Research Article

Open Access

Fabrizio Durante*, Juan Fernández-Sánchez, and Wolfgang Trutschnig

Solution to an open problem about a transformation on the space of copulas

Abstract: We solve a recent open problem about a new transformation mapping the set of copulas into itself. The obtained mapping is characterized in algebraic terms and some limit results are proved.

Keywords: Copula; Concordance; Quasi-copula

MSC: 62H05

DOI 10.2478/demo-2014-0005

Received September 12, 2014; accepted October 29, 2014

1 Introduction

Nowadays, the “copula industry” is actively producing a large number of methods in order to enlarge well-known families and/or construct novel copulas. Such investigations are usually motivated by the need to introduce more flexible stochastic models that go beyond traditional (and often unrealistic) assumptions related to the distribution of a multivariate random vector.

A large class of these methods is provided by transformations of copulas, i.e. mappings from the space of copulas (or some of its subsets) into itself that are usually employed to add parameters to some known families. Such mappings include, for instance, the distortion of copulas (see, e.g., [4, 9, 12, 16, 18, 22]), which represents one of the most extensively studied transformations, as well as other different constructions (see, e.g., [1, 5, 13, 14]).

In the current paper we are interested in the mapping transforming a function $C: [0, 1]^2 \rightarrow [0, 1]$ into another function C_λ defined on $[0, 1]^2$ by

$$C_\lambda(x, y) = \frac{C(x, y)}{1 + \lambda - \lambda C^*(x, y)}, \quad (1.1)$$

where C^* is the so-called *dual* of C , $C^*(x, y) = x + y - C(x, y)$, and $\lambda \in (0, 1]$ ($\lambda = 0$ corresponds to the identity transformation). Analogously, Eq. (1.1) can be rewritten in the form

$$C_\lambda(x, y) = \frac{C(x, y)}{1 + \lambda \bar{C}(x, y)}, \quad (1.2)$$

where $\bar{C}(x, y) = 1 - x - y + C(x, y)$ is the survival function associated with C .

As noted in [15], if C coincides with one of the Frechét–Hoeffding bounds $M(x, y) = \min(x, y)$ and $W(x, y) = \max(x + y - 1, 0)$ then C_λ of (1.1) is a copula. Moreover, if $C(x, y) = \Pi(x, y) = xy$ is the independence copula, then C_λ corresponds to the Ali–Mikhail–Haq family of copula with parameter $\lambda \in [-1, 0)$.

***Corresponding Author: Fabrizio Durante:** Faculty of Economics and Management, Free University of Bozen-Bolzano, I-39100 Bolzano (Italy), E-mail: fabrizio.durante@unibz.it

Juan Fernández-Sánchez: Grupo de Investigación de Análisis Matemático, Universidad de Almería, La Cañada de San Urbano, Almería, Spain, E-mail: juanfernandez@ual.es

Wolfgang Trutschnig: Department for Mathematics, University of Salzburg, Salzburg, Austria, E-mail: wolfgang@trutschnig.net

The seemingly natural question is, therefore, whether Eq. (1.1) defines a copula for any initial copula C (see [15, Problem 4.2] and also [5, section 4]). The answer to this question is affirmative as we show below, and the proof is based on a density argument in the space of copulas. Additionally, the new transformation is characterized in algebraic terms and some results concerning the limit behavior of the transformation (and a related transformation) are presented.

2 The main result

We start by considering how Eq. (1.1) acts in the class of quasi-copulas (for a definition, see [11]). Quasi-copulas are generalizations of copulas that are used, e.g., in finding bounds for several sub-classes of copulas [2, 20, 21]. The following result holds.

Proposition 2.1. *If Q is a quasi-copula, then Q_λ given by (1.1) is a quasi-copula for every $\lambda \in (0, 1]$.*

Proof. First, notice that Q_λ is well-defined since the denominator of Eq. (1.1) is bounded from below by 1 because every quasi-copula is bounded below by W . Moreover, Q_λ satisfies the boundary conditions for a quasi-copula, i.e.

$$Q_\lambda(x, 1) = Q_\lambda(1, x) = x$$

for all $x \in [0, 1]$. It can be easily checked that Q_λ is increasing in each variable (keeping the other fixed). Finally, in order to show that Q_λ is 1-Lipschitz continuous, consider $x, y, y + h \in [0, 1]$. Then we have

$$\begin{aligned} Q_\lambda(x, y + h) - Q_\lambda(x, y) &= \frac{\lambda h Q(x, y) + (Q(x, y + h) - Q(x, y))(1 + \lambda(1 - x - y))}{(1 + \lambda(1 - x - y - h + Q(x, y + h)))(1 + \lambda(1 - x - y + Q(x, y)))} \\ &\leq \frac{h}{(1 + \lambda(1 - x - y - h + Q(x, y + h)))} \leq h, \end{aligned}$$

from which the assertion follows. $\square$

Now, in order to prove that the transformation of Eq. (1.1) also maps copulas into copulas we need to show that it preserves the 2-increasing property, as is shown below by density arguments.

Proposition 2.2. *If C is a copula, then C_λ given by (1.1) is a copula for every $\lambda \in (0, 1]$.*

Proof. Suppose that C is an absolutely continuous copula that admits continuous mixed partial derivatives and set $\Sigma(x, y) = x + y$. Then the first derivative of C_λ with respect to first variable is given by

$$D_1 C_\lambda = \frac{\lambda C(1 - D_1 C)}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{D_1 C}{1 + \lambda - \lambda(\Sigma - C)}.$$

Thus, the mixed second partial derivative of C_λ is given by

$$\begin{aligned} D_{12} C_\lambda &= \frac{2\lambda^2 C(1 - D_1 C)(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^3} + \frac{\lambda D_2 C(1 - D_1 C) - \lambda C D_{12} C}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{\lambda(1 - D_2 C) D_1 C}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{D_{12} C}{1 + \lambda - \lambda(\Sigma - C)} \\ &= \frac{2\lambda^2 C(1 - D_1 C)(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^3} + \frac{\lambda D_2 C(1 - D_1 C) + \lambda D_1 C(1 - D_2 C)}{(1 + \lambda - \lambda(\Sigma - C))^2} + \frac{D_{12} C(1 + \lambda - \lambda \Sigma)}{(1 + \lambda - \lambda(\Sigma - C))^2}, \end{aligned}$$

which is non-negative since the first partial derivatives of a copula are bounded above by 1 wherever they exist. Now, the transformation of type (1.1) maps quasi-copulas into quasi-copulas and it is continuous with respect to the L^∞ norm. Taking into account that copulas with continuous mixed partial derivative are dense in the class of bivariate copulas with respect to the same norm (consider, for instance, Bernstein copulas [17]) and that we have shown that C_λ is a copula for every copula C with the properties defined as above, the desired assertion follows. $\square$

As a consequence, equality (1.1) defines, for every $\lambda \in (0, 1]$, a mapping $T_\lambda: \mathcal{C} \rightarrow \mathcal{C}$, where $\mathcal{C}$ is the class of bivariate copulas. In particular, for any λ , if μ_C denotes the measure induced by $C \in \mathcal{C}$, we have

$$T_\lambda(C)(x, y) = \frac{\mu_C([0, x] \times [0, y])}{1 + \lambda \mu_C([x, 1] \times [y, 1])} \quad (2.1)$$

For a fixed λ , the transformation T_λ is injective, since $T_\lambda(C_1) = T_\lambda(C_2)$ implies $C_1 = C_2$.

Remark 2.1. It is immediate that, for the copula W , $T_\lambda(W) = W$ holds for any $\lambda \in (0, 1]$. Moreover, W is the only fixed point of T_λ . In fact, if $C = C_\lambda$ for some $\lambda \in (0, 1]$, then, for every $(x, y) \in [0, 1]^2$, we must have

$$\lambda C(x, y)(1 - x - y + C(x, y)) = 0,$$

from which it follows that $C = W$. Notice that, here, a key role is played by the pointwise bounds in the space of (quasi-)copulas. In fact, consider the semi-copula S defined by $S(x, y) = 0$ on $(0, 1)^2$, and $S(x, y) = \min(x, y)$, otherwise. Then $S = S_\lambda$ for every $\lambda \in (0, 1]$.

Moreover, since W is invariant under T_λ , it also follows that, if C is a patchwork copula with basis copula given by W , then $T_\lambda(C)$ is also a copula of this type (see [8] for definition of patchwork).

The transformation of (1.1) preserves the concordance order between copulas. In fact, it can be easily checked that $C_1 \leq C_2$ pointwise implies $T_\lambda(C_1) \leq T_\lambda(C_2)$, from which we directly get the following consequences:

- the range of the transformation T_λ goes from $T_\lambda(W) = W$ to $T_\lambda(M) < M$. In other words, the T_λ -image of any family of copulas cannot describe perfect positive dependence.
- If ν denotes a measure of concordance (see [19]) between two random variables with copula C , then

$$\nu(T_\lambda(W)) = -1 \leq \nu(T_\lambda(C)) \leq \nu(T_\lambda(M)) \quad (2.2)$$

for every $\lambda \in (0, 1]$.

The upper bound in (2.2) can be calculated explicitly for concordance measures like Kendall's τ and Spearman's ρ . In fact, consider that, for any $\lambda \in (0, 1]$ the transformation of the comonotonicity copula M can be written as

$$T_\lambda(M)(x, y) = \frac{\min(x, y)}{1 + \lambda - \lambda \max(x, y)}, \quad (2.3)$$

from which it is apparent that it is a semilinear copula [6, 10] generated by $f_\lambda(t) = 1/(1 + \lambda - \lambda t)$. Moreover, formulas for measures of association of semilinear copulas are given in [6, Theorem 4]. By using them and by doing little algebra, the following result follows.

Proposition 2.3. *Let $\lambda \in (0, 1]$. Then the following inequalities hold:*

$$\begin{aligned} -1 \leq \tau(C_\lambda) &\leq \frac{4\lambda - \lambda^2 - 4 \ln(1 + \lambda)}{\lambda^2}, \\ -1 \leq \rho(C_\lambda) &\leq \frac{-12\lambda - 18\lambda^2 - 3\lambda^3 + 12(1 + \lambda)^2 \ln(1 + \lambda)}{\lambda^3}. \end{aligned}$$

In particular, $\tau(C_1) \leq 0.228$, while $\rho(C_1) \leq 0.272$.

Remark 2.2. In the case of Eq. (2.3), T_λ has transformed a copula which is purely singular into a copula with a singular component and an absolutely continuous component. See Figure 1.

Remark 2.3. It should be mentioned that the mapping $T_\lambda(C)$ is decreasing with respect to λ and the pointwise ordering between copulas. Thus, since $T_0(C) = C$, it follows that, for $\lambda \in (0, 1]$, $T_\lambda(C) \leq C$, so $T_\lambda(C)$ is at most as positively quadrant dependent as C .

Figure 1: Sample of size 10.000 from $(X, Y) \sim T_\lambda(M)$ for $\lambda = 1/2$, its two dimensional histogram and the corresponding marginal histograms.

Remark 2.4. The transformation of eq. (1.1) maps the diagonal section of C , δ_C , into the diagonal section $\delta_C(t)/(1 + \lambda - \lambda(2t - \delta_C(t)))$. As such, it changes the tail dependence coefficients of the corresponding copulas, where they exist (see, e.g., [7]). In particular, $UTDC(C_\lambda) = (1 - \lambda)UTDC(C)$, while $LTDC(C_\lambda) = LTDC(C)/(1 + \lambda)$, where UTDC and LTDC denote the upper and lower tail dependence coefficients, respectively.

Now, following the idea of linear constructions of copulas [13], we would like to characterize the transformation of type (1.1) in algebraic terms.

Proposition 2.4. Let $F : \mathbb{C} \rightarrow \mathbb{C}$ be a mapping given by

$$F(C)(x, y) = \frac{a_0 + a_1x + a_2y + a_3C(x, y)}{b_0 + b_1x + b_2y + b_3C(x, y)},$$

with $a_i, b_i \in \mathbb{R}$ for $(i = 0, \dots, 3)$. Then F is given by (1.1).

Proof. Since $F(C)$ must be a copula, the boundary conditions imply that for every $(x, y) \in [0, 1]^2$ we have

$$\frac{a_0 + (a_1 + a_3)x + a_2}{b_0 + (b_1 + b_3)x + b_2} = x, \quad \frac{a_0 + (a_2 + a_3)y + a_1}{b_0 + (b_2 + b_3)y + b_1} = y,$$

from which it is easily derived that

$$a_0 + a_2 = a_0 + a_1 = 0, \quad b_1 + b_3 = b_2 + b_3 = 0.$$

In particular, since $C(0, 0) = 0$, $a_0 = 0$, which, together with $C(1, 1) = 1$ and previous constraints, implies $a_1 = a_2 = 0$, and $a_3 = b_0 + b_1$. Thus, F is given by the formula

$$F(C)(x, y) = \frac{(b_0 + b_1)C(x, y)}{b_0 + b_1x + b_1y - b_1C(x, y)},$$

and the assertion follows by dividing numerator and denominator by $b_0 + b_1 \neq 0$. $\square$

In other words, transformations of type (1.1) are the only transformations that can be expressed as ratio of two linear functions involving the variables $x, y \in [0, 1]$ as well as $z = C(x, y)$.

3 The induced transformation and its iterations

The transformation of (1.1) induces a mapping T_λ in $\mathcal{C}$. It would be hence of interest to see what happens when the transformation is iterated. Interestingly, the iterations converge to the fixed point of T_λ , which is the countermonotonicity copula. Here, $T_\lambda^2 := T_\lambda \circ T_\lambda$ and, by recursion, for each $n \geq 3$, $T_\lambda^n := T_\lambda \circ T_\lambda^{n-1}$.

Proposition 3.1. *Let $\lambda \in (0, 1]$. For every $C \in \mathcal{C}$ we have $\lim_{n \rightarrow \infty} d_\infty(T_\lambda^n C, W) = 0$.*

Proof. First, we prove that

$$d_\infty(T_\lambda C, W) \leq \frac{d_\infty(C, W)}{1 + \lambda d_\infty(C, W)} \quad (3.1)$$

and distinguish two cases: (i) If $x + y - 1 \leq 0$ then we have $W(x, y) = 0$ so we get

$$\begin{aligned} T_\lambda C(x, y) - W(x, y) &= T_\lambda C(x, y) = \frac{C(x, y) - W(x, y)}{1 + \lambda(1 - x - y + C(x, y) - W(x, y))} \\ &\leq \frac{C(x, y) - W(x, y)}{1 + \lambda(C(x, y) - W(x, y))} \leq \frac{d_\infty(C, W)}{1 + \lambda d_\infty(C, W)}. \end{aligned}$$

(ii) If $x + y - 1 > 0$ then we have $W(x, y) = x + y - 1$ so we get

$$\begin{aligned} T_\lambda C(x, y) - W(x, y) &= \frac{C(x, y)}{1 + \lambda(1 - x - y + C(x, y))} - W(x, y) = \\ &= \frac{(C(x, y) - W(x, y))(1 - \lambda W(x, y))}{1 + \lambda(C(x, y) - W(x, y))} \leq \frac{d_\infty(C, W)}{1 + \lambda d_\infty(C, W)}. \end{aligned}$$

Finally, we simply iterate Eq. (3.1) to get

$$\begin{aligned} d_\infty(T_\lambda^n C, W) &\leq \frac{d_\infty(T_\lambda^{n-1} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-1} C, W)} \leq \frac{\frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-2} C, W)}}{1 + \lambda \frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + \lambda d_\infty(T_\lambda^{n-2} C, W)}} = \frac{d_\infty(T_\lambda^{n-2} C, W)}{1 + 2\lambda d_\infty(T_\lambda^{n-2} C, W)} \\ &\leq \dots \leq \frac{d_\infty(T_\lambda^{n-3} C, W)}{1 + 3\lambda d_\infty(T_\lambda^{n-3} C, W)} \leq \dots \leq \frac{d_\infty(C, W)}{1 + n\lambda d_\infty(C, W)}, \end{aligned}$$

from which we directly deduce the desired result. $\square$

Figure 2: Image plot of $T_\lambda^n(M)$ for $n \in \{0, 1, 5, 20\}$, whereby $\lambda = 1/2$.

There is another analytically very simple transformation $S_\lambda : \mathcal{C} \rightarrow \mathcal{C}$ that fulfills the analogous limit result with W replaced by M , i.e. $\lim_{n \rightarrow \infty} d_\infty(S_\lambda^n C, M) = 0$ for every $C \in \mathcal{C}$. In fact, defining

$$S_\lambda C(x, y) = \frac{C(x, y)(1 - \lambda x) + \lambda xy}{1 + \lambda(y - C(x, y))} \quad (3.2)$$

for every $\lambda \in (0, 1]$, the following result holds.

Proposition 3.2. *For every $\lambda \in (0, 1]$, the transformation S_λ given by (3.2) maps $\mathcal{C}$ into $\mathcal{C}$. Moreover, it is bijective and continuous, has M as unique fixed point and fulfills*

$$\lim_{n \rightarrow \infty} d_\infty(S_\lambda^n C, M) = 0 \quad \text{for every } C \in \mathcal{C}.$$

Proof. We prove the result by expressing S_λ in terms of T_λ . To do so, first define the isometry $\Phi : [0, 1]^2 \rightarrow [0, 1]^2$ by $\Phi(x, y) = (1 - y, x)$. Obviously the push-forward μ_C^Φ of a doubly stochastic measure μ_C via Φ is again doubly stochastic, so we can view Φ also as a transformation on $\mathcal{C}$ and write $\Phi(C)$ for the copula corresponding to μ_C^Φ . Actually, this transformation corresponds to the flipping operation studied in [3].

It is straightforward to verify that $\Phi(C)(x, y) = y - C(y, 1 - x)$ for all $x, y \in [0, 1]$, $\Phi(M) = W$, and $\Phi : \mathcal{C} \rightarrow \mathcal{C}$ is bijective and continuous. The inverse Φ^{-1} is given by $\Phi^{-1}(C)(x, y) = x - C(1 - y, x)$. Now, it holds that

$$\Phi^{-1} \circ T_\lambda \circ \Phi(C)(x, y) = x - T_\lambda \circ \Phi(C)(1 - y, x) = x - \frac{x - C(x, y)}{1 + \lambda(y - C(x, y))} = S_\lambda C(x, y)$$

from which all the assertions follow immediately. $\square$

Figures 2 and 3 illustrate an example of iteration of the transformations T_λ and S_λ , respectively.

Figure 3: Image plot of $S_\lambda^n(W)$ for $n \in \{0, 1, 5, 20\}$, whereby $\lambda = 1/2$.

Acknowledgement: The first author acknowledges the support of the Free University of Bozen-Bolzano, Faculty of Economics and Management, via the project MAGDA.

The first and second author have been supported by the Ministerio de Ciencia e Innovación (Spain), under Research Project MTM2011-22394.

References

- [1] C. Alsina, A. Damas, and J. J. Quesada-Molina. Some functionals for copulas. *Int. J. Math. Math. Sci.*, 14(1):45–54, 1991.
- [2] C. Bernard, X. Jiang, and S. Vanduffel. A note on “Improved Fréchet bounds and model-free pricing of multi-asset options” by Tankov (2011). *J. Appl. Probab.*, 49(3):866–875, 2012.
- [3] B. De Baets, H. De Meyer, J. Kalická, and R. Mesiar. Flipping and cyclic shifting of binary aggregation functions. *Fuzzy Sets Syst.*, 160(6):752–765, 2009.
- [4] E. Di Bernardino and D. Rulliére. On certain transformations of Archimedean copulas: Application to the non-parametric estimation of their generators. *Dependence Modeling*, 1:1–36, 2013.
- [5] A. Dolati and M. Úbeda-Flores. Constructing copulas by means of pairs of order statistics. *Kybernetika (Prague)*, 45(6):992–1002, 2009.
- [6] F. Durante. A new class of symmetric bivariate copulas. *J. Nonparametr. Stat.*, 18(7-8):499–510, (2007), 2006.
- [7] F. Durante, J. Fernández-Sánchez, and R. Pappadà. Copulas, diagonals and tail dependence. *Fuzzy Sets Syst.*, in press, 2015.
- [8] F. Durante, J. Fernández-Sánchez, and C. Sempì. Multivariate patchwork copulas: a unified approach with applications to partial comonotonicity. *Insurance Math. Econom.*, 53:897–905, 2013.
- [9] F. Durante, R. Foschi, and P. Sarkoci. Distorted copulas: constructions and tail dependence. *Comm. Statist. Theory Methods*, 39(12):2288–2301, 2010.
- [10] F. Durante, A. Kolesárová, R. Mesiar, and C. Sempì. Semilinear copulas. *Fuzzy Sets Syst.*, 159(1):63–76, 2008.
- [11] C. Genest, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, and C. Sempì. A characterization of quasi-copulas. *J. Multivariate Anal.*, 69(2):193–205, 1999.

- [12] E. P. Klement, R. Mesiar, and E. Pap. Transformations of copulas. *Kybernetika (Prague)*, 41(4):425–434, 2005.
- [13] A. Kolesárová and R. Mesiar. On linear and quadratic constructions of aggregation functions. *Fuzzy Sets Syst.*, in press, 2015.
- [14] A. Kolesárová, R. Mesiar, and J. Kalická. On a new construction of 1-Lipschitz aggregation functions, quasi-copulas and copulas. *Fuzzy Sets Syst.*, 226:19–31, 2013.
- [15] R. Mesiar and A. Stupňanová. Open problems from the 12th International Conference on Fuzzy Sets Theory and Its Applications. *Fuzzy Sets Syst.*, in press, 2015.
- [16] F. Michiels and A. De Schepper. How to improve the fit of Archimedean copulas by means of transforms. *Statist. Papers*, 53(2):345–355, 2012.
- [17] P. Mikusiński and M. D. Taylor. Some approximations of n -copulas. *Metrika*, 72(3):385–414, 2010.
- [18] P. M. Morillas. A method to obtain new copulas from a given one. *Metrika*, 61(2):169–184, 2005.
- [19] R. B. Nelsen. *An introduction to copulas*. Springer Series in Statistics. Springer, New York, second edition, 2006.
- [20] R. B. Nelsen and M. Úbeda-Flores. The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas. *C. R. Math. Acad. Sci. Paris*, 341(9):583–586, 2005.
- [21] P. Tankov. Improved Fréchet bounds and model-free pricing of multi-asset options. *J. Appl. Probab.*, 48(2):389–403, 2011.
- [22] E. A. Valdez and Y. Xiao. On the distortion of a copula and its margins. *Scand. Actuar. J.*, 4:292–317, 2011.