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MICROSCOPIC AND STRONGLY MICROSCOPIC SETS ON THE PLANE. FUBINI THEOREM AND FUBINI PROPERTY

Abstract. In this paper, we introduce the notions of microscopic and strongly microscopic sets on the plane and obtain a result analogous to Fubini Theorem.

In measure theory or in functional analysis, it is often proved that some property holds "almost everywhere", i.e. except on some set of Lebesgue measure zero, or "nearly everywhere", that is except on some set of the first Baire category. Both these families, sets of Lebesgue measure zero and sets of the first category (on the real line, or generally in $\mathbb{R}^n$), form $\sigma$-ideals, and moreover they are orthogonal to each other: there exist sets $A$ and $B$ such that $\mathbb{R} = A \cup B$, where $A$ is a set of the first category and $B$ is a nullset. Similarities and differences between these families are the main theme of the monograph "Measure and Category" of J. C. Oxtoby ([10]). The last part of these investigations is concerned with the Sierpiński–Erdös Duality Theorem. It leads to the Duality Principle, which allows us (assuming CH), in any proposition involving solely the notions of measure zero, first category, and notions of pure set theory, interchange the terms "nullset" and "set of the first category" whenever they appear. However, the extended principle, where the notions of measurability and the Baire property would be interchanged, is not true (see [10], Theorem 21.2 and Dual Statement).

Fubini Theorem presents a close connection between the measure of any plane measurable set and the linear measure of its sections perpendicular to an axis. In [10] one can find an elementary proof of the fact that if $E$ is a plane set of measure zero, then $E_x = \{y : (x, y) \in E\}$ is a linear nullset for all $x$ except a set of linear measure zero ([10], Theorem 14.2).

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Fubini Theorem has a category analogue. Kuratowski and Ulam, in 1932, proved (compare [9], p. 247, Corollary 1a) that if $E$ is a plane set of the first category, then $E_x$ is a linear set of first category for all $x$, except those belonging to a certain set of the first category (see also [10], Theorem 15.1).

In this note, we introduce the notions of microscopic and strongly microscopic sets on the plane. We study their properties and prove, among other facts, that these families are $\sigma$-ideals on the plane, situated between countable sets and sets of Lebesgue measure zero, and that these $\sigma$-ideals are orthogonal to the $\sigma$-ideal of sets of the first category on the plane.

One of the main theorems in this paper is a Fubini type result for microscopic sets (Theorem 16). We show also that our result implies the Kuratowski–Ulam Theorem.

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the real line, $\mathbb{R}^2$ the plane. Let $m$ and $m_2$ denote the Lebesgue measure on the real line and on the plane, respectively.

A set $A \subset \mathbb{R}$ is said to be microscopic if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of intervals such that $A \subset \bigcup_{n \in \mathbb{N}} I_n$ and $m(I_n) < \varepsilon^n$ for $n \in \mathbb{N}$. The notion of a microscopic set on the real line was introduced by J. Appell in 2001. The properties of these sets were investigated by J. Appell, E. D’Aniello and M. Väth in [1]. The authors proved there, among other facts, that the family of all microscopic sets on the real line is a $\sigma$-ideal situated between countable sets and sets of Lebesgue measure zero, which is essentially different from both these families. In fact, each microscopic set has Hausdorff dimension zero and there exists a microscopic set which is residual, so in particular uncountable. For example the set

\[ E = \bigcap_{i=1}^{\infty} \bigcup_{n=1}^{\infty} \left( r_n - \frac{1}{2^{i} n}, r_n + \frac{1}{2^{i} n} \right), \]

where $\{r_n\}_{n \in \mathbb{N}}$ is a sequence of all rational numbers, is microscopic and residual (compare [7], Lemma 2.2). On the other hand, the classical middle third Cantor set is an example of a non-microscopic set of Lebesgue measure zero. Using the notion of microscopic set, we proved among other facts (see [5], Corollary 12), that the set of all continuous nowhere monotone functions which are one-to-one on $[0, 1]$ except on some microscopic set, is residual in $C[0, 1]$. Clearly, this result is a generalization of the fact that the “typical” continuous function is one-to-one a.e. on $[0, 1]$ (see [2], Ex. 10:6.6, p. 471).

**Definition 1.** We shall say that $A \subset \mathbb{R}^2$ is a microscopic set if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of rectangles with sides, which are parallel to coordinate axes, such that $A \subset \bigcup_{n \in \mathbb{N}} I_n$ and $m_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$.
**Definition 2.** We shall say that $A \subset \mathbb{R}^2$ is a strongly microscopic set if for each $\varepsilon > 0$ there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of squares with sides, which are parallel to coordinate axes, such that $A \subset \bigcup_{n \in \mathbb{N}} I_n$ and $m_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$.

Denote by $\mathcal{M}_2$, the family of all microscopic sets in $\mathbb{R}^2$ and by $\mathcal{M}_{2s}$, the family of all strongly microscopic sets in $\mathbb{R}^2$. Obviously, each strongly microscopic set is microscopic, so $\mathcal{M}_{2s} \subset \mathcal{M}_2$.

In the sequel, the rectangle with sides which are parallel to coordinate axes will be called the interval.

**Theorem 3.** The families $\mathcal{M}_2$ and $\mathcal{M}_{2s}$ are the $\sigma$-ideals.

**Proof.** The proof of this fact is analogous to the proof in [1] that the family of microscopic sets on the real line is a $\sigma$-ideal, but we recall it for the convenience of the reader.

Obviously a subset of a microscopic set is also microscopic. Suppose that $A_k \in \mathcal{M}_2$ for $k \in \mathbb{N}$. Denote by $A$ the union of these sets. Given $\varepsilon \in (0, 1)$, put $\varepsilon_k = \varepsilon 2^k$ for $k \in \mathbb{N}$. Since each $A_k$ is microscopic, for any $k$ there exists a sequence $\{I_n^k\}_{n \in \mathbb{N}}$ of intervals such that

$$A_k \subset \bigcup_{n \in \mathbb{N}} I_n^k$$

and

$$m_2(I_n^k) < (\varepsilon k)^n$$

for each $n \in \mathbb{N}$. Define a map $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\psi(k, n) = 2^{k-1}(2n - 1)$. Clearly, $\psi$ is one-to-one function and every positive integer can be written in a unique way as product of an odd number and some power of 2, so $\psi$ is a bijection. Hence for each $m \in \mathbb{N}$, there exists a unique pair $(k, n) \in \mathbb{N} \times \mathbb{N}$ such that $\psi(k, n) = m$. Put

$$J_m = J_{\psi(k,n)} = I_n^k.$$  

Obviously $A \subset \bigcup_{m \in \mathbb{N}} J_m$. Moreover

$$m_2(J_m) = m_2(I_n^k) < (\varepsilon_k)^n = (\varepsilon 2^k)^n < \varepsilon^{2k-1}(2n-1) = \varepsilon_{\psi(k,n)} = \varepsilon^m$$

for each $m \in \mathbb{N}$. This shows that $A$ is a microscopic set. The proof for $\mathcal{M}_{2s}$ is analogous.  

**Theorem 4.** The following conditions are equivalent:

(i) $A$ is a microscopic set on the plane.

(ii) For each positive number $\eta$, there exists a sequence $\{J_n\}_{n \in \mathbb{N}}$ of intervals such that $A \subset \limsup_n J_n$ and $\sum_{k=n}^{\infty} m_2(J_k) < \eta^n$ for each $n \in \mathbb{N}$.

(iii) For each positive number $\delta$, there exists a sequence $\{I_n\}_{n \in \mathbb{N}}$ of intervals such that $A \subset \limsup_n I_n$ and $m_2(I_n) < \delta^n$ for each $n \in \mathbb{N}$.
Proof. The proof of this theorem is analogous to the proof of Theorem 2 in [3]. ■

The similar theorem holds also for strongly microscopic sets on the plane, if we replace intervals with squares with sides which are parallel to coordinate axes.

Let us denote by \( N_2 \) the family of all sets of Lebesgue measure zero on the plane and by \( P_2 \) the family of all countable subsets of the plane.

Obviously, each countable set is strongly microscopic and if \( A \) is microscopic, then \( A \) is of Lebesgue measure zero, so we have

\[
P_2 \subset M_2 \subset M_2 \subset N_2.
\]

Now we shall prove that all these inclusions are proper.

**Theorem 5.** The plane can be represented as the union of two disjoint sets \( A \) and \( B \) such that \( A \) is a set of the first category and \( B \) is a strongly microscopic set.

**Proof.** Let \( \{(x_i, y_i)\}_{i \in \mathbb{N}} \) be a sequence of all elements of the set \( \mathbb{Q} \times \mathbb{Q} \). For \( i, j \in \mathbb{N} \) put

\[
Q_{i,j} = \left( x_i - \frac{1}{2^{i+j+1}}, x_i + \frac{1}{2^{i+j+1}} \right) \times \left( y_i - \frac{1}{2^{i+j+1}}, y_i + \frac{1}{2^{i+j+1}} \right).
\]

Then \( m_2(Q_{i,j}) = \frac{1}{2^{i+j+1}} \cdot \frac{1}{2} < (\frac{1}{2})^i \) for every \( i, j \in \mathbb{N} \). Let

\[
G_j = \bigcup_{i \in \mathbb{N}} Q_{i,j},
\]

for \( j \in \mathbb{N} \). Clearly, \( \mathbb{Q} \times \mathbb{Q} \subset G_j \), so \( G_j \) is an open set dense in \( \mathbb{R}^2 \) for each \( j \in \mathbb{N} \). Consider the sets

\[
B = \bigcap_{j \in \mathbb{N}} G_j \quad \text{and} \quad A = \mathbb{R}^2 \setminus B.
\]

Obviously \( B \) is a residual (so also uncountable) set and \( A \) is of the first category. Moreover, \( B \) is a strongly microscopic set with cardinality continuum. Indeed, let \( \varepsilon > 0 \). There exists \( j_0 \in \mathbb{N} \) such that \( \frac{1}{2^{j_0}} < \varepsilon \). Obviously \( B \subset G_{j_0} = \bigcup_{i \in \mathbb{N}} Q_{i,j_0} \), where \( Q_{i,j_0} \) is the square such that \( m_2(Q_{i,j_0}) < (\frac{1}{2^{j_0}})^i < \varepsilon^i \) for each \( i \in \mathbb{N} \). ■

**Corollary 6.** There exists a strongly microscopic set \( B \subset \mathbb{R}^2 \), which is residual.

**Theorem 7.** There exists a set of plane measure zero, which is not microscopic.

**Proof.** Put \( A = \{(x, x) : x \in [0,1]\} \). Let \( \varepsilon > 0 \). Notice that if \( A \) is covered by the sequence \( \{I_n\}_{n \in \mathbb{N}} \) of intervals, with sides of lengths \( a_n \) and \( b_n \) such
that \(a_n \cdot b_n < \varepsilon^n\) for each \(n \in \mathbb{N}\), then \(A\) is covered by squares with sides of lengths \(\min\{a_n, b_n\}\). Hence, if \(A\) could be a microscopic set, it would be also a strongly microscopic set.

Suppose that \(A\) is a strongly microscopic set. Let \(\varepsilon = \frac{1}{16}\). There exists a sequence \(\{I_n\}_{n \in \mathbb{N}}\) of squares with sides of length \(a_n\), which are parallel to coordinate axes such that \(A \subset \bigcup_{n \in \mathbb{N}} I_n\) and \(m_2(I_n) < \left(\frac{1}{16}\right)^n\) for each \(n \in \mathbb{N}\). Hence \(a_n < \left(\frac{1}{4}\right)^n\) for \(n \in \mathbb{N}\). Since \(A \subset \bigcup_{n \in \mathbb{N}} I_n\), so
\[
1 \leq \sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{3},
\]
which gives a contradiction. So \(A\) is not a microscopic set on the plane. ■

**Theorem 8.** There exists a microscopic set \(A \subset \mathbb{R}^2\), which is not strongly microscopic.

**Proof.** Let \(A = [0, 1] \times \{0\}\) and \(\varepsilon > 0\). Putting \(I_1 = [0, 1] \times \left[-\frac{\varepsilon}{3}, \frac{\varepsilon}{3}\right]\), \(I_k = \emptyset\) for \(k > 1\), we have \(A \subset \bigcup_{n \in \mathbb{N}} I_n\) and \(m_2(I_n) < \varepsilon^n\) for each \(n \in \mathbb{N}\), so \(A\) is a microscopic set. By the same reasoning as in the last part of the proof of Theorem 7, we obtain that \(A\) can not be a strongly microscopic set. ■

Therefore we have
\[
\mathcal{P}_2 \subsetneq \mathcal{M}_{2s} \subsetneq \mathcal{M}_2 \subsetneq \mathcal{N}_2.
\]

Considering the sets from the proofs of Theorem 8 and Theorem 7, it is easy to see that the family \(\mathcal{M}_2\) is not invariant under rotation with respect to the origin. We shall prove that for the family \(\mathcal{M}_{2s}\), the situation is quite different.

**Theorem 9.** The set \(A \in \mathcal{M}_{2s}\) if and only if for each \(\varepsilon > 0\), there exists a sequence \(\{B_n\}_{n \in \mathbb{N}}\) of circles on the plane such that \(A \subset \bigcup_{n \in \mathbb{N}} B_n\) and \(m_2(B_n) < \varepsilon^n\) for each \(n \in \mathbb{N}\).

**Proof.** Let \(\varepsilon > 0\). If there exists a sequence \(\{Q_n\}_{n \in \mathbb{N}}\) of squares such that \(A \subset \bigcup_{n \in \mathbb{N}} Q_n\) and \(m_2(Q_n) < \left(\frac{\varepsilon}{n}\right)^n\) for \(n \in \mathbb{N}\) then there exists a sequence \(\{B_n\}_{n \in \mathbb{N}}\) of circles circumscribed on squares \(Q_n\), \(n \in \mathbb{N}\), respectively, such that \(A \subset \bigcup_{n \in \mathbb{N}} B_n\) and \(m_2(B_n) < \varepsilon^n\) for \(n \in \mathbb{N}\). Conversely, if \(\varepsilon > 0\) and there exists a sequence \(\{B_n\}_{n \in \mathbb{N}}\) of circles such that \(A \subset \bigcup_{n \in \mathbb{N}} B_n\) and \(m_2(B_n) < \left(\frac{\varepsilon}{n}\right)^n\) for \(n \in \mathbb{N}\) then there exists a sequence \(\{Q_n\}_{n \in \mathbb{N}}\) of squares circumscribed on circles \(B_n\), \(n \in \mathbb{N}\), respectively, such that \(A \subset \bigcup_{n \in \mathbb{N}} Q_n\) and \(m_2(Q_n) < \varepsilon^n\) for \(n \in \mathbb{N}\). ■

**Corollary 10.** The family \(\mathcal{M}_{2s}\) is invariant under rotation.

**Corollary 11.** If \(A \in \mathcal{M}_{2s}\) then the orthogonal projection of \(A\) onto any line is a microscopic set.
Proof. Let $\delta > 0$ and $\varepsilon \in (0, \frac{\delta^2}{4})$. From Theorem 9, there exists a sequence \(\{B_n\}_{n \in \mathbb{N}}\) of circles on the plane such that \(A \subset \bigcup_{n \in \mathbb{N}} B_n\) and \(m_2(B_n) < \varepsilon^n\) for each \(n \in \mathbb{N}\). Let \(P_V\) denote the orthogonal projection onto any line \(V\). Then
\[
m(P_V(B_n)) < \frac{2}{\sqrt{\pi}}(\sqrt{\varepsilon})^n < \frac{2}{2^n\sqrt{\pi}}\delta^n < \delta^n,
\]
for each \(n \in \mathbb{N}\). So \(P_V(A)\) is a microscopic set (on the line \(V\)). ■

If \(A \subset \mathbb{R}^2\) then \(\text{proj}_x A\) and \(\text{proj}_y A\) denote the orthogonal projection of \(A\) on \(Ox\)-axis and \(Oy\)-axis, respectively. From Corollary 11, it follows that if \(A \in \mathcal{M}_{2s}\) then both sets \(\text{proj}_x A\) and \(\text{proj}_y A\) are microscopic on the real line.

Now we shall prove that the inverse conclusion does not hold.

Observe that using argument analogous to the proof of Corollary 11, we obtain that if \(A \in \mathcal{M}_{2s}\) then the projection of \(A\) along the line \(y = x\) onto \(Ox\)-axis is a microscopic set on the real line. We shall denote this projection by \(\Phi(A)\).

**Theorem 12.** There exists a set \(A \subset \mathbb{R}^2\) such that \(\text{proj}_x A\) and \(\text{proj}_y A\) are microscopic sets on the real line and \(A\) is not strongly microscopic.

**Proof.** Let \(E\) be a microscopic set on the real line such that \(\mathbb{R}\setminus E\) is a set of the first category (see for example the set \(E\) defined by (1)). Put \(A = E \times E\).

Clearly, \(\text{proj}_x A = \text{proj}_y A = E\). Simultaneously,
\[
\Phi(E \times E) = E - E.
\]
Since \(E\) is residual, by theorem of S. Piccard (see [8], Theorem 2.9.1), \(E - E\) contains some interval. By (2), \(\Phi(E \times E)\) also contains some interval, so is not a microscopic set on the real line. From the above remark it follows that \(A \notin \mathcal{M}_{2s}\). ■

**Theorem 13.** If \(A \in \mathcal{M}_{2s}\) and \((\alpha, \beta) \in \mathbb{R}^2\) then

(a) \(A + (\alpha, \beta) = \{(x + \alpha, y + \beta) : (x, y) \in A\} \in \mathcal{M}_{2s}\),
(b) \(-A = \{(-x, -y) : (x, y) \in A\} \in \mathcal{M}_{2s}\),
(c) \((\alpha, \beta) \cdot A = \{(\alpha \cdot x, \beta \cdot y) : (x, y) \in A\} \in \mathcal{M}_{2s}\),
(d) if \(A \cap \{(x, y) : x \cdot y = 0\} = \emptyset\) then \(A^{-1} = \{(\frac{1}{x}, \frac{1}{y}) : (x, y) \in A\} \in \mathcal{M}_{2s}\).

**Proof.** The conditions (a) and (b) are obvious. Let \(A \in \mathcal{M}_{2s}\), \(\alpha, \beta \in \mathbb{R}\) and \(\varepsilon > 0\). To prove (c), we consider the following cases:

(i) \(|\alpha| \geq 1 \text{ and } \beta \neq 0\) or \(|\beta| \geq 1 \text{ and } \alpha \neq 0\).

Put \(\gamma = \max\{|\alpha|, |\beta|\}\). Since \(A \in \mathcal{M}_{2s}\), there exists a sequence of squares \(\{I_n\}_{n \in \mathbb{N}}\) with sides, which are parallel to coordinate axes, such that \(A \subset \bigcup_{n \in \mathbb{N}} I_n\) and \(m_2(I_n) < (\frac{\varepsilon}{\gamma})^n\) for each \(n \in \mathbb{N}\). Thus
$(\alpha, \beta) \cdot A \subset \bigcup_{n \in \mathbb{N}} ((\alpha, \beta) \cdot I_n) \subset \bigcup_{n \in \mathbb{N}} Q_n,$

where $Q_n$ is a square circumscribed on the interval $(\alpha, \beta) \cdot I_n$. Clearly $m_2(Q_n) = \gamma^2 m_2(I_n) \leq \varepsilon^n$ for each $n \in \mathbb{N}$, so $(\alpha, \beta) \cdot A \in \mathcal{M}_{2s}$.

(ii) $0 < |\alpha| < 1$ and $0 < |\beta| < 1$.

Put $\gamma = \max\{|\alpha|, |\beta|\}$. Since $A \in \mathcal{M}_{2s}$, there exists a sequence of squares $\{I_n\}_{n \in \mathbb{N}}$ such that $A \subset \bigcup_{n \in \mathbb{N}} I_n$ and $m_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$. Therefore $(\alpha, \beta) \cdot A \subset \bigcup_{n \in \mathbb{N}} ((\alpha, \beta) \cdot I_n) \subset \bigcup_{n \in \mathbb{N}} Q_n$, where $Q_n$ is a square circumscribed on the interval $(\alpha, \beta) \cdot I_n$. Clearly $m_2(Q_n) = \gamma^2 m_2(I_n) < \varepsilon^n$ for each $n \in \mathbb{N}$, so $(\alpha, \beta) \cdot A \in \mathcal{M}_{2s}$.

(iii) $\alpha = 0$ or $\beta = 0$.

If $\alpha = 0$ then $(\alpha, \beta) \cdot A \subset \{0\} \times (\beta \cdot \text{proj}_y A)$. From Corollary 11, the set $\text{proj}_y A$ is microscopic on the real line, so $\beta \cdot \text{proj}_y A$ is microscopic, too. It is easily seen that $\{0\} \times (\beta \cdot \text{proj}_y A) \in \mathcal{M}_{2s}$.

Now, to prove (d) we assume that $A \cap \{(x, y) : x \cdot y = 0\} = \emptyset$. Without loss of generality we can assume that $A \subset (0, 1]^2$. Let $\varepsilon > 0$. Obviously $(0, 1]^2 = \bigcup_{l,k \in \mathbb{N}} ([\frac{1}{k+1}, \frac{1}{k}] \times [\frac{1}{l+1}, \frac{1}{l}])$. Let $k, l \in \mathbb{N}$ and

$$A_{k,l} = A \cap \left(\left[\frac{1}{k+1}, \frac{1}{k}\right] \times \left[\frac{1}{l+1}, \frac{1}{l}\right]\right).$$

Since $A_{k,l} \in \mathcal{M}_{2s}$, there exists a sequence of squares $\{Q_n\}_{n \in \mathbb{N}}$ such that $Q_n = [a_n, b_n] \times [c_n, d_n] \subset [\frac{1}{k+1}, \frac{1}{k}] \times [\frac{1}{l+1}, \frac{1}{l}]$, and $A_{k,l} \subset \bigcup_{n \in \mathbb{N}} Q_n$ and $m_2(Q_n) < \left(\frac{\varepsilon}{(k+l)^4}\right)^n$ for each $n \in \mathbb{N}$. Moreover $\min\{a_n, c_n\} \geq \min\{\frac{1}{k+1}, \frac{1}{l+1}\} \geq \frac{1}{k+l}$. Putting $A_{k,l}^{-1} = \{(\frac{1}{x}, \frac{1}{y}) : (x, y) \in A_{k,l}\}$, we obtain $A_{k,l}^{-1} \subset \bigcup_{n \in \mathbb{N}} Q_n^{-1}$, where $Q_n^{-1} = [\frac{1}{b_n} - \frac{1}{a_n}, \frac{1}{a_n}] \times [\frac{1}{d_n} - \frac{1}{c_n}, \frac{1}{c_n}]$ for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $K_n$ be the smallest square such that the rectangle $Q_n^{-1}$ is contained in $K_n$. Obviously $A_{k,l}^{-1} \subset \bigcup_{n \in \mathbb{N}} K_n$.

Fix $n \in \mathbb{N}$. Let us assume that $\min\{a_n, c_n\} = a_n$. Then

$$m_2(K_n) = \left(\frac{b_n - a_n}{a_nb_n}\right)^2 \leq \frac{m_2(Q_n)}{a_n^2b_n^2} \leq \frac{m_2(Q_n)}{(\frac{1}{k+l})^4} = (k+l)^4m_2(Q_n) < \varepsilon^n.$$

If $\min\{a_n, c_n\} = c_n$, the proof is analogous. Since $M_{2s}$ is a $\sigma$-ideal, it follows that $A^{-1} \in \mathcal{M}_{2s}$. □

**Remark 14.** If $A \in \mathcal{M}_2$ and $(\alpha, \beta) \in \mathbb{R}^2$ then

(a) $A + (\alpha, \beta) = \{(x + \alpha, y + \beta) : (x, y) \in A\} \in \mathcal{M}_2$,

(b) $-A = \{(-x, -y) : (x, y) \in A\} \in \mathcal{M}_2$,

(c) $(\alpha, \beta) \cdot A = \{(\alpha \cdot x, \beta \cdot y) : (x, y) \in A\} \in \mathcal{M}_2$,

(d) if $A \cap \{(x, y) : x \cdot y = 0\} = \emptyset$ then $A^{-1} = \{(\frac{1}{x}, \frac{1}{y}) : (x, y) \in A\} \in \mathcal{M}_2$. 

The proof of above Remark is analogous to the proof of Theorem 13 for the family \( \mathcal{M}_{2s} \).

For strongly microscopic sets, the property stronger than the condition (d) in the last theorem also holds. We have

**Theorem 15.** Let \( G \subset \mathbb{R}^2 \) be an open set and let \( f : G \to \mathbb{R}^2 \) be a function fulfilling local Lipschitz condition. If \( A \in \mathcal{M}_{2s} \) and \( A \subset G \) then \( f(A) \in \mathcal{M}_{2s} \).

**Proof.** From Lindelöf theorem, it follows that there exists a sequence \( \{G_n\}_{n \in \mathbb{N}} \) of open sets and a sequence \( \{L_n\}_{n \in \mathbb{N}} \) of positive real numbers such that \( G = \bigcup_{n \in \mathbb{N}} G_n \) and \( f|_{G_n} \) fulfills Lipschitz condition with a constance \( L_n \) for each \( n \in \mathbb{N} \). We may assume that \( L_n \geq \frac{1}{2} \) for \( n \in \mathbb{N} \). Obviously, \( A = \bigcup_{n \in \mathbb{N}} (A \cap G_n) \) and \( A \cap G_n \) is strongly microscopic set for each \( n \in \mathbb{N} \). Let \( n \) be an arbitrary (fixed) positive integer and let \( \varepsilon > 0 \). Since \( A \cap G_n \in \mathcal{M}_{2s} \), there exists a sequence of squares \( \{I_k\}_{k \in \mathbb{N}} \) with sides which are parallel to coordinate axes such that \( A \cap G_n \subset \bigcup_{k \in \mathbb{N}} I_k \) and \( m_2(I_k) < \left( \frac{\varepsilon}{8L_n^2} \right)^k \) for each \( k \in \mathbb{N} \). Then the diameter of \( I_k \) is less than \( \left( \frac{\varepsilon}{L_n^{\sqrt{8}}} \right)^k \cdot \sqrt{2} \) for \( k \in \mathbb{N} \). Hence

\[
\text{diam}_f(I_k) \leq \left( \frac{\varepsilon}{L_n^{\sqrt{8}}} \right)^k \cdot \sqrt{2} \cdot L_n,
\]

for \( k \in \mathbb{N} \). So for each \( k \in \mathbb{N} \), there exists a square \( Q_k \) with sides which are parallel to coordinate axes such that \( f(I_k) \subset Q_k \) and

\[
m_2(Q_k) \leq \left( \frac{\varepsilon}{8L_n^2} \right)^k \cdot 8L_n^2.
\]

Consequently, \( f(A \cap G_n) \subset \bigcup_{k \in \mathbb{N}} f(I_k) \subset \bigcup_{k \in \mathbb{N}} Q_k \) and

\[
m_2(Q_k) \leq \frac{\varepsilon^k}{(8L_n^2)^{k-1}} \leq \varepsilon^k,
\]

for each \( k \in \mathbb{N} \), so \( f(A \cap G_n) \in \mathcal{M}_{2s} \) for each \( n \in \mathbb{N} \). Since \( \mathcal{M}_{2s} \) is a \( \sigma \)-ideal, \( f(A) \) is also strongly microscopic. \( \blacksquare \)

Put \( G = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x \cdot y = 0 \} \). Obviously, a function \( f : G \to \mathbb{R}^2 \), \( f(x, y) = (\frac{1}{x}, \frac{1}{y}) \) fulfills local Lipschitz condition. Hence, the condition (d) in Theorem 13 follows from Theorem 15. Simultaneously, the result analogous to the last theorem for microscopic sets on the plane does not hold. From Theorem 8 and 7, it follows that the family \( \mathcal{M}_2 \) is not invariant under rotation with respect to the origin, which is an isometry, so it fulfills Lipschitz condition.

By the definition, it follows that if \( A \in \mathcal{M}_{2s} \) then the \( \alpha \)-dimensional Hausdorff measure of \( A \) is equal to zero for arbitrary \( \alpha > 0 \). Consequently, each strongly microscopic set on the plane is a set of Hausdorff dimension zero.
For microscopic sets on the plane, the analogous property does not hold. The set $A$ from Theorem 8 is microscopic, but the 1-dimensional Hausdorff measure is equal to 1, so the Hausdorff dimension of $A$ is at least 1.

It is easy to see that for strongly microscopic sets, the theorem analogous to Fubini theorem also holds. It is not difficult to observe because there is a close connection between the area of the square and the length of its side. Now we shall prove that the result analogous to Fubini theorem for microscopic sets on the plane is also valid.

If $E \subseteq X \times Y$ and $x \in X$, the set

$$E_x = \{y \in Y : (x, y) \in E\}$$

is called the x-section of $E$.

**Theorem 16.** Let $E \subseteq \mathbb{R}^2$ be a microscopic set on the plane. Then $E_x$ is a microscopic set on the real line for each $x \in \mathbb{R}$ except on some microscopic set on the real line, i.e. the set $\{x \in \mathbb{R} : E_x \text{ is not a microscopic set on the real line}\}$ is microscopic on $\mathbb{R}$.

**Proof.** Let $E$ be a microscopic set on the plane. Without loss of generality, we can assume that $E \subseteq [0, 1]^2$. Put $\varepsilon_k = (\frac{1}{k+1})^{2^k}$, $k = 1, 2, \ldots$. From our assumption, for each $k \in \mathbb{N}$ and a number $(\varepsilon_k)^2$, there exists the sequence $\{P_n^{(k)}\}_{n \in \mathbb{N}}$ of intervals such that $E \subseteq \bigcup_{n \in \mathbb{N}} P_n^{(k)}$ and $m_2(P_n^{(k)}) < (\varepsilon_k)^n$ for each $n \in \mathbb{N}$. Clearly, if $m_2(P_n^{(k)}) < (\varepsilon_k)^{2n}$ then $m_1(\text{proj}_x P_n^{(k)}) < (\varepsilon_k)^n$ or $m_1(\text{proj}_y P_n^{(k)}) < (\varepsilon_k)^n$. Put

$$N_k^* = \{n \in \mathbb{N} : m_1(\text{proj}_x P_n^{(k)}) < (\varepsilon_k)^n\}$$

and

$$N_k^{**} = \mathbb{N} \setminus N_k^*.$$

Obviously, $m_1(\text{proj}_y P_n^{(k)}) < (\varepsilon_k)^n$ for $n \in N_k^{**}$. Put

$$A_k = \bigcup_{n \in N_k^*} \text{proj}_x P_n^{(k)} \quad \text{and} \quad B_k = \bigcup_{n \in N_k^{**}} \text{proj}_y P_n^{(k)},$$

for $k \in \mathbb{N}$. Clearly,

$$E \subseteq (A_k \times [0, 1]) \cup ([0, 1] \times B_k),$$

for each $k \in \mathbb{N}$. Consequently,

$$E \subseteq \bigcap_{k \in \mathbb{N}} ((A_k \times [0, 1]) \cup ([0, 1] \times B_k)) \subseteq \limsup_k ((A_k \times [0, 1]) \cup ([0, 1] \times B_k)) = (\limsup_k A_k \times [0, 1]) \cup ([0, 1] \times \limsup_k B_k).$$
Put

\[ A = \limsup_{k} A_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} A_k, \quad \text{and} \quad B = \limsup_{k} B_k = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} B_k. \]

Clearly,

\[(4) \quad E \subset (A \times [0, 1]) \cup ([0, 1] \times B).\]

We shall prove that \( A \) is a microscopic set on the real line. Let \( \varepsilon > 0 \). Then there exists \( k_0 \in \mathbb{N} \) such that \( \frac{1}{k_0 + 1} < \varepsilon \). Obviously, \( A \subset \bigcup_{k=k_0}^{\infty} A_k \) and for \( k \geq k_0 \) each \( A_k \) is a union of intervals \( I_n^{(k)} \), \( n \in \mathbb{N} \), with length less than \( \left( \left( \frac{1}{k+1} \right)^2 \right)^n \) respectively, where \( I_n^{(k)} = \text{proj}_x P_n^{(k)} \) for \( n \in N_k^{*} \) and if \( n \notin N_k^{*} \) then we put \( I_n^{(k)} = \emptyset \). Using (3), we obtain that \( A \) is covered by the intervals

\[
\begin{align*}
I_1^{(k_0)},& \quad I_2^{(k_0)}, \ldots \\
I_1^{(k_0+1)},& \quad I_2^{(k_0+1)}, \ldots \\
& \ldots
\end{align*}
\]

such that \( m_1(I_n^{(k)}) < (\varepsilon_k)^n \) for each \( n \in \mathbb{N} \). We put all these intervals \( I_n^{(k)} \), where \( n \in \mathbb{N} \) and \( k \geq k_0 \) into a sequence, using a function \( \psi : \{k \in \mathbb{N} : k \geq k_0\} \times \mathbb{N} \to \mathbb{N} \) defined as follows

\[ \psi(k, n) = 2^{k-k_0}(2n - 1). \]

Clearly, \( \psi \) is a bijection. Now consider the sequence of intervals \( \{J_m\}_{m \in \mathbb{N}} \), where \( J_m = I_n^{(k)} \) with \( \psi(k, n) = m \). Then \( A \subset \bigcup_{m=1}^{\infty} J_m \) and

\[
\begin{align*}
m_1(J_m) &= m_1(I_n^{(k)}) < (\varepsilon_k)^n = \left( \left( \frac{1}{k+1} \right)^2 \right)^n = \left( \frac{1}{k+1} \right)^{2k^2 - 1.2n} \\
&< \left( \frac{1}{k+1} \right)^{2k^2 - k_0.2n - 1} \psi(k, n) = \left( \frac{1}{k+1} \right)^m = \left( \frac{1}{k+1} \right)^m \\
&\leq \left( \frac{1}{k_0 + 1} \right)^m < \varepsilon^m,
\end{align*}
\]

for \( m \in \mathbb{N} \). Consequently, \( A \) is a microscopic set on the real line. Analogously, we can prove that \( B \) is also a microscopic set on the real line. From (4) it follows that

\[ \{x \in [0, 1] : E_x \text{ is not a microscopic set on the real line} \} \subset A \]

because if \( x \notin A \), then \( E_x = B \). \( \blacksquare \)

The set \( A \) from Theorem 7 shows that the converse of Fubini theorem for microscopic set is not true.
Observe that from the last theorem, we can obtain Kuratowski–Ulam theorem. The proof of this fact is analogous to the considerations in the book of J. C. Oxtoby ([10]) that Kuratowski–Ulam Theorem can be reduced to Fubini Theorem. Really, let $E$ be a closed and nowhere dense subset of the plane. Then each section $E_x$ is either nowhere dense set or it contains an interval. Let $I_1, I_2, \ldots$ be the sequence of all open intervals on the real line with rational endpoints. Put

$$F_i = \{x : E_x \supset I_i\},$$

for each $i \in \mathbb{N}$ and

$$A = \bigcup_{i \in \mathbb{N}} F_i.$$ 

Obviously, $F_i$ is a closed subset of the real line for $i \in \mathbb{N}$, so $A$ is a set of type $F_\sigma$ and

$$\{x \in \mathbb{R} : E_x \text{ is not of the first category}\} = A.$$ 

From Theorem 10 in [6], it follows that there exists a product homeomorphism $h = f \times g$ of the plane onto itself such that $h(E)$ is a microscopic set. For each $x \in A$, the section $E_x$ contains some interval $I_i$. Obviously, the section $(h(E))_{f(x)}$ contains the interval $g(I_i)$, so it is not a microscopic set. Consequently,

$$f(A) \subset \{x : (h(E))_x \text{ is not a microscopic set}\}.$$ 

Using Theorem 16, we obtain that $f(A)$ is a microscopic set. Hence $f(A)$ is a microscopic set of type $F_\sigma$. From [6], Theorem 5, $A$ is a set of the first category.

**Theorem 17.** A product set $A \times B$ is microscopic on the plane if and only if at least one of the sets $A$ or $B$ is microscopic on the real line.

**Proof.** If $A$ is a microscopic set on the real line then $A \times [k, k+1]$ for each $k \in \mathbb{Z}$ is a microscopic set on the plane and hence $A \times \mathbb{R} = \bigcup_{k \in \mathbb{Z}} (A \times [k, k+1])$ is also microscopic. Consequently, $A \times B$ as a subset of $A \times \mathbb{R}$ is also a microscopic set on the plane.

If $A \times B$ is a microscopic set on the plane then from Theorem 16 analogous to Fubini theorem, we obtain that the set $\{x \in \mathbb{R} : (A \times B)_x \text{ is not a microscopic set on } \mathbb{R}\}$ is microscopic on the real line. Let us suppose that $B$ is not a microscopic set on the real line. Obviously, $(A \times B)_x = B$ for each $x \in A$, so $A = \{x \in \mathbb{R} : (A \times B)_x \text{ is not a microscopic set on } \mathbb{R}\}$. Consequently, $A$ is microscopic on $\mathbb{R}$. ■

**Corollary 18.** If $\text{proj}_x E \in \mathcal{M}$ or $\text{proj}_y E \in \mathcal{M}$ then $E \in \mathcal{M}_2$. 

The converse theorem is not true. Let
\[ E = ([0,1] \times \{0\}) \cup (\{0\} \times [0,1]). \]
Clearly \( E \in \mathcal{M}_2 \). Simultaneously, \( \text{proj}_x E = \text{proj}_y E = [0,1] \notin \mathcal{M} \).

**Theorem 19.** A set \( E \subset \mathbb{R}^2 \) is microscopic if and only if there exists a set \( U \) microscopic on the real line such that
\[ E \subset (U \times \mathbb{R}) \cup (\mathbb{R} \times U). \]

**Proof.** If \( E \) is microscopic on the plane then (5) follows from (4). The inverse conclusion follows from Theorem 17. ■

R. Ger in [4] (see also Lemma 17.5.3 in [8]) considered two families of sets connected with some proper linearly invariant (in abbr.p.l.i.) ideal \( \mathcal{I} \) of subsets of a group \( G \). He said that two proper linearly invariant ideals \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) in \( G \) and \( G^2 = G \times G \), respectively, are conjugate if and only if for every \( M \in \mathcal{I}_2 \), there exists a set \( U \in \mathcal{I}_1 \) such that the set \( M_x \) belongs to \( \mathcal{I}_1 \), whenever \( x \notin U \).

It is easily seen (Lemma 1 and Corollary 1 in [4]) that if \( \mathcal{I} \) is a p.l.i. ideal in \( G \) then the families
\[ \Pi(\mathcal{I}) = \{ M \subset G^2 : \exists U \in \mathcal{I} \text{ s.t. } (U \times G) \cup (G \times U) \} \]
and
\[ \Omega(\mathcal{I}) = \{ M \subset G^2 : \exists U \in \mathcal{I} \forall x \in G \setminus U M_x \in \mathcal{I} \} \]
yield proper linearly invariant ideals in \( G^2 \), both conjugate with \( \mathcal{I} \). Moreover, \( \Omega(\mathcal{I}) \) is the greatest one in the family of all p.l.i. ideals in \( G^2 \) which are conjugate with \( \mathcal{I} \).

Clearly, the analogous considerations can be carried if \( \mathcal{I} \) is a \( \sigma \)-ideal and the similar results are obtained.

If we put \( G = \mathbb{R} \) and \( \mathcal{I} = \mathcal{M} \), then according to Theorem 19
\[ \Pi(\mathcal{M}) = \mathcal{M}_2. \]

Additionally, the set \( A \) from Theorem 7 belongs to \( \Omega(\mathcal{M}) \setminus \Pi(\mathcal{M}) \).

Obviously, the \( \sigma \)-ideal \( \mathcal{M} \) of microscopic sets on the real line is proper (i.e. \( \mathbb{R} \notin \mathcal{M} \)), contains all singletons and has a basis consisting of Borel sets, because every set from \( \mathcal{M} \) is contained in some microscopic set of type \( G_\delta \) (see [7], Lemma 2.3). Simultaneously, not every set from \( \mathcal{M} \) is covered by some microscopic set of type \( F_\sigma \), because there are microscopic sets which are residual (see [7], Lemma 2.2). It means that the \( \sigma \)-ideal \( \mathcal{M} \) is not \( \Sigma^0_2 \) supported, analogously as the \( \sigma \)-ideal of sets of Lebesgue measure zero.
In [11], the authors introduced the notion of Fubini Property for the pair \((\mathcal{I}, \mathcal{J})\) of two \(\sigma\)-ideals, in a following way. Let \(\mathcal{I}\) and \(\mathcal{J}\) be two \(\sigma\)-ideals on Polish spaces \(X\) and \(Y\), respectively. The pair \((\mathcal{I}, \mathcal{J})\) has the Fubini Property (FP) if for every Borel subset \(B\) of \(X \times Y\), if all its vertical sections \(B_x = \{y \in Y : (x, y) \in B\}\) are in \(\mathcal{J}\), then the set of all \(y \in Y\) for which horizontal section \(B^y = \{x \in X : (x, y) \in B\}\) does not belong to \(\mathcal{I}\), is a set from \(\mathcal{J}\).

A Borel set \(B \subset X \times Y\) is a \(0-1\) counterexample to (FP) for the pair \((\mathcal{I}, \mathcal{J})\) if \(B_x \in \mathcal{J}\) for each \(x \in X\) and \(X \setminus B^y \in \mathcal{I}\) for each \(y \in Y\).

Now let \(X = Y = \mathbb{R}\). Let \(\mathcal{N}\) and \(\mathcal{K}\) denote the \(\sigma\)-ideal of sets of Lebesgue measure zero and sets of the first category on the real line, respectively. From Fubini Theorem and Kuratowski–Ulam Theorem, it follows that the pairs \((\mathcal{N}, \mathcal{N})\) and \((\mathcal{K}, \mathcal{K})\) satisfy (FP). Simultaneously, neither \((\mathcal{K}, \mathcal{N})\) nor \((\mathcal{N}, \mathcal{K})\) satisfies (FP) (see [11], Example 3.6).

The aim of the last part of the paper is to decide which pairs \((\mathcal{I}, \mathcal{J})\) satisfy (FP), if at least one of the \(\sigma\)-ideals \(\mathcal{I}\) or \(\mathcal{J}\) coincides with the \(\sigma\)-ideal \(\mathcal{M}\) of microscopic sets on the real line.

Analogously as in [11], (Example 3.6), we have.

**Theorem 20.** Neither the pair \((\mathcal{K}, \mathcal{M})\) nor \((\mathcal{M}, \mathcal{K})\) satisfies (FP).

**Proof.** Let \(M\) be a microscopic set such that \(K = \mathbb{R} \setminus M\) is a set of the first category (see (1)). Put

\[
B = \{(x, y) \in \mathbb{R}^2 : x + y \in M\}.
\]

For each \(x \in \mathbb{R}\), we have \(B_x = M - x\), so \(B_x \in \mathcal{M}\). Simultaneously, \(B^y = M - y\) for each \(y \in \mathbb{R}\) and \(M\) is a residual set, so \(B^y \not\in \mathcal{K}\) for \(y \in \mathbb{R}\). Moreover, \(\mathbb{R} \setminus B^y \in \mathcal{K}\) for each \(y \in \mathbb{R}\), hence \(B\) is a \(0-1\) counterexample to (FP) for the pair \((\mathcal{K}, \mathcal{M})\).

Clearly, the set \(\{(x, y) \in \mathbb{R}^2 : x + y \in K\}\) is a \(0-1\) counterexample to (FP) for the pair \((\mathcal{M}, \mathcal{K})\).

**Theorem 21.** The pair \((\mathcal{M}, \mathcal{N})\) does not satisfy (FP).

**Proof.** Let \(C\) be the ternary Cantor set. Put

\[
B = \{(x, y) \in \mathbb{R}^2 : x + y \in C\}.
\]

For each \(x \in \mathbb{R}\), we have \(B_x = C - x\), so \(B_x \in \mathcal{N}\). Simultaneously, \(B^y = C - y\) for each \(y \in \mathbb{R}\) and \(C\) is not microscopic, so \(B^y \not\in \mathcal{M}\) for \(y \in \mathbb{R}\).

**Problem.** Does the pair \((\mathcal{N}, \mathcal{M})\) or \((\mathcal{M}, \mathcal{M})\) satisfy (FP)?

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References


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