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GENERALIZATIONS OF OPIAL-TYPE INEQUALITIES IN SEVERAL INDEPENDENT VARIABLES

Abstract. In this paper, we consider Willett's and Rozanova's generalizations of Opial's inequality and extend them to inequalities in several independent variables. Also, we present some new Opial-type inequalities in several independent variables.

1. Introduction

In 1960, the Polish mathematician Zdzisław Opial [6] proved next integral inequality, known in literature as the Opial inequality:

Let $x(t) \in C^1[0,h]$ be such that $x(0) = x(h) = 0$ and $x(t) > 0$ for $t \in (0,h)$. Then

$$\int_0^h \left|x(t)\,x'(t)\right| dt \le \frac{h}{4} \int_0^h \left(x'(t)\right)^2 dt, \tag{1.1}$$

where constant $h/4$ is the best possible.

This integral inequality, containing the derivative of the function, is recognized as fundamental result in the analysis of qualitative properties of solution of differential equations (see [1]). Over the last five decades, an enormous amount of work has been done on Opial's inequality. Many papers, which deal with new proofs, various generalizations, extensions and discrete analogues, have appeared in the literature (see [1, 5] and the references cited therein).

One such inequality is the next one involving $x^{(n)}$, $n \geq 1$, given in [2] (it's actually an extension of Willett's inequality [9], [1, page 128]):

Theorem 1.1. *Let f be a convex function on $[0,\infty)$ with $f(0) = 0$. Further, let $x \in AC^n[a,b]$ be such that $x^{(i)}(a) = x^{(i)}(b) = 0$, $i = 0,\dots,n-1$, $n \geq 1$.*

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If f is a differentiable function, then the following inequality holds

$$\int_a^b f'(|x(t)|)\,|x^{(n)}(t)|\,dt \;\leq\; \frac{2(n-1)!}{(b-a)^n}\int_a^b f\left(\frac{(b-a)^n\,|x^{(n)}(t)|}{2(n-1)!}\right)dt. \tag{1.2}$$

Following inequality is an extension of Rozanova's inequality ([8], [1, page 82]), given also in [2]:

Theorem 1.2. *Let f be a convex function on $[0,\infty)$ with $f(0)=0$. Let g be convex, nonnegative and increasing on $[0,\infty)$. Let $w(t)\geq 0$, $w'(t)>0$, $t\in[a,b]$ with $w(a)=0$. Further, let $x\in AC^n[a,b]$ be such that $x^{(i)}(a)=0$, $i=0,\ldots,n-1$, $n\geq 1$. If f is a differentiable function, then the following inequality holds*

$$\begin{aligned}\int_a^b w'(t)\,g\left(\frac{(b-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)f'\left(w(t)\,g\left(\frac{|x(t)|}{w(t)}\right)\right)dt\\ \leq\frac{1}{b-a}\int_a^b f\left((b-a)\,w'(t)\,g\left(\frac{(b-a)^{n-1}}{(n-1)!}\frac{|x^{(n)}(t)|}{w'(t)}\right)\right)dt.\end{aligned} \tag{1.3}$$

Next theorem comes from [4] and presents inequality in several independent variables. It uses the following notation:

Let $\Omega=\prod_{j=1}^m[a_j,b_j]$. Let $t=(t_1,\ldots,t_m)$ be a general point in Ω, $\Omega_t=\prod_{j=1}^m[a_j,t_j]$ and $dt=dt_1\ldots dt_m$. Further, let $Du(x)=\frac{d}{dx}u(x)$, $D_ku(t_1,\ldots,t_m)=\frac{\partial}{\partial t_k}u(t_1,\ldots,t_m)$ and $D^ku(t_1,\ldots,t_m)=D_1\cdots D_ku(t_1,\ldots,t_m)$, $1\leq k\leq m$.

Theorem 1.3. *Let $m\geq 2$ and let x_i, D^jx_i, $i=1,\ldots,p$, $j=1,\ldots,m$ be real-valued continuous functions on Ω with*

$$x_i(t)|_{t_j=a_j}=0,\quad i=1,\ldots,p,\; j=1,\ldots,m$$

or

$$x_i(t)|_{t_1=a_1}=D^1x_i(t)|_{t_2=a_2}=\ldots=D^{m-1}x_i(t)|_{t_m=a_m}=0,\quad i=1,\ldots,p.$$

Let f be a nonnegative and differentiable function on $[0,\infty)^p$ with $f(0,\ldots,0)=0$ such that D_if, $i=1,\ldots,p$ are nonnegative, continuous and nondecreasing on $[0,\infty)^p$. Then the integral inequality

$$\begin{aligned}\int_\Omega\left(\sum_{i=1}^p D_if\left(|x_1(t)|,\ldots,|x_p(t)|\right)|D^mx_i(t)|\right)dt\\ \leq f\left(\int_\Omega|D^mx_1(t)|\,dt,\ldots,\int_\Omega|D^mx_p(t)|\,dt\right)\end{aligned} \tag{1.4}$$

holds.

The aim of this paper is to generalize Opial-type integral inequalities from Theorem 1.1 and Theorem 1.2, following the idea of Theorem 1.3 for the case of several independent variables. As a consequence, we obtain more general result for the Theorem 1.3. Hence, we introduce further notation:

Let $\Omega' = \prod_{j=2}^{m}[a_j, b_j]$ and $dt' = dt_2 \ldots, dt_m$. Let $vol(\Omega) = \prod_{j=1}^{m}(b_j - a_j)$ and $\bar{\Omega}_t = \prod_{j=1}^{m}[t_j, b_j]$. Let $D^{jl}u(t_1, \ldots, t_m) = \frac{\partial^{jl}}{\partial t_1^l \ldots \partial t_j^l} u(t_1, \ldots, t_m)$, $1 \le j \le m$, $1 \le l \le n$.

Also by $C^{mn}(\Omega)$, we denote the space of all functions u on Ω which have continuous derivatives $D^{jl}u$ for $j = 1, \ldots, m$ and $l = 1, \ldots, n$. Further, $AC(\Omega)$ is the space of all absolutely continuous functions on Ω. By $AC^{mn}(\Omega)$, we denote the space of all functions $u \in C^{m(n-1)}(\Omega)$ with $D^{m(n-1)}u \in AC(\Omega)$.

Finally, next lemma about convex function of several variables will be used in our proofs ([7, page 11]).

Lemma 1.4. *Suppose f is defined on the open convex set $U \subset \mathbb{R}^n$. If f is convex (strictly) on U and the gradient vector $f'(x)$ exists throughout U, then f' is (strictly) increasing on U.*

2. Main results

First theorem is a generalization of Theorem 1.3.

Theorem 2.1. *Let $m, n, p \in \mathbb{N}$. Let f be a nonnegative and differentiable function on $[0, \infty)^p$, with $f(0, \ldots, 0) = 0$. Further, for $i = 1, \ldots, p$, let $x_i \in AC^{mn}(\Omega)$ be such that $D^{jl}x_i(t)|_{t_j = a_j} = D^{jl}x_i(t)|_{t_j = b_j} = 0$, where $j = 1, \ldots, m$ and $l = 0, \ldots, n-1$. Also, let $D_i f$, $i = 1, \ldots, p$, be nonnegative, continuous and nondecreasing on $[0, \infty)^p$. Then the following inequality holds*

$$\int_{\Omega} \Big(\sum_{i=1}^{p} D_i f\left(|x_1(t)|, \ldots, |x_p(t)| \right) |D^{mn} x_i(t)| \Big) dt \tag{2.1}$$
$$\le \frac{2\left((n-1)!\right)^m}{(vol(\Omega))^{n-1}} f\Bigg(\frac{(vol(\Omega))^{n-1}}{2\left((n-1)!\right)^m} \int_{\Omega} |D^{mn} x_1(t)|\, dt, \ldots,$$
$$\frac{(vol(\Omega))^{n-1}}{2\left((n-1)!\right)^m} \int_{\Omega} |D^{mn} x_p(t)|\, dt \Bigg).$$

Proof. We extend technique used in [2, Theorem 2.1] on several independent variables. Let $c = (c_1, \cdots, c_m) \in \Omega$ and let

$$y_i(t) = \int_{\Omega_t} \int_{\Omega_{t,1}} \ldots \int_{\Omega_{t,m-1}} |D^{mn} x_i(s)|\, ds\, dt_{1,1} \ldots dt_{m,n-1} \tag{2.2}$$
$$= \frac{1}{((n-1)!)^m} \int_{\Omega_t} \prod_{j=1}^{m} (t_j - s_j)^{n-1} |D^{mn} x_i(s)|\, ds,$$

for $t \in \Omega_c$, $i = 1, \dots, p$. Hence $D^{mn} y_i(t) = |D^{mn} x_i(t)|$ and $y_i(t) \geq |x_i(t)|$. It is easy to conclude that for each $l = 0, \dots, n-1$, we have $D^{jl} y_i(t) \geq 0$ and nondecreasing on Ω_c ($i = 1, \dots, p$ and $j = 1, \dots, m$). From $D^{jl} y_i(t)|_{t_j = a_j} = 0$, it follows

$$y_i(t) \leq \frac{(vol\Omega_c)^{n-1}}{((n-1)!)^m} D^{m(n-1)} y_i(t), \quad t \in \Omega_c,$$

and also

$$y_i(t) \leq \frac{(vol\Omega)^{n-1}}{((n-1)!)^m} D^{m(n-1)} y_i(t), \quad t \in \Omega_c.$$

Define

$$u_i(t) = \frac{(vol\Omega)^{n-1}}{((n-1)!)^m} D^{m(n-1)} y_i(t),$$

for $t \in \Omega_c$ and $i = 1, \dots, p$. Since $D_i f$ are nonnegative, continuous and nondecreasing on $[0, \infty)^p$, we have

$$(2.3) \quad \int\limits_{\Omega_c} \Big[\sum_{i=1}^{p} D_i f\left(|x_1(t)|, \dots, |x_p(t)| \right) |D^{mn} x_i(t)| \Big] dt \leq \int\limits_{\Omega_c} \Big[\sum_{i=1}^{p} D_i f\left(y_1(t), \dots, y_p(t) \right) D^{mn} y_i(t) \Big] dt.$$

Consequently,

$$\begin{aligned}
(2.4) \quad & \int\limits_{\Omega_c} \Big[\sum_{i=1}^{p} D_i f\left(|x_1(t)|, \dots, |x_p(t)| \right) |D^{mn} x_i(t)| \Big] dt \\
&\leq \int\limits_{\Omega_c} \Bigg[\sum_{i=1}^{p} D_i f \left(\frac{(vol\Omega_c)^{n-1}}{((n-1)!)^m} D^{m(n-1)} y_1(t), \dots, \frac{(vol\Omega_c)^{n-1}}{((n-1)!)^m} D^{m(n-1)} y_p(t) \right) \\
&\qquad D^{mn} y_i(t) \Bigg] dt \\
&\leq \int\limits_{\Omega_t} \Big[\sum_{i=1}^{p} D_i f\left(u_1(t_1, c_2, \dots, c_m), \dots, u_p(t_1, c_2, \dots, c_m) \right) D^{mn} y_i(t) \Big] dt \\
&= \int_{a_1}^{c_1} \Bigg[\sum_{i=1}^{p} D_i f\left(u_1(t_1, c_2, \dots, c_m), \dots, u_p(t_1, c_2, \dots, c_m) \right) \\
&\qquad \int\limits_{\Omega_c'} D^{mn} y_i(t)\, dt' \Bigg] dt_1
\end{aligned}$$

$$\leq \int_{a_1}^{c_1} \Bigg[\sum_{i=1}^{p} D_i f\,(u_1(t_1, c_2, \ldots, c_m), \ldots, u_p(t_1, c_2, \ldots, c_m)) \frac{((n-1)!)^m}{(vol(\Omega_c))^{n-1}} D_1 u_i(t_1, c_2 \ldots, c_m)\Bigg] dt_1$$

$$= \frac{((n-1)!)^m}{(vol(\Omega_c))^{n-1}} \int_{a_1}^{c_1} \frac{d}{dt_1} \left[f\,(u_1(t_1, c_2, \ldots, c_m), \ldots, u_p(t_1, c_2, \ldots, c_m))\right] dt_1$$

$$= \frac{((n-1)!)^m}{(vol(\Omega_c))^{n-1}} f\,(u_1(c_1, c_2, \ldots, c_m), \ldots, u_p(c_1, c_2, \ldots, c_m))$$

$$= \frac{((n-1)!)^m}{(vol(\Omega_c))^{n-1}} f\Bigg(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m} \int\limits_{\Omega_c} |D^{mn} x_1(t)|\, dt, \ldots, \frac{(vol(\Omega))^{n-1}}{((n-1)!)^m} \int\limits_{\Omega_c} |D^{mn} x_p(t)|\, dt \Bigg).$$

For $t \in \bar{\Omega}_c$ and $i = 1 \ldots, p$, we have

$$y_i(t) = \int\limits_{\bar{\Omega}_t} \int\limits_{\bar{\Omega}_{t,1}} \cdots \int\limits_{\bar{\Omega}_{t,m-1}} |D^{mn} x_i(s)|\, ds\, dt_{1,1} \ldots dt_{m,n-1} = \frac{1}{((n-1)!)^m} \int\limits_{\bar{\Omega}_t} \prod_{j=1}^{m} (s_j - t_j)^{n-1} |D^{mn} x_i(s)|\, ds, \tag{2.5}$$

from which analogously, we obtain

$$\int\limits_{\bar{\Omega}_c} \Bigg[\sum_{i=1}^{p} D_i f\,(|x_1(t)|, \ldots, |x_p(t)|)\, |D^{mn} x_i(t)| \Bigg] dt \leq \frac{((n-1)!)^m}{(vol(\Omega))^{n-1}} f\Bigg(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m} \int\limits_{\bar{\Omega}_c} |D^{mn} x_1(t)|\, dt, \ldots, \frac{(vol(\Omega))^{n-1}}{((n-1)!)^m} \int\limits_{\bar{\Omega}_c} |D^{mn} x_p(t)|\, dt \Bigg). \tag{2.6}$$

Let $c \in \Omega$ be such that for every $i = 1, \ldots, p$

$$\int\limits_{\Omega_c} |D^{mn} x_i(t)|\, dt = \int\limits_{\bar{\Omega}_c} |D^{mn} x_i(t)|\, dt = \frac{1}{2} \int\limits_{\Omega} |D^{mn} x_i(t)|\, dt. \tag{2.7}$$

Now from (2.4), (2.6) and (2.7), it follows (2.1). ∎

Remark 2.2. For $n = 1$, the inequality (2.4) becomes the inequality (1.4), requiring boundary conditions only on a_j, $j = 1, \ldots, m$.

Next, we follow with inequality for convex function f.

Theorem 2.3. *Let $m, n, p \in \mathbb{N}$. Let f be a convex and differentiable function on $[0,\infty)^p$ with $f(0,\ldots,0) = 0$. Further for $i = 1,\ldots,p$, let $x_i \in AC^{mn}\Omega$ be such that $D^{jl}x_i(t)|_{t_j=a_j} = D^{jl}x_i(t)|_{t_j=b_j} = 0$, where $j = 1,\ldots,m$ and $l = 0,\ldots,n-1$. Then the following inequality holds*

$$
\begin{aligned}
(2.8)\quad &\int_\Omega \Big(\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\ldots,|x_p(t)|\right)|D^{mn}x_i(t)|\Big)dt \\
&\leq \frac{2\left((n-1)!\right)^m}{(vol(\Omega))^n}\int_\Omega f\Bigg(\frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_1(t)|,\ldots, \\
&\qquad \frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_p(t)|\Bigg)dt.
\end{aligned}
$$

Proof. As in the proof of the previous theorem, we obtain (2.1) with the difference of applying Lemma 1.4 in (2.3) since f is a convex function. Then, from Jensen's inequality [7, page 51], we have

$$
\begin{aligned}
&\int_\Omega \Big[\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\ldots,|x_p(t)|\right)|D^{mn}x_i(t)|\Big]dt \\
&\leq \frac{2\left((n-1)!\right)^m}{(vol(\Omega))^{n-1}} f\Bigg(\frac{(vol(\Omega))^{n-1}}{2\left((n-1)!\right)^m}\int_\Omega |D^{mn}x_1(t)|\,dt,\ldots, \\
&\qquad \frac{(vol(\Omega))^{n-1}}{2\left((n-1)!\right)^m}\int_\Omega |D^{mn}x_p(t)|\,dt\Bigg) \\
&= \frac{2\left((n-1)!\right)^m}{(vol(\Omega))^{n-1}} f\Bigg(\frac{1}{vol(\Omega)}\int_\Omega \frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_1(t)|\,dt,\ldots, \\
&\qquad \frac{1}{vol(\Omega)}\int_\Omega \frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_p(t)|\,dt\Bigg) \\
&\leq \frac{2\left((n-1)!\right)^m}{(vol(\Omega))^n}\int_\Omega f\Bigg(\frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_1(t)|,\ldots, \\
&\qquad \frac{(vol(\Omega))^n}{2\left((n-1)!\right)^m}|D^{mn}x_p(t)|\Bigg)dt. \quad \blacksquare
\end{aligned}
$$

Remark 2.4. As a special case for $p = 1$ and $m = 1$, Theorem 1.1 is reobtained.

Next theorem is a generalization of Theorem 1.2 for several independent variables.

Theorem 2.5. *Let $m, n, p \in \mathbb{N}$. Let f be a convex and differentiable function on $[0,\infty)^p$ with $f(0,\ldots,0) = 0$. Let g_i be convex, nonnegative and increasing on $[0,\infty)$ for $i = 1,\ldots,p$. For $i = 1,\ldots,p$, let $h_i : \Omega \to [0,\infty)$ be such that $D^m h_i$ is nonnegative with $D^{j-1}h_i(t)|_{t_j=a_j} = 0$, $j = 1,\ldots,m$. Further for $i = 1,\ldots,p$, let $x_i \in AC^{mn}\Omega$ be such that $D^{jl}x_i(t)|_{t_j=a_j} = 0$, where $j = 1,\ldots,m$ and $l = 0,\ldots,n-1$. Then the following inequality holds*

$$\begin{aligned}(2.9)\qquad &\int\limits_\Omega \left(\sum_{i=1}^{p} D_i f\left(h_1(t)g_1\left(\frac{|x_1(t)|}{h_1(t)}\right),\ldots,h_p(t)g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\right.\\ &\left.\times D^m h_i(t) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_i(t)|}{D^m h_i(t)}\right)\right)dt\\ &\le \frac{1}{vol(\Omega)}\int\limits_\Omega f\left(vol(\Omega)\,D^m h_1(t)\,g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_1(t)|}{D^m h_1(t)}\right),\ldots,\right.\\ &\left. vol(\Omega)\,D^m h_p(t)\,g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_p(t)|}{D^m h_p(t)}\right)\right)dt.\end{aligned}$$

Proof. As in the proof of Theorem 2.1, for $i = 1,\ldots,p$, $t \in \Omega$ we have $D^{mn}y_i(t) = |D^{mn}x_i(t)|$, $y_i(t) \ge |x_i(t)|$ and

$$y_i(t) \le \frac{(vol\Omega)^{n-1}}{((n-1)!)^m} D^{m(n-1)}y_i(t).$$

From Jensen's inequality, monotonicity and convexity of each g_i ($i=1,\ldots,p$), we have

$$\begin{aligned} g_i\left(\frac{|x_i(t)|}{h_i(t)}\right) &\le g_i\left(\frac{y_i(t)}{h_i(t)}\right) \le g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{m(n-1)}y_i(t)}{h_i(t)}\right)\\ &= g_i\left(\frac{\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\int_{\Omega_t} D^m h_i(s)\frac{|D^{mn}x_i(s)|}{D^m h_i(s)}ds}{\int_{\Omega_t} D^m h_i(s)ds}\right)\\ &\le \frac{1}{h_i(t)}\int_{\Omega_t} D^m h_i(s) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_i(s)}{D^m h_i(s)}\right)ds.\end{aligned}$$

Define

$$U_i(s) = D^m h_i(s) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_i(s)}{D^m h_i(s)}\right),$$

for $t \in \Omega$ and $i = 1,\ldots,p$. Hence,

$$\begin{aligned}(2.10)\qquad &\int\limits_\Omega \left[\sum_{i=1}^{p} D_i f\left(h_1(t)g_1\left(\frac{|x_1(t)|}{h_1(t)}\right),\ldots,h_p(t)g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\right.\\ &\left.\times D^m h_i(t) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}|x_i(t)|}{D^m h_i(t)}\right)\right] dt\end{aligned}$$

$$
\begin{aligned}
&\le \int_{\Omega}\Bigg[\sum_{i=1}^{p} D_i f\left(\int_{\Omega_t} D^m h_1(s) g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_1(s)}{D^m h_1(s)}\right)ds,\dots,\right.\\
&\qquad \left.\int_{\Omega_t} D^m h_p(s) g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_p(s)}{D^m h_p(s)}\right)ds\right)\\
&\qquad \times D^m h_i(t) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_i(t)}{D^m h_i(t)}\right)\Bigg]dt\\
&= \int_{\Omega}\Bigg[\sum_{i=1}^{p} D_i f\left(\int_{\Omega_t} U_1(s)\,ds,\dots,\int_{\Omega_t} U_p(s)\,ds\right)U_i(t)\Bigg]dt\\
&= f\left(\int_{\Omega} U_1(t)\,dt,\dots,\int_{\Omega} U_p(t)\,dt\right)\\
&= f\left(\int_{\Omega} D^m h_1(t) g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_1(t)}{D^m h_1(t)}\right)dt,\dots,\right.\\
&\qquad \left.\int_{\Omega} D^m h_p(t) g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}y_p(t)}{D^m h_p(t)}\right)dt\right)\\
&= f\left(\int_{\Omega} D^m h_1(t) g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_1(t)|}{D^m h_1(t)}\right)dt,\dots,\right.\\
&\qquad \left.\int_{\Omega} D^m h_p(t) g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_p(t)|}{D^m h_p(t)}\right)dt\right).
\end{aligned}
$$

Finally, by Jensen's inequality, we obtain

$$
\begin{aligned}
&\int_{\Omega}\Bigg[\sum_{i=1}^{p} D_i f\left(h_1(t) g_1\left(\frac{|x_1(t)|}{h_1(t)}\right),\dots,h_p(t) g_p\left(\frac{|x_p(t)|}{h_p(t)}\right)\right)\\
&\qquad \times D^m h_i(t) g_i\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{D^{mn}|x_i(t)|}{D^m h_i(t)}\right)\Bigg]dt\\
&\le f\left(\int_{\Omega} D^m h_1(t) g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_1(t)|}{D^m h_1(t)}\right)dt,\dots,\right.\\
&\qquad \left.\int_{\Omega} D^m h_p(t) g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_p(t)|}{D^m h_p(t)}\right)dt\right)\\
&= f\left(\frac{1}{vol(\Omega)}\int_{\Omega} vol(\Omega)\, D^m h_1(t)\, g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_1(t)|}{D^m h_1(t)}\right)dt,\dots,\right.\\
&\qquad \left.\frac{1}{vol(\Omega)}\int_{\Omega} vol(\Omega)\, D^m h_p(t)\, g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_p(t)|}{D^m h_p(t)}\right)dt\right)\\
&\le \frac{1}{vol(\Omega)}\int_{\Omega} f\Bigg(vol(\Omega)\, D^m h_1(t)\, g_1\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_1(t)|}{D^m h_1(t)}\right),\dots,\\
&\qquad vol(\Omega)\, D^m h_p(t)\, g_p\left(\frac{(vol(\Omega))^{n-1}}{((n-1)!)^m}\frac{|D^{mn}x_p(t)|}{D^m h_p(t)}\right)\Bigg)dt. \blacksquare
\end{aligned}
$$

Remark 2.6. Theorem 1.2 follows for $p = 1$ and $m = 1$. Also, the inequality (2.10) is an extension of the inequality given in [3, Theorem 1].

References

[1] R. P. Agarwal, P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1995.

[2] M. Andrić, A. Barbir, J. Pečarić, *On Willett's, Godunova–Levin's and Rozanova's Opial type inequalities with related Stolarsky type means*, Math. Notes, (2014), to appear.

[3] I. Brnetić, J. Pečarić, *Note on generalization of Godunova–Levin–Opial inequality*, Demonstratio Math. 3(30) (1997), 545–549.

[4] I. Brnetić, J. Pečarić, *Note on the generalization of the Godunova–Levin–Opial type inequality in several independent variables,* J. Math. Anal. Appl. 215 (1997), 274–283.

[5] D. S. Mitrinović, J. Pečarić, A. M. Fink, *Inequalities Involving Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1991.

[6] Z. Opial, *Sur une inégalité*, Ann. Polon. Math. 8 (1960), 29–32.

[7] J. E. Pečarić, F. Proschan, Y. C. Tong, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Inc., 1992.

[8] G. I. Rozanova, *Integral inequalities with derivatives and with arbitrary convex functions*, Uc. Zap. Mosk. Gos. Ped. In-ta im. Lenina 460 (1972), 58–65.

[9] D. Willett, *The existence–uniqueness theorem for an n-th order linear ordinary differential equation*, Amer. Math. Monthly 75 (1968), 174–178.

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