

The geoid or quasigeoid – which reference surface should be preferred for a national height system?

Research Article

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Abstract:

Most European states use M. S. Molodensky's concept of normal heights for their height systems with a quasigeoid model as the reference surface, while the rest of the world rely on orthometric heights with the geoid as the zero-level. Considering the advances in data caption and theory for geoid and quasigeoid determinations, the question is which system is the best choice for the future. It is reasonable to assume that the latter concept, in contrast to the former, will always suffer from some uncertainty in the topographic density distribution, while Molodensky's approach to quasigeoid determination has a convergence problem. On the contrary, geoid and quasigeoid models computed by analytical continuation (e.g., rcr technique or KTH method) have no integration problem, and the quasigeoid can always be determined at least as accurate as the geoid. As the numerical instability of the analytical continuation is better controlled in the KTH method vs. the rcr method, we propose that any future height system be based on normal heights with a quasigeoid model computed similar to or directly based on the KTH method (Least squares modification of Stokes formula with additive corrections).

Keywords:

analytical continuation • geoid • normal height • orthometric height • quasigeoid

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1. Introduction

Today the Earth's surface and its geometric height above the selected reference ellipsoid (the geodetic height) can be regarded as known quantities as determined by satellite positioning, and, in particular from Global Navigation Satellite System (GNSS) data. However, the geoid, which for most countries is the primary vertical reference surface, cannot be provided from such data. For those countries the orthometric height (primarily determined by levelling) defines the height system as the height above the defined geoid model. For many countries the national height networks are rather sparse, and densifying heights by GNSS-levelling is a versatile technique. This technique needs the adopted geoid model, and along with that GNSS positioning improves, the quality of the geoid model has usually become the limiting factor for such height determination. The world-wide task of the geodetic community today to determine "the 1-cm geoid" is not easy, in particu-

lar in mountainous regions, as there is a major problem stemming from the geoid dependence on the only partly known topographic mass distribution, and this problem occurs also in determining orthometric heights.

In 1945 M. S. Molodensky (Molodensky et al. 1962) suggested substituting the geoid and orthometric height by the concepts of the quasigeoid and normal height, a brilliant idea to avoid the above problem with the topographic mass distribution. Since then former Soviet Union states and most of additional European states have changed their height systems to normal heights related with a quasigeoid surface.

In many countries today the qualities of the geoid and quasi-geoid models limit the use of GNSS-levelling, i.e. the technique to do levelling by GNSS and a gravimetric geoid model. Looking into the future, we may expect large improvements in GNSS technology for accurate positioning, which will increase the need for even more accurate geoid models. In parallel, the global and regional gravity

field data will be known with much higher accuracy and density than today, which will be of great benefit for accurate geoid and quasi-geoid modelling. This is particularly the case if such development goes hand in hand with the necessary improvements in geoid computational techniques with thorough corrections for all types of errors and optimal combination of various data types (cf. Ågren and Sjöberg 2012).

Let us now imagine that all such errors are reduced to nearly zero, but the uncertainty in the topographic density distribution remains. Is the normal height concept attached with the quasi-geoid then the right way to go? Or are there also other limitations in the two approaches that make this choice less evident? According to Vanicek et al. (2012) it is so, as Molodensky's method is based on a geometric technique of integrating gravity over the Earth's surface (or, more precisely a smoothed version of it; the telluroid), which is too rough a surface in many regions to be a practical tool for accurate quasigeoid determination. On the other hand, there could also be other, more attractive, alternatives to compute the quasi-geoid.

Below we will discuss the two future alternatives for a precise vertical reference surface: the geoid or the quasi-geoid? The answer to the question is definitely very important to most countries, as the longer they wait to change system (if this is the choice), the more complicated and costly it will certainly be. And, in any case, each country deserves being confident in its choice of height system for the future.

2. The geoid

The geoid is an important tool for geophysical interpretation, etc., but this aspect is not at stake in this article. Here we will only consider it as a technical reference surface for height systems.

Gravimetric geoid modelling is based on Stokes' formula (Stokes 1849), which integrates the gravity anomaly on a sphere, approximating the Earth's surface, to provide the geoid height, i.e. the height of the geoid above the reference ellipsoid. In modern techniques, Stokes' formula is modified in various ways, primarily for taking into account an Earth Gravitational Model (EGM) for representing the long wavelengths of the geoid, while, more or less, only short wavelengths are determined by surface gravity (e.g., Sjöberg 2003b).

The application of Stokes' formula requires a number of corrections to the primarily determined surface gravity anomaly data. As the formula is performed on a sphere, and it does not allow masses external to the sphere, the data must be a) corrected for the topographic effect on gravity anomaly (direct topographic effect on gravity anomaly), b) corrected for the atmospheric effect (direct atmospheric effect) and c) reduced from the Earth's surface to the sphere (the downward continuation, dwc, effect). After these reductions to the gravity anomaly, Stokes' formula yields the *co-geoid*, which needs to be corrected for the removed topographic and atmospheric effects (indirect topographic and atmospheric effects) and for the approximation of sea level by a sphere rather than

an ellipsoid (ellipsoidal effect) to provide the geoid height. It is important to remember that also in the modified Stokes formula, the EGM reduction to the sphere needs corrections for direct and indirect topographic and atmospheric effects.

The above technique for removing the effects of the external masses was originally called *regularization* of the geoid, resulting in the *co-geoid* or *regularized geoid* (Heiskanen and Moritz 1967, p. 289). Due to the mass shifts, the level surfaces have changed, which calls for two corrections (the primary and secondary indirect effects) to the co-geoid to finally reach the geoid. Today, this type of geoid determination is included in the more general term *remove-compute-restore (rcr) technique* for geoid determination. Although there are different versions of the rcr geoid estimator, it can generally be presented as follows:

$$\begin{aligned} \tilde{N}^{L,M} = & \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) [\Delta g^M + \delta\Delta g_{dir}^T + \delta\Delta g_{dir}^a \\ & + \delta\Delta g^{dwc} + \delta\Delta g_{dir}^e] d\sigma + c \sum_{n=0}^M \left(\frac{2}{n-1} \right) \Delta g_n^{EGM} \\ & + \delta N_{dir,M}^T + \delta N_l^T + \delta N_l^a + \delta N_l^e \end{aligned} \quad (1a)$$

where R is the mean Earth radius, γ_0 is normal gravity on the reference ellipsoid, $S^L(\psi)$ is the modified Stokes formula with geocentric angle ψ as argument, σ_0 is the cap of integration around the computational point,

$$\Delta g^M = \Delta g - \sum_{n=2}^M \Delta g_n^{EGM} \quad (1b)$$

and the remaining correction terms in the bracket $[\]$ are in order the direct gravity anomaly effects of the topography, atmosphere and ellipsoid-to-sphere, $\delta N_{dir,M}^T$ is the direct topographic effect on the geoid of the EGM, and the last three terms are the indirect effects on the geoid of the topography, atmosphere and sphere-to-ellipsoid correction. In practice, some of these terms are usually neglected. [Note that in Eq. (1) the direct topographic effect refers to the gravity anomaly and not to the gravity attraction, which is more common in the rcr models. This leads to that the commonly added secondary indirect effect does not apply.] The indices M and L are the maximum degrees of the EGM and the modification of Stokes formula, respectively.

To keep the direct and indirect topographic effects small, they usually do not act on the complete attraction of the topographic masses, but on its difference to a compensation mass model located on or below sea level. By this trick one achieves both the original need for the DTE in removing the effect of the forbidden topographic masses, and a reduction of its magnitude. This reduction is dependent on the choice of compensation model, e.g., Helmert's 2nd method of condensation. To avoid systematic errors in the geoid process, the same compensation model must be used also for the indirect topographic effects.

Alternatively, the modified Stokes formula can be computed preliminary without the gravity anomaly corrections above, and each correction is added afterward as a combined effect of direct and indirect effects. This is the case in the KTH approach called *Least Squares Modification of Stokes formula with Additive corrections* (LSMSA; Sjöberg 2003a and 2003b):

$$\tilde{N}^{L,M} = \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) \Delta g d\sigma + c \sum_{n=0}^M (Q_n^L + s_n) \Delta g_n^{EGM} + \delta N_{comb}^T + \delta N_{dwc} + \delta N_{tot}^a + \delta N_{tot}^e, \quad (2a)$$

where s_n and Q_n^L are so-called modification parameters (chosen to minimize the error of the geoid estimator) and the Molodensky truncation coefficients, respectively. Moreover,

$$\delta N_{comb}^T = \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) \delta \Delta g_{dir}^T d\sigma + \delta N_i^T \quad (2b)$$

$$\delta N_{dwc} = \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) \delta \Delta g_{dwc} d\sigma \quad (2c)$$

$$\delta N_{tot}^a = \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) \delta \Delta g_{dir}^a d\sigma + \delta N_i^a \quad (2d)$$

$$\delta N_{tot}^e = \frac{R}{4\pi\gamma_0} \iint_{\sigma_0} S^L(\psi) \delta \Delta g_{dir}^e d\sigma + \delta N_i^e \quad (2e)$$

are the additive corrections. Hence, the first row of Eq. (2a) is the modified Stokes formula, which uses the original EGM and gravity anomaly data. The second row of the equation consists of the additive corrections. For more details, see, e.g., Sjöberg (2003b).

Some advantages of the LSMSA method are that

- it uses least squares to minimize the effects of errors in the data and truncation.
- the method becomes more flexible, as the additive corrections can be added whenever the data for them becomes available. (It means that the repetition of the main computational steps can be avoided.)
- some additive corrections are much easier to compute than the corresponding direct and indirect effects.
- The direct and indirect effects are consistently combined into an effect on the geoid height. (This means also that the topographic compensation is meaningless.)

Assuming a constant topographic density ρ (e.g., 2.67 g/cm³) the combined topographic effect on the geoid height (the direct plus the indirect topographic effects), called the topographic bias by Sjöberg (2007a), can be determined by the simple formula

$$dN_{comb}^T = -\frac{2\pi G\rho}{\gamma_0} \left(H^2 + \frac{2}{3} \frac{H^3}{R} \right), \quad (3)$$

For Mount Everest this effect is of the order of 9.8 m.

Sjöberg (2004) used Eq. (3) to estimate the error in the geoid determination caused by the uncertainty in the topographic density. As the formula is directly proportional to the density, an uncertainty in this parameter of 10% may range to about 1 m for the highest mountain. At the elevations of 1 and 3 km the uncertainties become 1 and 11 cm, respectively. Although the assumption of a 10% error in density variation from the normal mean value may be pessimistic, these figures still show that this error source could be significant already for the 1-cm geoid model in mountainous regions. These error estimates are larger than those reported by Martinec (1998) and Vanicek et al. (2012), and the reason for this discrepancy is that our estimates are based on the total topographic effect, while the cited estimates only refer to the uncertainty in either the DTE or the (primary) indirect effect, which are only part of the error and they vary with the chosen method for reduction of gravity.

Theoretically, the computational results in the rcr and LSMA techniques should agree, but numerically significant differences may occur. For a comparative discussion, see Sjöberg (2005). The most important difference for this study is that the additive correction for the dwc effect to the geoid height by the LSMSA method is numerically much more stable than the dwc effect on the gravity anomaly used in the rcr technique, as discussed in Sect. 4.

Interestingly, the LSMSA technique initially uses a Stokes integral with surface gravity anomaly, and this integral is the same as the zero-order solution in the quasigeoid determination by Molodensky's technique (Molodensky et al. 1962).

3. The determination of the quasi-geoid

3.1. Molodensky's approach

Molodensky's original approach to quasigeoid height determination (Molodensky et al. 1962) is a geometric-physical method that deals with solving a Fredholm integral equation of the second kind over the known surface of the Earth by successive approximations, an iterative procedure whose convergence is doubtful unless the Earth's surface is sufficiently smoothed. For the real topography the solution is definitely divergent. Moritz (1980, Sect. 47) concludes that the Molodensky series can be regarded as an asymptotic series, implying that the iteration improves up to some high order, say n_{max} , beyond which the series diverges. If this is the case, the truncation error below order n_{max} could well be negligible, suggesting that Molodensky's approach is practical. On the other hand, the Earth's surface is partly too rough to allow a sufficiently smooth integration, making the series divergent also at low

orders of iteration. This is in agreement with the arguments used by Vanicek et al. (2012).

3.2. Remove-compute-restore technique

The quasi-geoid can also be determined by the rcr technique, which goes back to the method of analytical dwc of the gravity anomaly to sea level (Bjerhammar 1962 and 1963). The main difference to the geoid determination is that Stokes formula is replaced by the extended Stokes formula $S(r_P, \psi)$ (e.g., Heiskanen and Moritz 1967, p. 320). The resulting formula for the height anomaly (ζ) becomes:

$$\begin{aligned} \tilde{\zeta}^{L,M} = & \frac{R}{4\pi\gamma} \iint_{\sigma_0} S^L(r_P, \psi) [\Delta g^M + \delta\Delta g_{dir}^T + \delta\Delta g_{dir}^a \\ & + \delta\Delta g^{dwc} + \delta\Delta g_{dir}^e] d\sigma + c \sum_{n=0}^M \left(\frac{2}{n-1} \right) \left(\frac{R}{r_P} \right)^{n+2} \Delta g_n^{EGM} \\ & + \delta\zeta_{dir}^{T,M} + \delta\zeta_l^T + \delta\zeta_l^a + \delta\zeta_l^e \end{aligned} \quad (4)$$

where $S^L(r_P, \psi)$ is the extended, modified Stokes formula and γ is normal gravity at normal height.

As the integration now is carried out over a sphere (in contrast to Molodensky's approach above), there is no convergence problem of the integral. The only substantial problem is the same as with the geoid determination, namely to compute the dwc effect of the gravity anomaly from the surface to the internal (Bjerhammar) sphere. Thus we conclude that from a practical point of view this computational method is at least as simple as that for geoid determination by the rcr technique.

3.3. LSMSA technique

The LSMSA technique can also be used for computing the quasi-geoid height by the formula

$$\begin{aligned} \tilde{\zeta}^{L,M} = & \frac{R}{4\pi\gamma} \iint_{\sigma_0} S^L(r_P, \psi) \Delta g d\sigma + c \sum_{n=0}^M (Q_n^L + S_n) \left(\frac{R}{r_P} \right)^{n+2} \Delta g_n^{EGM} \\ & + \delta\tilde{\zeta}_{dwc} + \delta\zeta_{tot}^a + \delta\zeta_{tot}^e \end{aligned} \quad (5)$$

Note that there is no topographic effect in this equation, because the direct and indirect topographic effects cancel each other. Alternatively, the formula is written as a Stokes integral at point level (Ågren 2004, Sjöberg 2007b, Ågren et al. 2009):

$$\begin{aligned} \tilde{\zeta}^{L,M} = & \frac{R}{4\pi\gamma} \iint_{\sigma_0} S^L(\psi) \Delta g d\sigma + c \sum_{n=0}^M (Q_n^L + S_n) \left(\frac{R}{r_P} \right)^{n+2} \Delta g_n^{EGM} \\ & + \delta\tilde{\zeta}_{dwc} + \delta\zeta_{tot}^a + \delta\zeta_{tot}^e \end{aligned} \quad (6)$$

which uses the original (modified) Stokes formula and a slightly different dwc effect, denoted $\delta\tilde{\zeta}_{dwc}$. For further details, see the next section, where the dwc problem is paid a special attention.

4. The dwc problem

Here we discuss the downward continuation effects by the rcr and LSMSA methods.

4.1. The dwc effect in the rcr technique

For the rcr technique the dwc problem is exactly the same for geoid and quasigeoid determination, as the problem is that of analytically continuing the surface gravity anomaly to sea level approximated by the computational sphere used in Stokes' formula.

The downward continuation of the gravity anomaly to the sphere can be approximated by a Taylor series, but this approach has limited use as the surface radial derivatives can hardly be determined sufficiently accurate. (One way to achieve the derivatives is by Proposition 1 of the Appendix.)

A mathematical rigorous model for the dwc problem is given by Poisson's integral formula, as first proposed by Bjerhammar (1962) and (1963). As the downward continued gravity anomaly on the sphere is the unknown under the integral, the formula is a Fredholm integral equation of the first kind and the problem is improperly posed (e.g., Payne 1975 and Hansen 1998). In the present case this implies that a detailed solution will be sensitive to errors in the data, and the propagated uncertainty will increase with topographic elevation and the resolution requested for the solution. Although the rcr technique uses topographically reduced gravity anomaly data as input (while Bjerhammar used original surface gravity anomaly data), the numerical ill-conditioning of the solution is a severe problem for a high resolution discretized grid step size, e.g., of 30"x 60" as demonstrated by Martinec (1998, Sect. 8.6.4) using data from the Canadian Rocky Mountains. However, as suggested by Novak et al. (2001) and demonstrated by Kingdon and Vanicek (2011), the ill-conditioning can be mitigated through regularization, and even without regularization Huang (2002) claimed that the geoid error only reaches a few centimetres in regions with elevations over 3 kilometres. Goli and Najafi (2011) showed that planar approximation of the Poisson equation can considerably speed up the laborious computational process at the prize of about 1 cm additional uncertainty in the geoid height.

4.2. The dwc problem by the LSMSA method

In the LSMSA approach the downward continuation effect on the gravity anomaly is directly estimated on the geoid or height anomaly. This approach is advantageous from the point of view that it avoids the ill-conditioning in the dwc of the gravity anomaly by Poisson's integral. Here we limit the discussion to that on the height anomaly when considering the original approach with a global set of gravity anomalies and no contribution from an EGM.

The preliminary estimator of the height anomaly is Eq. (5) with the dwc error (here considered for the limiting case with $\sigma_0 = \sigma$; for a

more detailed study, see Sjöberg 2007b)

$$d\zeta_{dwc} = \frac{R}{4\pi\gamma} \iint_{\sigma} S(r_P, \psi) [\Delta g^* - \Delta g] d\sigma, \quad (7)$$

which can be split into "the spherical shell effect" ($d\zeta_{dwc}^1$) and "the terrain effect" ($d\zeta_{dwc}^2$):

$$d\zeta_{dwc} = d\zeta_{dwc}^1 + d\zeta_{dwc}^2, \quad (8a)$$

where

$$d\zeta_{dwc}^1 = \frac{R}{4\pi\gamma} \iint_{\sigma} S(r_P, \psi) [\Delta g^* - \Delta g(r_P, Q)] d\sigma_Q \quad (8b)$$

and

$$d\zeta_{dwc}^2 = \frac{R}{4\pi\gamma} \iint_{\sigma} S(r_P, \psi) [\Delta g(r_P, Q) - \Delta g] d\sigma_Q. \quad (8c)$$

Here Δg^* is the downward continued surface gravity anomaly. By expanding $S(r_P, \psi)$, Δg^* and $\Delta g(r_P, Q)$ as Laplace harmonic series and substituting into Eq. (7) and considering the following relation between the height anomaly harmonic (ζ_n) and gravity anomaly harmonic (Δg_n^*):

$$\zeta_n = \frac{R\Delta g_n^*}{\gamma(n-1)} \left(\frac{R}{r_P}\right)^{n+1}, \quad (9)$$

the spherical shell effect becomes

$$d\zeta_{dwc}^1 = \sum_{n=2}^{\infty} \zeta_n \left[1 - \left(\frac{R}{r_P}\right)^{n+2} \right]. \quad (10)$$

This formula can be applied numerically by utilizing a high degree EGM such as EGM2008 for estimating ζ_n . Alternatively one may expand the term $(R/r_P)^{n+2}$ as a Taylor to end up with the (truncated) series (Sjöberg 2007b)

$$d\zeta_{dwc}^1 \approx \frac{H_P \Delta g_P}{\gamma} + 3\zeta_P \frac{H_P}{r_P} + \frac{H_P^2}{2\gamma} \left(\frac{\partial \Delta g}{\partial H}\right)_P - \frac{H_P^2 \Delta g_P}{r_P \gamma} - 3\zeta_P \left(\frac{H_P}{r_P}\right)^2, \quad (11)$$

where the second and last terms are not significant at the 1 cm and one mm levels, respectively.

The terrain effect can also be estimated by a Taylor series (here limited to first order):

$$d\zeta_{dwc}^2 \approx \frac{R}{4\pi\gamma} \iint_{\sigma} S(r_P, \psi) (H_P - H_Q) \left(\frac{\partial \Delta g}{\partial H}\right)_Q d\sigma_Q \quad (12)$$

Eqs. (10)-(12) can be applied numerically without problems with respect to the roughness of the terrain. However, what could be required in rough terrain is that the Taylor series in Eq. (12) is extended to second or higher order (which should not imply a problem of convergence). An integral formula for obtaining higher derivatives of the gravity anomaly is given in Appendix.

We now consider the dwc effect in the slightly different approach for height anomaly by Eq. (6). Again we limit the study here to the limiting case with a global Stokes integration. Then the dwc effect can be written (Sjöberg 2007b)

$$d\tilde{\zeta}_{dwc} = \frac{R}{4\pi\gamma} \iint_{\sigma} [S(r_P, \psi) \Delta g^* - S(\psi) \Delta g] d\sigma, \quad (13)$$

which can be split into the two components

$$\begin{aligned} d\tilde{\zeta}_{dwc}^1 &= \frac{R}{4\pi\gamma} \iint_{\sigma} [S(r_P, \psi) \Delta g^* - S(\psi) \Delta g(r_P, Q)] d\sigma_Q \\ &= \frac{H_P}{r_P} \zeta_P, \end{aligned} \quad (14a)$$

and

$$\begin{aligned} d\tilde{\zeta}_{dwc}^2 &= \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi) [\Delta g(r_P, Q) - \Delta g] d\sigma_Q \\ &\approx \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi) (H_P - H_Q) \left(\frac{\partial \Delta g}{\partial H}\right)_Q d\sigma_Q. \end{aligned} \quad (14b)$$

Slightly different forms of these equations were derived by Ågren (2004), who substituted the factor γ^{-1} by the approximation $\gamma_0^{-1} (1 + 2H_P/r_P)$ in Eq. (6) and the subsequent equations. See Ågren et al. (2011).

5. Conclusions and final remarks

It is obvious that Molodensky's geometric/physical approach to determine the height anomaly cannot be applied with high accuracy in rough terrain due to convergence problems in the Molodensky series.

The rcr technique has a problem in the analytical downward continuation of the gravity anomaly in rough terrain. Hence, this is a problem both in geoid and quasigeoid determination.

In contrast to the rcr technique, the LSMSA method computes directly the dwc effect on the geoid or quasigeoid height. This procedure is much more stable than computing the dwc effect on the gravity anomaly. In Sect. 4 we have shown that the dwc effect on the height anomaly can be practically handled without any convergence problem related with the terrain. Considering also that the determination of the geoid (in contrast to the quasigeoid) requires information on the topographic density distribution (which is frequently not well known), we conclude that the height anomaly can

be determined more accurately than the geoid. As there is a similar problem for the uncertainty in the topographic density distribution in determining orthometric heights (but not for normal heights), we conclude that a normal height system is the best choice for a future height system. Once the normal gravity field is defined, the normal heights and the quasigeoid can be determined without any error stemming from the topographic mass distribution, and the quasigeoid can be estimated more precisely than the geoid as the reference surface.

One aspect on orthometric heights vs. normal heights is also that the latter are smoother (as they are geopotential numbers divided by mean gravity and mean normal gravity, respectively, and normal gravity is the smoother of the two quantities), which makes sense, e.g., when interpolating between data points. On the other hand, the geoid, being an equipotential surface of the Earth's gravity field, is smoother than the quasigeoid surface.

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The following proposition was derived by Sjöberg (2007b).

APPENDIX

From Heiskanen and Moritz (1967, p. 38) one obtains the radial derivative of the disturbing potential T (or any harmonic function) on the sphere of radius R by the following surface integral:

$$\left(\frac{\partial T}{\partial r}\right)_P = -\frac{T_P}{R} + \frac{R^2}{2\pi} \iint_{\sigma} \frac{T - T_P}{l_0^3} d\sigma, \quad (\text{A1})$$

where

$$l_0 = 2R \sin(\psi/2). \quad (\text{A2})$$

Notation: $g^{(k)} = \partial^k \Delta g / \partial r^k$.

Proposition .1.

Under spherical approximation it holds that

$$g_P^{(k+1)} = -\frac{k+1}{r_P} g_P^k + \frac{1}{16\pi r_P} \iint_{\sigma} \frac{g^{(k)} - g_P^{(k)}}{\sin^3(\psi/2)} d\sigma. \quad (\text{A3})$$

Proof. Under spherical approximation it holds that

$$\Delta g = -\frac{\partial T}{\partial r} - 2\frac{T}{r}, \quad (\text{A4})$$

from which follows that $U = r^k g^{(k)}$ is a harmonic function. Hence by putting $U = T$ and $R = r_P$ in Eq. (A1) the proposition follows. \square