote the set of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \text{im } P_n \rightarrow \text{im } P_n$ for which the sequence $(A_n P_n)$ converges * -strongly to some operator $W(\mathbf{A})$ on H . Provided with element-wise defined operations and the supremum norm, $\mathcal{F}^{\mathcal{P}}$ becomes a C^* -algebra, the set $\mathcal{G}^{\mathcal{P}}$ of all sequences in $\mathcal{F}^{\mathcal{P}}$ which converge to 0 in the norm is a closed ideal of $\mathcal{F}^{\mathcal{P}}$, and the mapping W , also called the consistency map of the filtration $\mathcal{P}$, is a * -homomorphism from $\mathcal{F}^{\mathcal{P}}$ to the algebra $L(H)$ of the bounded linear operators on H .

We prepare the proof of Theorem 4 below by an assertion of independent interest.

Proposition 3. *Let $\mathcal{P} = (P_n)$ be a filtration on a Hilbert space H . Then the ideal $\mathcal{G}^{\mathcal{P}}$ is contained in the smallest closed subalgebra $\mathcal{J}$ of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_n K P_n)$ with K compact if and only if $\mathcal{P}$ is injective.*

Proof. The "only if"-part of the assertion is evident. For the "if"-part we are going to show that, for each $n_0 \in \mathbb{N}$, there is a sequence (G_n) in $\mathcal{J}$ with $G_{n_0} \neq 0$ and $G_n = 0$ for all $n \neq n_0$. Since the matrix algebras $\mathbb{C}^{k \times k}$ are simple, this fact already implies that each sequence (G_n) with arbitrarily prescribed $G_{n_0} \in L(\text{im } P_{n_0})$ and $G_n = 0$ for $n \neq n_0$ belongs to $\mathcal{J}$. Since $\mathcal{G}^{\mathcal{P}}$ is generated by sequences of this special form, the assertion follows. For $n_0 \in \mathbb{N}$, put

$$\mathbb{N}_< := \{n \in \mathbb{N} : \text{im } P_n \cap \text{im } P_{n_0} \text{ is a proper subspace of } \text{im } P_{n_0}\},$$

and set $\mathbb{N}_> := \mathbb{N} \setminus (\{n_0\} \cup \mathbb{N}_<)$. The set $\mathbb{N}_<$ is at most countable, and none of the closed linear spaces $\text{im } P_n \cap \text{im } P_{n_0}$ has interior points relative to $\text{im } P_{n_0}$. By the Baire category theorem, $\bigcup_{n \in \mathbb{N}_<} (\text{im } P_n \cap \text{im } P_{n_0})$ is a proper subset of $\text{im } P_{n_0}$. Choose a unit vector

$$f \in \text{im } P_{n_0} \setminus \bigcup_{n \in \mathbb{N}_<} (\text{im } P_n \cap \text{im } P_{n_0}).$$

Then $\|P_n f\| < 1$ for all $n \in \mathbb{N}_<$ by the Pythagoras theorem. (Indeed, otherwise $\|P_n f\| = 1$, and the equality $1 = \|f\|^2 = \|P_n f\|^2 + \|f - P_n f\|^2$ implies $f = P_n f$, whence $f \in \text{im } P_n$.)

Let $Q_n := I - P_n$. If $n \in \mathbb{N}_>$, then $\text{im } P_n \cap \text{im } P_{n_0} = \text{im } P_{n_0}$ by the definition of $\mathbb{N}_>$. Thus, $\text{im } P_{n_0} \subseteq \text{im } P_n$, and since no two of the projections P_n coincide, this implies that $\text{im } P_{n_0}$ is a proper subspace of $\text{im } P_n$ and $\text{im } Q_n$ is a proper subspace of $\text{im } Q_{n_0}$ for $n \in \mathbb{N}_>$. Again by the Baire category theorem, $\bigcup_{n \in \mathbb{N}_>} \text{im } Q_n$ is a proper subset of $\text{im } Q_{n_0}$. Choose a unit vector

$$g \in \text{im } Q_{n_0} \setminus \bigcup_{n \in \mathbb{N}_>} \text{im } Q_n.$$

As above, $\|Q_n g\| < 1$ for all $n \in \mathbb{N}_>$. Consider the operator $K : x \mapsto \langle x, g \rangle f$ on H . Its adjoint is the operator $x \mapsto \langle x, f \rangle g$, and

$$P_n K Q_n K^* P_n x = \langle P_n x, f \rangle \langle Q_n g, g \rangle P_n f = \langle x, P_n f \rangle \|Q_n g\|^2 P_n f.$$

If $n \in \mathbb{N}_<$ then $\|P_n f\| < 1$, and if $n \in \mathbb{N}_>$ then $\|Q_n g\| < 1$ by construction. In both cases, $\|P_n K Q_n K^* P_n\| < 1$. In case $n = n_0$, the operator $P_n K Q_n K^* P_n x = \langle x, f \rangle f$ is an orthogonal projection of norm 1, which we call P . The sequence $\mathbf{K} := (P_n K Q_n K^* P_n)$ belongs to the algebra $\mathcal{J}$ since

$$(P_n K Q_n K^* P_n) = (P_n K K^* P_n) - (P_n K P_n) (P_n K^* P_n).$$

As $r \rightarrow \infty$, the powers $\mathbf{K}^r$ converge in the norm of $\mathcal{F}^{\mathcal{P}}$ to the sequence (G_n) with $G_{n_0} = P \neq 0$ and $G_n = 0$ if $n \neq n_0$. Indeed, since $P_n \rightarrow I$ strongly, one has $\|Q_n g\| < 1/2$ for n large enough, whence $\|P_n K Q_n K^* P_n\| < 1/2$ for these n , whereas $\|P_n K Q_n K^* P_n\| < 1$ for the remaining (finitely many) n as seen above. Since $\mathbf{K}^r \in \mathcal{J}$ and $\mathcal{J}$ is closed, the sequence (G_n) has the claimed properties. $\square$

The goal in this section is to study the FSD of TTO with respect to the filtration $\mathcal{P}_u$. In accordance with Theorem 1 (e), we define the corresponding (full) algebra of the FSD as the smallest closed subalgebra $\mathcal{S}(\mathcal{T}_u(C))$ of $\mathcal{F}^{\mathcal{P}_u}$ which contains all sequences $(P_{u_n}(T_u(a) + K)P_{u_n})_{n \geq 1}$ with $a \in C(\mathbb{T})$ and $K \in K(K_u^2)$.

Theorem 4. $\mathcal{S}(\mathcal{T}_u(C))$ consists of all sequences $(P_{u_n}(T_u(a) + K)P_{u_n} + G_n)$ with $a \in C(\mathbb{T})$, $K \in K(K_u^2)$ and $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$.

Proof. The proof runs parallel to that of Theorem 1.53 in [6]; so we address to some main steps only.

For a moment, let $\mathcal{S}_1$ denote the set of all sequences of the mentioned form. The sequences $(P_{u_n}(T_u(a) + K)P_{u_n})$ are contained in $\mathcal{S}(T_u(C))$ by definition, and since the filtration $\mathcal{P}_u$ is injective, we conclude from Proposition 3 that the ideal $\mathcal{G}^{\mathcal{P}_u}$ of the zero sequences is also contained in $\mathcal{S}(T_u(C))$. Thus, $\mathcal{S}_1 \subseteq \mathcal{S}(T_u(C))$.

For the reverse inclusion, we prove that $\mathcal{S}_1$ is a closed subalgebra of $\mathcal{S}(T_u(C))$. If $a, b \in C(\mathbb{T})$ then, by Theorem 2 (Widom's identity) and Corollary 2,

$$\begin{aligned} P_{u_n} T_u(a) P_{u_n} \cdot P_{u_n} T_u(b) P_{u_n} &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - R_{u_n} H(\tilde{a}) H(b) R_{u_n}^* \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} (H(a) H(\tilde{b}) + H(M_u) H(\tilde{a}) H(b) H(M_u)^*) P_{u_n} + G_n \\ &= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} K P_{u_n} + G_n \end{aligned}$$

with a compact operator K and a sequence $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$. Thus,

$$(P_{u_n} T_u(a) P_{u_n}) (P_{u_n} T_u(b) P_{u_n}) \in \mathcal{S}_1.$$

It follows now in a standard way that $\mathcal{S}_1$ is an algebra. To prove that $\mathcal{S}_1$ is closed, let $((P_{u_n}(T_u(a_m) + K_m)P_{u_n} + G_n^m)_{n \geq 1})_{m \geq 1}$ be a sequence in $\mathcal{S}_1$ which converges in the norm of $\mathcal{F}^{\mathcal{P}_u}$. Let W denote the consistency map of the filtration $\mathcal{P}_u$, i.e., $W((A_n)) = s\text{-}\lim A_n P_{u_n}$. Then

$$(W((P_{u_n}(T_u(a_m) + K_m)P_{u_n} + G_n^m)_{n \geq 1}))_{m \geq 1} = (T_u(a_m) + K_m)_{m \geq 1}$$

is a Cauchy sequence in $T_u(C)$. Since $T_u(C)$ is a closed algebra, this sequence converges in $T_u(C)$. The limit of this sequence is of the form $T_u(a) + K$ with $a \in C(\mathbb{T})$ and K compact by Theorem 1 (e). (But note that the representation of the limit in that form is not unique by Theorem 1 (d).) It is now easy to see that

$$(P_{u_n}(T_u(a_m) + K_m)P_{u_n})_{n \geq 1} \rightarrow (P_{u_n}(T_u(a) + K)P_{u_n})_{n \geq 1}$$

in the norm of $\mathcal{F}^{\mathcal{P}_u}$ as $m \rightarrow \infty$. Then, finally, the sequence $((G_n^m)_{n \geq 1})_{m \geq 1}$ converges; its limit is in $\mathcal{G}^{\mathcal{P}_u}$. $\square$

Theorem 5. A sequence $\mathbf{A} = (A_n) \in \mathcal{S}(T_u(C))$ is stable if and only if the operator $W(\mathbf{A}) = s\text{-}\lim A_n P_{u_n}$ is invertible.

Proof. By Theorem 4, we have to show that the sequence $\mathbf{A} := (P_{u_n}(T_u(a) + K)P_{u_n})$ (with $a \in C(\mathbb{T})$ and $K \in K(K_u^2)$) is stable if and only if the operator $A := T_u(a) + K$ is invertible. Since the stability of $\mathbf{A}$ implies the invertibility of A by Polski's theorem, we are left with the reverse implication.

So let A be invertible. By inverse closedness of C^* -algebras, $A^{-1} \in T_u(C)$; hence, $A^{-1} = T_u(b) + L$ with a certain function $b \in C(\mathbb{T})$ and a compact operator L by Theorem 1 (e). Using Widom's identity as in the proof of the previous theorem and employing assertion (a) of Theorem 1, we conclude that there are compact operators R_1, R_2 and a sequence $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$ such that

$$\begin{aligned} P_{u_n} T_u(a) P_{u_n} \cdot P_{u_n} T_u(b) P_{u_n} - P_{u_n} &= P_{u_n} (T_u(ab) - I) P_{u_n} - P_{u_n} R_1 P_{u_n} + G_n \\ &= P_{u_n} R_2 P_{u_n} - P_{u_n} R_1 P_{u_n} + G_n. \end{aligned}$$

Thus, the sequence $(P_{u_n} T_u(b) P_{u_n})$ is a right inverse of the sequence $(P_{u_n} T_u(a) P_{u_n})$ modulo the ideal

$$\mathcal{J} := \{(P_{u_n} K P_{u_n} + G_n) : K \in K(K_u^2), (G_n) \in \mathcal{G}^{\mathcal{P}_u}\}$$

of $\mathcal{S}(T_u(C))$. A similar computation shows that it is a left inverse modulo $\mathcal{J}$, too. Then $(P_{u_n} T_u(b) P_{u_n})$ is also an inverse of $(P_{u_n} A P_{u_n})$ modulo $\mathcal{J}$. Now the assertion follows from the Lifting Theorem 5.37 in [6] in its simplest form, i.e., with $\mathcal{J}/\mathcal{G}^{\mathcal{P}_u}$ consisting of one elementary ideal only. $\square$

The following is certainly the most important consequence of Theorem 5. For the definition of a fractal algebra see Section 1.6 in [6].

Corollary 3. *The algebra $\mathcal{S}(\mathcal{T}_u(C))$ is fractal.*

Note that the consistency map of $\mathcal{P}_u$ is fractal; so the assertion of the corollary follows from Theorems 1.69 in [6] and 5. $\square$

Sequences in fractal algebras are distinguished by their excellent convergence properties. To mention only a few of them, let $\sigma(a)$ denote the spectrum of an element a of a C^* -algebra with identity element e , write $\sigma_2(a)$ for the set of the singular values of a , i.e., $\sigma_2(a)$ is the set of all non-negative square roots of elements in the spectrum of a^*a and finally, for $\varepsilon > 0$, let $\sigma^{(\varepsilon)}(a)$ refer to the ε -pseudospectrum of a , i.e. to the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not invertible or $\|(a - \lambda e)^{-1}\| \geq 1/\varepsilon$. Let further

$$d_H(M, N) := \max \left\{ \max_{m \in M} \min_{n \in N} |m - n|, \max_{n \in N} \min_{m \in M} |m - n| \right\}$$

denote the Hausdorff distance between the non-empty compact subsets M and N of the complex plane.

Theorem 6. *Let (A_n) be a sequence in $\mathcal{S}(\mathcal{T}_u(C))$ with strong limit A . Then the following set-sequences converge with respect to the Hausdorff distance as $n \rightarrow \infty$:*

(a) $\sigma(A_n) \rightarrow \sigma(A)$ if (A_n) is self-adjoint;

(b) $\sigma_2(A_n) \rightarrow \sigma_2(A)$;

(c) $\sigma^{(\varepsilon)}(A_n) \rightarrow \sigma^{(\varepsilon)}(A)$.

The proof follows immediately from the stability criterion in Theorem 5 and from Theorems 3.20, 3.23 and 3.33 in [6]. Note that in general one cannot remove the assumption $(A_n) = (A_n)^*$ in assertion (a), whereas (c) holds without this assumption. $\square$

The notion of a Fredholm sequence was introduced in [2]; see also Chapter 6 of [6]. In the present setting, the Fredholm property of a sequence $(A_n) \in \mathcal{S}(\mathcal{T}_u(C))$ means nothing but the invertibility of the coset $(A_n) + \mathcal{J}$ in the quotient algebra $\mathcal{S}(\mathcal{T}_u(C))/\mathcal{J}$, and the results of [2] specify as follows. Let $\sigma_1(a) \leq \dots \leq \sigma_n(A) = \|A\|$ denote the singular values of the $n \times n$ -matrix A .

Theorem 7. *Let (A_n) be a sequence in $\mathcal{S}(\mathcal{T}_u(C))$ with strong limit A . Then*

(a) (A_n) is a Fredholm sequence if and only if A is a Fredholm operator.

(b) If A is a Fredholm operator and $\dim \ker A = k$, then

$$\lim_{n \rightarrow \infty} \sigma_k(A_n) = 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \sigma_{k+1}(A_n) > 0.$$

Assertion (b) allows the numerical determination of the kernel dimension of a Fredholm operator $A \in \mathcal{T}_u(C)$.

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