New solutions for the geoid potential W_0 and the Mean Earth Ellipsoid dimensions

Research Article

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Abstract:

Earth Gravitational Models (EGMs) describe the Earth's gravity field including the geoid, except for its zero-degree harmonic, which is a scaling parameter that needs a known geometric distance for its calibration. Today this scale can be provided by the absolute geoid height as estimated from satellite altimetry at sea.

On the contrary, the above technique cannot be used to determine the geometric parameters of the Mean Earth Ellipsoidal (MEE), as this problem needs global data of both satellite altimetry and gravimetric geoid models, and the standard technique used today leads to a bias for the unknown zero-degree harmonic of the gravimetric geoid height model. Here we present a new method that eliminates this problem and simultaneously determines the potential of the geoid (W_0) and the MEE axes. As the resulting equations are non-linear, the linearized observation equations are also presented.

Keywords:

Geoid datum • global vertical datum • Mean Earth Ellipsoid • reference ellipsoid

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1. Introduction

The level surface of the Earth's gravity field defined by the undisturbed sea level is the Gauss-Listing definition of the geoid (Gauss 1828; Listing 1873). Choosing the geoid as the global vertical datum (GVD) implies that the datum is defined by the potential (W_0) of this particular level surface of the Earth's gravity field.

Frequently, the normal potential U_1 at the selected reference ellipsoid; e.g., Geodetic Reference System 1980 (GRS80), is defined to be equal to that of the geoid, which is not well known. Then the problem is that U_1 will not be precise enough for today's need for defining W_0 , as the data has considerably improved and mean sea level has been increasing by the order of 1.7 mm/year in the mean during the 1900s and accelerating to more than 3 mm/yr. today (e.g., Nicholls and Cazenave 2010). That is, although U_1 may be kept fixed to that of GRS80 reference system, the geopotential at the geoid (W_0) frequently needs a realization that better agrees with the Gauss-Listing definition.

The event of satellite altimetry in the 1970s provided a tool for the realization of a GVD as being the equipotential surface of the Earth's gravity field that minimizes the sea surface topography (SST) all over the oceans in a least squares sense (Mather 1978). This leads to Approach I below, which implies a direct integration of satellite altimetry derived sea surface topography (SST; frequently also denoted Dynamic Ocean Topography) combined with the potential of an Earth Gravitational Model (EGM) all over the oceans. In contrast, Approach II consists in using the same data to first determine the size of the axes of the globally best fitting ellipsoid to the geoid surface (called the Mean Earth Ellipsoid; MEE; Heiskanen and Moritz 1967, p. 214), followed by determining W_0 from the result. A major problem with Approach II is that satellite altimetry is only successful over the oceans, while the method requires global data. Sanchez (2012) reviews the development in the field with lots of references.

In Sections 2 and 3 the two approaches will be reviewed and a short discussion is provided on some of their problems, and in Sect. 4 the second approach is presented under the consideration that that

the zero-degree harmonic for the EGM derived geoid model is unknown. This implies also that while the above approaches assume that both the geocentric gravitational constant and the Earth's mean daily angular velocity are known (fixed), we will assume that the former constant is only approximately known. Sect. 5 concludes the paper.

2. Approach I: Direct determination of W_0 from satellite altimetry and an EGM

2.1. Geometric and gravimetric geoid heights

By satellite positioning the geodetic height h of the Earth's surface above the reference ellipsoid can be determined. Assuming also that the orthometric height (H) is known, the geometric determination of the geoid height becomes

$$N^h = h - H. \quad (1)$$

For land areas this technique for geoid height determination is usually called GNSS/levelling, where h is determined by GNSS technology and H is the orthometric height determined by levelling and gravity. At sea h is the geodetic height of mean sea level determined from satellite altimetry, while H is the SST, which practically is either ignored, derived by some oceanographic method or estimated from satellite altimetry and a preliminary geoid model. Importantly, for land applications Eq. 1 suffers from inherited systematic errors, primarily biases, in the levelling networks, which make the formula less useful (or even useless) for solving the GVD problem (but useful for transformations from the GVD to local height systems; e.g., Sjöberg 2011). For ocean areas Eq. 1 is most important despite of the fact that the SST is frequently unknown and simply neglected. This is because, except for the long-wavelength gravity field features as determined by satellite data, there is very sparse gravity related data available from other sources at sea.

As an alternative, the geoid height can also be estimated *gravimetrically* from an EGM, such as EGM2008 (Pavlis et al. 2012). Neglecting the atmosphere, the Earth's gravity potential outside the topographic masses can be represented as an external type series of spherical harmonics:

$$\begin{aligned} \tilde{W}(r, \theta, \lambda) = \\ \frac{GM_1}{r} \left[1 + \sum_{n=2}^{n_{\max}} \sum_{m=-n}^n A_{nm} \left(\frac{R}{r} \right)^{n+1} Y_{nm}(\theta, \lambda) \right] + \Phi(r, \theta, \lambda), \end{aligned} \quad (2)$$

where (r, θ, λ) are the geocentric radius, co-latitude and longitude of the computational point, GM_1 is an adopted value for the geocentric gravitational constant, Φ is Earth's rotational potential, each spectral potential component A_{nm} is determined from a global set of gravity related data by harmonic analysis up to the chosen degree and order $n_{\max}$ at the reference radius R and $Y_{nm}(\theta, \lambda)$ is the

surface spherical harmonic of degree n and order m . One notices that there are no first-degree terms in Eq. (2), which implies that the origin of the coordinate system is selected at the Earth's gravity centre. In a similar way a normal gravity field potential can also be expressed as a harmonic series of a gravitational potential plus the rotation potential:

$$\begin{aligned} U(r, \theta, \lambda) = \\ \frac{GM_1}{r} \left[1 + \sum_{n=2}^{n_{\max}} \left(\frac{R}{r} \right)^n \sum_{m=-n}^n B_{nm} Y_{nm}(\theta, \lambda) \right] + \Phi(r, \theta, \lambda), \end{aligned} \quad (3)$$

whose potential is constant (say, U_1) on a chosen (level) reference ellipsoid with mass M_1 . Then one obtains the following series for the *disturbing potential*:

$$\begin{aligned} T(r, \theta, \lambda) &= \tilde{W}(r, \theta, \lambda) - U(r, \theta, \lambda) \\ &= \frac{GM_1}{r} \sum_{n=2}^{\infty} \left(\frac{R}{r} \right)^n \sum_{m=-n}^n C_{nm} Y_{nm}(\theta, \lambda), \end{aligned} \quad (4)$$

where $C_{nm} = A_{nm} - B_{nm}$. In practice, one tries to choose the constant GM_1 as the best available estimate of the geocentric gravitational constant. From now on the disturbing potential estimate from the EGM in Eq. (4) will be denoted by T^{EGM} .

A direct way to estimate the geoid potential (W_0) is to apply Eq. (2) at the radius vector r_g of the geoid (e.g., Dayoub et al. 2012):

$$\hat{W}_g = \hat{W}_0 = \tilde{W}(r_g, \theta, \lambda), \quad (5)$$

where

$$r_g = r_g(\theta, \lambda) = r_1(\theta) + N(\theta, \lambda). \quad (6)$$

Here $r_1(\theta)$ is the geocentric radius vector of the defined reference ellipsoid and $N(\theta, \lambda)$ is the related geoid height (which we frequently will abbreviate with N), which can be estimated geometrically from satellite altimetry in ocean areas according to Eq. (1).

Alternatively, one may start from applying Bruns' formula for the normal potential at the geoid (U_g ; Hesikainen and Moritz 1967, p. 84):

$$T_g = W_0 - U_g = W_0 - (U_1 - \gamma_1 N), \quad (7)$$

which leads to (see also Sacerdote and Sanz 2004)

$$W_0 = U_1 - \gamma_1 N + T_g, \quad (8)$$

where U_1 (= constant) and γ_1 are the normal potential and gravity on the reference ellipsoid, respectively. [Actually, $\gamma_1 = \gamma_1(\theta)$, but we will frequently just use the short notation. In practise, $T^{EGM} = T^{EGM}(r_1(\theta), \theta, \lambda)$ can be used for representing T_g ,

i.e. the approximation error of using $r_1(\theta)$ for r_g is usually negligible.] From Eq. (8) it thus follows that the geoid height above the reference ellipsoid is given by

$$N = \frac{T^{EGM}}{\gamma_1} - \frac{\Delta W_0}{\gamma_1}, \quad (9)$$

where $\Delta W_0 = W_0 - U_1$. Eq. (9) shows that in general the geoid height

$$N^{EGM} = \frac{T^{EGM}}{\gamma_1}, \quad (10)$$

determined by the EGM, lacks the unknown correction $-\Delta W_0/\gamma_1$, which must be determined from geometric data (e.g., by satellite altimetry at sea; see Sect. 2.2). For a detailed discussion on the determination of the absolute geoid height from an EGM, see Smith (1998).

It is important to remember that on the continents T_g is the disturbing potential inside the topographic masses, and its computation therefore needs a correction for the topography. That is, the harmonic series for the geoid height in Eq. 2a needs a correction for the analytical downward continuation error or topographic bias of T_g^{EGM} (Sjöberg 1977 and 2007; Martinec 1998, Sects. 7.3-7.4; Ågren 2004), which is -5 cm for the zero-degree harmonic (Sjöberg 2001).

2.2. Direct estimates of W_0

Equations (5) and (8) are the bases for the direct determination of the geoid potential. As one estimator $\hat{W}_g$ of Eq. (5) is directly averaged over the ocean areas. In two other applications one may take advantage of the geodetic height determined by satellite altimetry by Eq. (1) and the disturbing potential (determined by an EGM) at the surface point on the undisturbed sea-level (assumed to be the geoid surface), possibly corrected for SST, yielding the following result for a first order Taylor expansion (e.g., Sacerdote and Sansó 2004):

$$W_0 = U_g + T_g = U_1 - \gamma h_g + T_g^{EGM}, \quad (11)$$

where $h_g = h - SST = N^h$ is the geoid height determined from satellite altimetry. By taking the mean value over the ocean area (σ_1) of such point-wise estimates for W_0 , one obtains the following estimators of the geopotential at the geoid:

1. Averaged geopotential on the ocean

$$\hat{W}_0 = U_1 + \frac{1}{\sigma_1} \iint_{\sigma_1} (T_g^{EGM} - \gamma_1 h_g) d\sigma \quad (12)$$

and

2. Minimized SST (Sacerdote and Sansó 2004):

$$\hat{W}_0 = U_1 + \frac{\iint_{\sigma_1} \left[\frac{T_g^{EGM} - \gamma h_P}{\gamma_P^2} \right] d\sigma}{\iint_{\sigma_1} \left[\frac{1}{\gamma_P^2} \right] d\sigma}, \quad (13)$$

where subscript P refers to surface point P and $\bar{\gamma}_P$ is the mean normal gravity along the normal height at P .

In a similar way the estimator of Eq. (5) can be averaged over the ocean areas.

2.3. Discussion

If the integration area of Eq. (12) were the whole sphere, the disturbing potential determined by the EGM would vanish and the estimator would be U_1 minus the global average of the geometric geoid height (times γ). In all other cases the solution depends on the EGM, which includes both commission and omission errors. For instance, when σ_1 is the area of the oceans, one can expect that the rms geoid error of EGM2008 complete to degree 2159 is about 5-6 cm (Pavlis et al. 2012), corresponding to an uncertainty in W_0 of about $0.5 - 0.6 \text{ (m/s)}^2$, and this value is in agreement with nowaday's accuracy in determining W_0 (Sanches 2012).

3. Approach II: Joint determination of W_0 and the MEE parameters

3.1. Introduction

Here the geometric approach to determine W_0 will be presented under the assumption that the mean angular velocity of the Earth's daily rotation (ω) is known, and the problem is to estimate both the dimensions [i.e., semi-major and -minor axes a and b (or eccentricity e) of the globally best fitting ellipsoid (= MEE)] as well as the geoid potential W_0 in a joint adjustment from the available geoid surface estimates derived from satellite altimetry and an EGM. The ideal normal potential U_0 of the MEE is given by four parameters, namely a , b , GM and ω (e.g., Heiskanen and Moritz 1967, Eq. 2-61):

$$U_0 = \frac{GM}{\sqrt{a^2 - b^2}} \arctan \frac{\sqrt{a^2 - b^2}}{b} + \frac{\omega^2 a^2}{3}, \quad (14)$$

or, by using the substitution $\arctan \left(e/\sqrt{1 - e^2} \right) = \arcsin(e)$, one obtains

$$U_0 = \frac{GM}{ae} \arcsin(e) + \frac{\omega^2 a^2}{3}, \quad (15)$$

and this potential is also the best choice for W_0 . As will be shown, the geocentric gravitational constant GM is not part of the adjustment, (but it could be indirectly estimated from the adjustment results). The general computational procedure is outlined below.

Similar with Sect. 2, first the normal potential U_1 of a preliminary reference ellipsoid with geometric parameters a_1 and $b_1 = a_1 \sqrt{1 - e_1^2}$ and geocentric gravitational constant GM_1 is given by:

$$U_1 = \frac{GM_1}{a_1 e_1} \arcsin(e_1) + \frac{\omega^2 a_1^2}{3}, \quad (16)$$

where all parameters are chosen. The radius vector of the surface of this reference ellipsoid can be written

$$r_1(\beta) = a_1 \sqrt{1 - e_1^2 \sin^2 \beta}, \quad (17)$$

where β is the reduced latitude.

According to Heiskanen and Moritz (1967, p. 214), the MEE, is the ellipsoid, whose mass is the same as that of the real Earth (requires that M_1 is assumed to be the Earth's mass), and the axes are such that the global mean square of the geoid height (N) is a minimum:

$$\frac{1}{4\pi} \iint_{\sigma} N^2 d\sigma = \min.(a, e), \quad (18)$$

where σ is the unit sphere. The best known value for GM has a standard error of the order of $0.8 \text{ m}^3 \text{ s}^{-2}$ (Groten 2004), which corresponds to uncertainties in W_0 and the geoid height of $0.1 \text{ m}^2 \text{ s}^{-2}$ and 1 cm, respectively. As the present uncertainty in W_0 is about $0.5 \text{ m}^2 \text{ s}^{-2}$ (e.g. Sanchez 2012), the fixing of GM to the best known value may improve W_0 by adjusting just for the ellipsoidal geometry parameters by the following geometric approach. This implies that once a and e have been fixed, U_0 (and thereby $W_0 = U_0$) follows from Eq. (15).

However, to be more precise in the approach to follow, it is not the integral in Eq. (18) that is to be minimized, but it is the mean square discrepancy between the radius vector of the geoid surface estimated by $r_1(\beta) + N$, where $r_1(\beta)$ is the radius vector of the reference ellipsoid related with the geoid estimate N , and the radius vector $r_E(a, e, \beta)$ of a general reference ellipsoid that should be optimized. Hence, mathematically the problem could be expressed:

$$J = \frac{1}{4\pi} \iint_{\sigma} [r_1(\beta) + N - r_E(a, e, \beta)]^2 d\sigma = \min.(a, e), \quad (19a)$$

where

$$r_E(\beta) = a \sqrt{1 - e^2 \sin^2 \beta}. \quad (19b)$$

Once the ellipsoidal parameters a and e of the MEE have been fixed by solving Eq. (19a), the normal potential at the MEE, i.e. U_0 of Eq. (15), can be computed, provided that GM is (sufficiently well) known, and this value should also be the estimate for the geopotential value at the geoid, i.e.

$$W_0 = U_0. \quad (20)$$

However, one problem with this approach is that the present-day uncertainty in GM contributes to about 20% of the uncertainty in W_0 (see Groten 2004). In addition, the main problem to optimize the target function J is that the absolute geoid height is not well known globally, but there are only relative geoid models, such

as those expressed by an EGM. However, as stated above, satellite altimetry can provide a geometric estimate of the absolute geoid height over the oceans under the assumption that the SST is known with sufficient accuracy, but such an incomplete integration area for J in Eq. (19a) would only lead to the best fitting reference ellipsoid and W_0 estimated for the ocean areas. Dayoub et al. (2012) applied the above technique, and they compared the preliminary geoid surfaces given by EGM2008 and a satellite altimetry model in coastal areas and concluded that the two surfaces agree well (without specifying the magnitude of the agreement), and they directly filled-in the geoid heights for the land areas by EGM2008 geoid heights. However, the EGM derived geoid height needs the correction $-\Delta W_0/\gamma_1$ of Eq. (9), which must be estimated. If the correction is fixed to a preliminary value (which apparently was the case in Dayoub 2012, Dayoub private com.), it means that also $W_0 = U_1 + \Delta W_0$ has been (more or less) fixed. However, to avoid fixing W_0 to an a priori value, the problem can be solved by augmenting the target function of Eq. (19a) by the unknown parameter ΔW_0 . This implies that a , e and ΔW_0 are determined in a combined adjustment.

3.2. The combined adjustment approach

In the previous section the target function J was based on the assumptions that the geocentric gravitational constant is known and agrees with that of the normal potential U_1 , and the estimated geoid surface is continuous and known all over the Earth. In reality, we have seen that neither of these assumptions is correct. From satellite altimetry the geoid height is known only over the oceans, and the EGM geoid height lacks the term $\Delta W_0/\gamma_1$ as presented in Eq. (9). In the approach that follows we are not primarily concerned with the unknown GM , but it is sufficient to consider the extra unknown $x = -\Delta W_0 = U_1 - W_0$. We will assume that the reference ellipsoids for N^h and N^{EGM} are the same (with geometric parameters a_1 and e_1), and radius vector given by Eq. (17). Then the augmented target function reads:

$$I = p I_1(a, e) + (1 - p) I_2(x, a, e) + I_3(x, a, e) = \min(x, a, e), \quad (21a)$$

where

$$I_1(a, e) = \iint_{\sigma_1} [N^h + r_1(\beta) - r_E(a, e, \beta)]^2 d\sigma \quad (21b)$$

$$I_2(x, a, e) = \iint_{\sigma_1} [x/\gamma_1 + N^{EGM} + r_1(\beta) - r_E(a, e, \beta)]^2 d\sigma \quad (21c)$$

$$I_3(x, a, e) = \iint_{\sigma_2} [x/\gamma_1 + N^{EGM} + r_1(\beta) - r_E(a, e, \beta)]^2 d\sigma. \quad (21d)$$

Here σ_1 and σ_2 are those parts of the unit sphere that are covered by ocean and land, respectively, $x = -\Delta W_0 = U_1 - W_0$ (see

Eq. 9) and $r_E(a, e, \beta)$, given by Eq. (20), is the radius vector of the MEE, whose parameters a and e are unknown and p is a fixed number in the range $0 \leq p \leq 1$ that weighs the contributions from N^h and N^{EGM} over the ocean areas.

If p is set to 0, implying that only EGM data are used, it is shown in Sect. 4.1.1 that the solution is singular. This is obvious, as in this case one tries to solve the problem with only relative geoid heights given by the EGM. Alternatively, if $p = 1$ (i.e. only the satellite altimetry derived geoid height is employed over the oceans, while the EGM is utilized only over land), the solution discards the information from the EGM over the oceans. On the contrary, below we suggest that in the application of Eqs. (21a) - (21d) the choice of p should be based on the a priori variances κ_1^2 and κ_2^2 of the satellite altimetry and EGM derived geoid heights, respectively, yielding $p = \kappa_2^2 / (\kappa_1^2 + \kappa_2^2)$.

The least squares condition for the unknowns x , a and e , as specified by Eq. (21a), is satisfied by the three equations

$$\frac{\partial I}{\partial x} = 0, \frac{\partial I}{\partial a} = 0 \text{ and } \frac{\partial I}{\partial e^2} = 0, \quad (22)$$

from which the unknowns can be determined, provided that the equations are independent.

If $\hat{x}$ is the solution for x , and U_1 is the a priori value for the ellipsoidal normal potential, the geoid potential estimate finally follows from

$$\hat{W}_0 = U_1 - \hat{x}, \quad (23)$$

and by re-inserting the estimates for $W_0 = U_0$, a and e into Eq. (15), a new estimate for GM becomes

$$\begin{aligned} G\hat{M} &= \left(\hat{W}_0 - \hat{a}^2 \omega^2 / 3 \right) \frac{\hat{a} \hat{e}}{\arcsin(\hat{e})} \\ &\approx \left(\hat{W}_0 - \hat{a}^2 \omega^2 / 3 \right) \hat{a} \left(1 - \frac{\hat{e}^2}{6} \right), \end{aligned} \quad (24)$$

but this estimate is probably poor compared to estimates based on satellite laser ranging, etc. (e.g. Ries et al. 1992).

This concludes the principle of the combined approach.

The assumptions, data, the problem and its solution can be summarized as follows:

Given (fixed parameters): ω and a reference/level ellipsoid with dimensions a_1 and $b_1 = a_1 \sqrt{1 - e_1^2}$ as well as a normal gravity field with normal potential $U_1 = \text{constant}$ and normal gravity $\gamma_1 = \gamma_1(\beta)$ on the surface of the ellipsoid.

Observations: N^{EGM} (globally) and N^h (on the oceans; σ_1); both types of geoid heights refer to the defined reference ellipsoid.

Problem: Minimize the global mean square difference between radius vectors of the surfaces of the geoid and an arbitrary reference ellipsoid (with parameters a and e) by varying these parameters and the additional unknown $x (= U_1 - W_0)$ until the minimum is reached. As there are two types of differences available in the

region σ_1 , these differences must be weighted in one way or another. The mathematical formulation of the problem is given by the target function described in Eqs. (21a) - (21d).

Solution: The solution follows from Eqs. (22).

4. Approach II: practical solutions by linearization and iteration

Below we present two explicit solutions (by linearization and iteration) to the system of equations given by Eqs. (22).

4.1. Solution by linearization

A first order solution of Eq. (22) is obtained by linearization of the radius vector r_E to

$$r_E(\beta) \approx a - g \sin^2 \beta, \quad (25)$$

where

$$g = ae^2/2. \quad (26)$$

Introducing the abbreviate notations

$$R^h = r_1 + N^h \quad (27a)$$

and

$$R^{EGM} = r_1 + N^{EGM}, \quad (27b)$$

one obtains the following system of equations (to order e^2) by taking the derivatives of I w.r.t. x , a , and g and equating to zero:

$$\begin{aligned} c_{11}x + c_{12}a + c_{13}g &= f_1 \\ c_{21}x + c_{22}a + c_{23}g &= f_2 \\ c_{31}x + c_{32}a + c_{33}g &= f_3, \end{aligned} \quad (28)$$

where

$$\begin{aligned} c_{11} &= (1-p) \iint_{\sigma_1} \frac{d\sigma}{\gamma_1^2} + \iint_{\sigma_2} \frac{d\sigma}{\gamma_1^2}, \\ c_{12} &= (1-p) \iint_{\sigma_1} \frac{d\sigma}{\gamma_1} + p \iint_{\sigma_2} \frac{d\sigma}{\gamma_1} = c_{21}; \end{aligned} \quad (29a)$$

$$c_{13} = (1-p) \iint_{\sigma_1} \frac{\sin^2 \beta d\sigma}{\gamma_1} + \iint_{\sigma_2} \frac{\sin^2 \beta d\sigma}{\gamma_1} = c_{31}; \quad c_{22} = 4\pi; \quad (29b)$$

$$\begin{aligned} c_{23} &= - \iint_{\sigma} \sin^2 \beta d\sigma = -\frac{2\pi}{3} = c_{32}; \\ c_{33} &= \iint_{\sigma} \sin^4 \beta d\sigma = \frac{2\pi}{5} \end{aligned} \quad (29c)$$

$$f_1 = - \iint_{\sigma_1} \frac{(1-p)R^h + pR^{EGM}}{\gamma_1} d\sigma;$$

$$f_2 = \iint_{\sigma_1} [(1-p)R^h + pR^{EGM}] d\sigma + \iint_{\sigma_2} R^{EGM} d\sigma \quad (29d)$$

and

$$f_3 = - \iint_{\sigma_1} \sin^2 \beta [pR^h + (1-p)R^{EGM}] d\sigma. \quad (29e)$$

By solving Eq. (28) the solutions for x , a and e are obtained. However, one should bear in mind that the linear solution will only give a rough solution to the problem.

4.1.1. Special case

Consider the case with $p = 0$. Then

$$c_{13} = \iint_{\sigma} \frac{\sin^2 \beta}{\gamma_1} d\sigma. \quad (30)$$

From Heiskanen and Moritz (1967; Eq. 2-126), we realize that $1/\gamma_1$ can be approximated to first order by

$$\frac{1}{\gamma_1} \approx k_0 + k_2 \sin^2 \beta, \quad (31)$$

where k_0 and k_2 are constants, and therefore

$$c_{13} \approx -k_0 c_{23} + k_2 c_{33}. \quad (32)$$

Hence, the coefficients for the unknown g of Eq. (28) are linearly dependent, implying that the system of equations is singular. Therefore the system has no solution for $p = 0$.

4.2. Solution by iteration

The linear solution in Sect. 4.1 can provide the initial values for x , a and e (denoted x , A and E) in an iterative solution. Alternatively, the initial values for A and E are taken from a Geodetic Reference System, e.g., GRS1980, and x is initially set to zero. Then the MEE radius

$$r_E = a\sqrt{1 - e^2 \sin^2 \beta} \quad (33)$$

can be expanded to first order as

$$r_E = r_0 + r_a dA + r_e dE, \quad (34a)$$

where

$$r_0 = A\sqrt{1 - E^2 \sin^2 \beta} \quad (34b)$$

$$r_a = \left[\frac{\partial r_E}{\partial a} \right]_{e=E} = \sqrt{1 - E^2 \sin^2 \beta} \quad (34c)$$

and

$$r_e = \left[\frac{\partial r_E}{\partial e} \right]_{a=A, e=E} = -\frac{AE \sin^2 \beta}{\sqrt{1 - E^2 \sin^2 \beta}}. \quad (34d)$$

Introducing the residual geoid heights from satellite altimetry and the EGM:

$$dR^h = r_1 + N^h - r_0 \text{ and } dR^{EGM} = r_1 + N^{EGM} - r_0, \quad (35)$$

respectively, the target function I of Eq. (21a) can be written

$$I = pI_1(dA, dE) + (1-p)I_2(x, dA, dE) + I_3(x, dA, dE) \\ = \min(x, dA, dE), \quad (36a)$$

where

$$I_1 = \iint_{\sigma_1} [dR^h - r_a dA - r_e dE]^2 d\sigma \quad (36b)$$

$$I_2 = \iint_{\sigma_1} [dR^{EGM} + x/\gamma_1 - r_a dA - r_e dE]^2 d\sigma \quad (36c)$$

and

$$I_3 = \iint_{\sigma_2} [dR^{EGM} + x/\gamma_1 - r_a dA - r_e dE]^2 d\sigma. \quad (36d)$$

The least squares solution for I is obtained by equating its derivatives w.r.t. x , dA and dE to zero. This leads to the following matrix system of equations:

$$\begin{pmatrix} F_{xx} & F_{xa} & F_{xe} \\ F_{xa} & F_{aa} & F_{ae} \\ F_{xe} & F_{ae} & F_{ee} \end{pmatrix} \begin{pmatrix} x \\ dA \\ dE \end{pmatrix} = \begin{pmatrix} f_x \\ f_a \\ f_e \end{pmatrix}, \quad (37)$$

where

$$F_{xx} = (1-p) \iint_{\sigma_1} \frac{d\sigma}{\gamma_1^2} + \iint_{\sigma_2} \frac{d\sigma}{\gamma_1^2};$$

$$F_{xa} = - \iint_{\sigma_1} \frac{r_a}{\gamma_1} d\sigma - p \iint_{\sigma_1} \frac{r_a}{\gamma_1} d\sigma \quad (38a)$$

$$F_{xe} = - \iint_{\sigma_1} \frac{r_e}{\gamma_1} d\sigma - p \iint_{\sigma_1} \frac{r_e}{\gamma_1} d\sigma;$$

$$F_{aa} = \iint_{\sigma} r_a^2 d\sigma;$$

$$F_{ee} = \iint_{\sigma} r_e^2 d\sigma \quad (38b)$$

$$F_{ae} = \iint_{\sigma} r_a r_e d\sigma = -\frac{4\pi}{3} AE \quad (38c)$$

and

$$f_x = - \iint_{\sigma} \frac{dR^{EGM}}{\gamma_1} d\sigma,$$

$$f_a = \iint_{\sigma_1} r_a [p dR^h + (1-p) dR^{EGM}] d\sigma + \iint_{\sigma_2} r_a dR^{EGM} d\sigma \quad (38d)$$

$$f_e = \iint_{\sigma_1} r_e [p dR^h + (1-p) dR^{EGM}] d\sigma + \iint_{\sigma_2} r_e dR^{EGM} d\sigma. \quad (38e)$$

After each iteration, A and E are updated to the next step by

$$A = A + dA \text{ and } E = E + dE. \quad (39)$$

After the iteration is stopped, the final estimate for W_0 is given by Eq. (24), i.e.

$$\hat{W}_0 = U_1 - \hat{x} \quad (40)$$

with $\hat{x}$ being the final estimate for x .

5. Conclusions

The geoid potential W_0 can be directly estimated as a correction to the normal potential of the reference ellipsoid and the oceanic average of the difference between geoid heights from satellite altimetry and an EGM. Systematic errors in the data propagate into biased solutions, which are therefore sensitive to the chosen data. Typical systematic error sources are lacking SST information for the satellite altimetry (which locally may reach several decimetres to metre) and truncation error in the EGM derived geoid height. The effects of these error types will change with the area of integration. The latter error source would vanish, if the averaging is extended to the whole Earth, but that would require geometric geoid heights in continental regions, which are prone to other types of biases. The above technique cannot be used to estimate the axes of the MEE, but their relations can be conditioned by Eq. (16) once $U_0 = W_0$ has been fixed.

The alternative technique to solve for W_0 is to first determine the axes of the MEE and then find the geoid potential. The main problem with this technique is that it propagates a bias from the unknown zero-degree harmonic of the gravimetric geoid height into the solution. This problem is solved by the new technique discussed in Sect. 4, where W_0 (or its correction from the potential of the preliminary reference ellipsoid) is included as an extra unknown. Again, a main problem is due to the more or less unknown SST corrections in the ocean areas.

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