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Chaos synchronization of a fractional nonautonomous system

Abstract: In this paper we investigate the dynamic behavior of a nonautonomous fractional-order biological system. With the stability criterion of active nonlinear fractional systems, the synchronization of the studied chaotic system is obtained. On the other hand, using a Phase-Locked-Loop (PLL) analogy we synchronize the same system. The numerical results demonstrate the effectiveness of the proposed methods.

Keywords: Chaos; Fractional-order system; Active control; PLL; Synchronization

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1 Introduction

The study of nonlinear dynamical systems and chaos have become a subject of great interest and it has attracted enormous research interest after the earlier numerical demonstration of chaos by Lorenz [16]. Chaotic systems are associated with complex dynamical behaviors that possess some special features including bounded trajectories with positive Lyapunov exponents; and sensitive dependence on initial conditions. The study of chaotic dynamics has found applications in various fields of scientific and engineering disciplines, including meteorology, physics, chemistry, engineering, economics, biology, etc [24?]. On the other hand, the development of models based on fractional-order differential systems has recently received growing attention in the investigation of dynamical systems. The interest in the study of fractional-order nonlinear systems lies in the fact that fractional derivatives provide an excellent tool for the description of memory and hereditary properties, which are not taken into account in the classical integer-order models. Further fractional versions of many systems were investigated ([3, 11, 12, 24, 30]). Owing to its interesting applications in secure communication of analog and digital signals, cryptographic systems and control processing, time series analysis, modelling brain and cardiac rhythmic activity as well as earthquake dynamics [25], chaos synchronization has been an interesting research area since the pioneering work of Carroll and Pecora [23]. Nowadays, different techniques and methods have been proposed to achieve chaos synchronization such as linear and nonlinear feedback control [17], adaptive control [29], sliding mode control [10], backstepping nonlinear control approach [21], etc. In this work we consider a nonautonomous fractional-order dynamical system arising in biology [5, 14, 15]. The dynamical behaviour of the system is studied. Using two synchronization schemes we force the slave trajectories to track the master trajectories.

The remainder of the paper is organized as follows: After this introduction, preliminaries regarding fractional calculus and numerical scheme are described in Section 2. In Section 3, the fractional-ordered model is described and its dynamical behaviour is studied. In Section 4 synchronization via two methods is presented, numerical simulations of the results are presented and discussed. Finally, the paper is concluded in Section 5.

2 Preliminaries

2.1 Fractional Calculus

We briefly recall here some basic necessary tools on fractional calculus. For more details on the subject and applications, we refer the reader to the references [4] [19] and [26].

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$ if there exists a real number $\lambda > \mu$ such that $f(x) = x^\lambda g(x)$, where $g(x) \in C[0, \infty)$ and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$ for $m \in \mathbb{N}$.

Definition 2. The Riemann-Liouville fractional integral operator of order q of a real function $f(x) \in C_\mu$, $\mu \geq -1$, is defined as

$$J^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} f(t) dt, \quad q > 0, x > 0 \quad \text{and} \quad J^0 f(x) = f(x). \quad (1)$$

The operators J^q has some proprieties, for $q, p, \xi \geq 0$:

- $J^q J^p f(x) = J^{q+p} f(x)$,
- $J^q J^p f(x) = J^p J^q f(x)$,
- $J^q x^\xi = \frac{\Gamma(\xi+1)}{\Gamma(q+\xi+1)} x^{q+\xi}$.

Where Γ is the Euler's Gamma function.

Definition 3. The Caputo fractional derivatives D^q of a function $f(x)$ of any real number q such that $m-1 < q \leq m$, $m \in \mathbb{N}$, for $x > 0$ and $f \in C_{-1}^m$ in the terms of J^q is defined as

$$D^q f(x) = J^{m-q} D^m f(x) = \frac{1}{\Gamma(m-q)} \int_0^x (x-t)^{m-q-1} f^{(m)}(t) dt, \quad (2)$$

and has the following proprieties for $m-1 < q \leq m$, $m \in \mathbb{N}$, $\mu \geq -1$ and $f \in C_\mu^m$,

- $D^q J^q f(x) = f(x)$,
- $J^q D^q f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \frac{x^k}{k!}$, for $x > 0$.

Definition 4. The Laplace transform of the Caputo fractional derivative is given by

$$L \{ D^q f(t) \} = s^q L \{ f(t) \} - \sum_{k=0}^{m-1} s^{q-1-k} f^{(k)}(0), \quad (3)$$

where L means the Laplace transform and s is a complex variable. Upon considering the initial conditions to zero, Eq.(3) reduces to

$$L \{ D^q f(t) \} = s^q L \{ f(t) \}. \quad (4)$$

3 Stability conditions

Theorem 1 (see [18]). For a given fractional order system

$$D^q X(t) = AX(t), \quad X(0) = X_0, \quad (5)$$

with $X(t) = (x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n$, $A = (a_{ij}) \in \mathbb{R}^{n \times m}$ and $q = (q_1, q_2, \dots, q_n)^T$, where $0 < q_i \leq 2$ ($i = 1, 2, \dots, n$). Assume m to be the lowest common multiple of the denominators u_i of q_i satisfying

$$q_i = \frac{v_i}{u_i}, \quad (u_i, v_i) = 1, \quad u_i, v_i \in \mathbb{Z}_+.$$

Set $\gamma = \frac{1}{m}$. Define the following characteristic equation

$$\det(\text{diag}(\lambda^{mq_1}, \lambda^{mq_2}, \dots, \lambda^{mq_n}) - A) = 0. \quad (6)$$

Then the steady state of system (5) is globally asymptotically stable if all roots λ_i of equation (6) satisfy

$$|\arg(\lambda_i)| > \frac{\pi}{2}\gamma. \quad (7)$$

For studying the stability of fractional order systems, we will need the following final value theorem.

Theorem 2. Let $F(s)$ be the Laplace transform of a function $f(t)$. If all poles of $sF(s)$ are in the open left half plane, then

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s).$$

3.1 Adams-Bashforth (PECE) algorithm

Consider the fractional-order initial value problem

$$\begin{cases} D_t^q y = f(t, y(t)) & 0 \leq t \leq T, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \dots, m-1. \end{cases} \quad (8)$$

It is equivalent to the Volterra integral equation

$$y(t) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) ds. \quad (9)$$

Diethelm et al. have given a predictor-correctors scheme (see [6][7]), based on the Adams-Bashforth-Moulton algorithm to integrate Equation (9). By applying this scheme to the fractional-order system (8), and setting

$$h = \frac{T}{N}, \quad t_n = nh, \quad n = 0, 1, \dots, N,$$

Equation (9) can be discretized as follows:

$$y_h(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{h^q}{\Gamma(q+2)} f(t_{n+1}, y_h^p(t_{n+1})) + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, y_h(t_j)), \quad (10)$$

where

$$\begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0, \\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \leq j \leq n \\ 1, & j = n+1, \end{cases} \quad (11)$$

and the predictor is given by

$$y_h^p(t_{n+1}) = \sum_{k=0}^{[q]-1} y_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^n b_{j,n+1} f(t_j, y_h(t_j)), \quad (12)$$

where $b_{j,n+1} = \frac{h^q}{q}((n+1-j)^q - (n-j)^q)$. The error estimate of the above scheme is

$$\max_{j=0,1,\dots,N} \{|y(t_j) - y_h(t_j)|\} = O(h^p),$$

in which $p = \min(2, 1+q)$.

3.2 Synchronization of two fractional order non-autonomous systems

Consider the master-slave synchronization scheme of two non-autonomous fractional order systems

$$\begin{cases} D_t^q X_1 = f(t, X_1(t)) & (Master) \\ D_t^q X_2 = g(t, X_2(t)) + U(t) & (Slave). \end{cases} \quad (13)$$

Where q is the fractional-order of derivation, $X_1, X_2 \in \mathbb{R}^n$ are the states of the master and the slave systems respectively and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are the vector fields of the master and the slave systems respectively.

The main goal is to take a suitable line control function $U(t) = (u_1(t), \dots, u_n(t))$ such that the states of the master and the slave systems are synchronized, i.e, $\lim_{t \rightarrow \infty} \|X_1 - X_2\| = 0$, where $\|\cdot\|$ denotes the Euclidean norm.

4 A nonautonomous fractional-order biological system

We give a brief description of the biological model that we will study in the present paper; it consists of enzyme-substrate system with ferroelectric behaviour in brain waves [8, 9, 14, 15]. It is described by the second order nonautonomous differential equation of the activated enzyme molecules [13, 21]

$$\ddot{x} - \mu(1 - x^2 + \alpha x^4 - \beta x^6)\dot{x} + x = E \cos(\omega t), \quad (14)$$

where an overdot represents differentiation with respect to time t , α is the parameter of nonlinearity and x is the concentration of the biological system. The quantities α, β are positive real parameters while E and ω are respectively the amplitude and the frequency of the external excitation.

In [21], the authors convert Equation (14) to the following system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu(1 - x^2 + \alpha x^4 - \beta x^6)y - x + E \cos(\omega t). \end{cases} \quad (15)$$

They proposed recursive and adaptive backstepping nonlinear controllers to control and synchronize the biological system (15).

In this paper, we consider the fractional commensurate ordered model of (15), which takes the form

$$\begin{cases} D_t^q x = y, \\ D_t^q y = \mu(1 - x^2 + \alpha x^4 - \beta x^6)y - x + E \cos(\omega t), \end{cases} \quad (16)$$

where $q \in (0, 1]$, $\alpha = 2.55$, $\beta = 1.70$, $\mu = 2.001$, $\omega = 3.465$ and $E = 8.27$.

It is easy to check that (16) possesses one equilibrium point $(0, 0)$. The Jacobian matrix at this point is

$$J|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix},$$

and the corresponding eigenvalues are

$$\lambda_1 = \frac{\mu + \sqrt{\mu^2 - 4}}{2}, \quad \lambda_2 = \frac{\mu - \sqrt{\mu^2 - 4}}{2}.$$

Since λ_1 is positive, then

$$\arg \lambda_1 < \frac{\pi}{2} q.$$

Therefore the equilibrium point O is unstable.

Fig. 1 depicts the chaotic behaviour of system (16) by considering : $q = 0.9$ and the initial conditions : $(x(0), y(0)) = (0, 1)$.

Fig. 1. Chaotic phase portrait of (16) for $q = 0.9$, $\alpha = 2.55$, $\beta = 1.70$, $\mu = 2.001$, $\omega = 3.465$, $E = 8.27$ and $(x(0), y(0)) = (0, 1)$.

5 Synchronisation of the NAF OBS

In this section, we aim to achieve the chaos synchronization of two identical fractional-order biological systems by using two methods, namely the active control synchronization and the phase-locked-loop analogy. A master-slave synchronization scheme is illustrated in Fig.2.

Fig. 2. The master-slave synchronization scheme.

5.1 Active control method

Chaos synchronization using active control was proposed by Bai and Lonngren [1] and has recently been widely accepted as an efficient technique for the synchronization of chaotic systems.

To demonstrate the effectiveness of the used methods, we will give some numerical simulations which are conducted by a PECE scheme, using Maple 17 software with a time-step $h = 10^{-2}$. For convenience, the parameters of system (16) are specified as $\alpha = 2.55$, $\beta = 1.70$, $\mu = 2.001$, $\omega = 3.465$, $E = 8.27$.

Now consider the master-slave synchronization scheme of two non-autonomous identical fractional-order systems as follows

$$(Master) \begin{cases} D_t^q x_1 = y_1, \\ D_t^q y_1 = \mu(1 - x_1^2 + \alpha x_1^4 - \beta x_1^6)y_1 - x_1 + E \cos(\omega t), \\ (x_{1,0}, y_{1,0}). \end{cases} \quad (17)$$

and

$$(Slave) \begin{cases} D_t^q x_2 = y_2 + u_1(t), \\ D_t^q y_2 = \mu(1 - x_2^2 + \alpha x_2^4 - \beta x_2^6)y_2 - x_2 + E \cos(\omega t) + u_2(t), \\ (x_{2,0}, y_{2,0}). \end{cases} \quad (18)$$

Where u_1 and u_2 are unknown active control functions to be determined. Note that the initial conditions $(x_{1,0}, y_{1,0})$ and $(x_{2,0}, y_{2,0})$ are different and we want to synchronize the signals in spite of discrepancy between the initial conditions. So let us define the error vector $\mathbf{e}(t)$ as

$$\begin{cases} e_1 = x_2 - x_1, \\ e_2 = y_2 - y_1. \end{cases} \quad (19)$$

The subtraction of (17) from (18) and the use of (19) gives

$$\begin{cases} D^q e_1 = e_1 + u_1, \\ D^q e_2 = -e_1 - \mu(y_2 x_2^2 - y_1 x_1^2) + \mu\alpha(y_2 x_2^4 - y_1 x_1^4) - \mu\beta(y_2 x_2^6 - y_1 x_1^6) + u_2(t). \end{cases} \quad (20)$$

Let

$$\begin{cases} u_1 = -2e_1 - e_2, \\ u_2 = e_1 - 2\mu e_2 + \mu(y_2 x_2^2 - y_1 x_1^2) - \mu(y_2 x_2^4 - y_1 x_1^4) + \mu\beta(y_2 x_2^6 - y_1 x_1^6). \end{cases} \quad (21)$$

Thus, the fractional-order error dynamical system is reduced to

$$\begin{cases} D^q e_1 = -e_1, \\ D^q e_2 = -2\mu e_2. \end{cases} \quad (22)$$

Theorem 3. For any initial conditions, the master and slave defined by the synchronization scheme (17),(18) are globally asymptotically synchronized with the control law (21).

Proof. Applying Laplace transform to system (22), letting $\mathcal{E}_1(s) = L(e_1(t))$ and $\mathcal{E}_2(s) = L(e_2(t))$, we obtain

$$\begin{cases} s^q \mathcal{E}_1(s) - s^{q-1} e_1(0) - s^{q-2} e_1^{(1)}(0) = -\mathcal{E}_1(s), \\ s^q \mathcal{E}_2(s) - s^{q-1} e_2(0) - s^{q-2} e_2^{(1)}(0) = -2\mu \mathcal{E}_2(s). \end{cases} \quad (23)$$

It follows from Equation (23) that

$$\begin{cases} \mathcal{E}_1(s) = \frac{s^{q-1} e_1(0) + s^{q-2} e_1^{(1)}(0)}{s^q + 1}, \\ \mathcal{E}_2(s) = \frac{s^{q-1} e_2(0) + s^{q-2} e_2^{(1)}(0)}{s^q + 2\mu}. \end{cases} \quad (24)$$

From the final-value theorem of the Laplace transform, we have

$$\begin{cases} \lim_{t \rightarrow \infty} e_1(t) = \lim_{s \rightarrow 0} s \mathcal{E}_1(s) = \lim_{s \rightarrow 0} \frac{s^{q-1} e_1(0) + s^{q-2} e_1^{(1)}(0)}{s^q + 1} = 0, \\ \lim_{t \rightarrow \infty} e_2(t) = \lim_{s \rightarrow 0} s \mathcal{E}_2(s) = \lim_{s \rightarrow 0} \frac{s^{q-1} e_2(0) + s^{q-2} e_2^{(1)}(0)}{s^q + 2\mu} = 0. \end{cases} \quad (25)$$

Therefore the synchronization of (17),(18) is achieved under the control law (21). $\square$

Fig. 3. Time series : (a) $x_1(t)$ and (b) $x_2(t)$ of (17),(18).

Fig. 4. Time series : (a) $y_1(t)$ and (b) $y_2(t)$ of (17),(18).

Fig. 5. Evolution of error functions: (a) $e_1(t)$, (b) $e_2(t)$ of (17),(18).

Figs.3, 4 and 5 show the simulation result of synchronization of the chaotic systems (17),(18) with initial conditions: (0.7, 1), (0.2, 5) and $q = 0.98$. We remark that the error dynamic system (22) rapidly settles to zero, which means chaos synchronization.

5.2 Phase-Locked-Loop analogy

Phase-Locked-Loops (PLL for short) have been applied in various fields. They are increasingly popular in telecommunication systems, wireless systems, power engineering and biological systems. Recently Pham et al. [22] introduced a new model of the FOPLL (Fractional Order Phase-Locked-Loop) and propose control and synchronization methods for it. Motivated by their work, we consider a master-slave synchronization, for system (16) as follows

$$\begin{cases} (Master) & D_t^q X_1 = \mathbf{A}X_1 + \mathbf{B}g(X_1) + \mathbf{C}(t), & X_{1,0}. \\ (Slave) & D_t^q X_2 = \mathbf{A}X_2 + \mathbf{B}g(X_1) + \mathbf{C}(t) + \mathbf{K}e(t), & X_{2,0}. \end{cases} \quad (26)$$

where $X_i = \begin{pmatrix} x_i(t) \\ y_i(t) \end{pmatrix}$ for $i = 1, 2$, and

$$\begin{cases} \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{K} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \\ \mathbf{e}(t) = \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix}, \mathbf{g}(X_1) = \begin{pmatrix} 0 \\ -\mu x_1^2 y_1 + \mu \alpha x_1^4 y_1 - \beta \mu x_1^6 y_1 \end{pmatrix} \text{ and } \mathbf{C}(t) = \begin{pmatrix} 0 \\ E \cos(\omega t) \end{pmatrix}. \end{cases}$$

where $\mathbf{e}(t)$ is the error synchronization and $\mathbf{K}$ is a matrix to be determined .
The components of the vector error are

$$\begin{cases} e_1 = x_1 - x_2, \\ e_2 = y_1 - y_2. \end{cases} \quad (27)$$

Then the dynamical synchronization error of (26) can be written as

$$D^q \mathbf{e} = (\mathbf{A} - \mathbf{K})\mathbf{e}. \quad (28)$$

Theorem 4. *Systems (26) will approach global synchronization for any initial conditions, with the following control law*

$$\mathbf{K}\mathbf{e} = \begin{pmatrix} e_1 + e_2 \\ e_1 + 4e_2 \end{pmatrix}, \quad (29)$$

and $\mu < 4$.

Proof. From (28) we have

$$\begin{cases} D^q e_1 = -k_{11}e_1 + (1 - k_{12})e_2, \\ D^q e_2 = -(1 + k_{21})e_1 + (\mu - k_{22})e_2. \end{cases} \quad (30)$$

The above system has one equilibrium point (0, 0), and the Jacobian matrix at this point is

$$\begin{pmatrix} -k_{11} & 1 - k_{12} \\ -1 - k_{21} & \mu - k_{22} \end{pmatrix}. \quad (31)$$

The eigenvalues of the above matrix are:

$$\lambda_{1,2} = \frac{1}{2} \left(\mu - k_{22} - k_{11} \pm \sqrt{\mu^2 - 2\mu k_2 + 2k_{11}\mu + k_{22}^2 - 2k_{11}k_{22} + k_{11}^2 - 4 + 4k_{12} - 4k_{21} + 4k_{21}k_{12}} \right). \quad (32)$$

The synchronization of (26) occurs when the matrix $\mathbf{K}$ is chosen appropriately such that

$$\lim_{t \rightarrow \infty} \|\mathbf{e}(t)\| = 0,$$

hence, we use the condition of stability (7) to find $\mathbf{K}$. Then $\mathbf{K}$ is achieved as

$$\mathbf{K} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad (33)$$

and the eigenvalues of (31) become

$$\lambda_1 = -1, \quad \lambda_2 = \mu - 4. \quad (34)$$

If $\mu < 4$, according to Theorem 5, it is direct to see that the error dynamics converge to the manifold $(e_1, e_2) = (0, 0)$ as $t \rightarrow \infty$. Consequently the synchronization between two identical systems (26) is achieved via the control law (29). $\square$

The effectiveness of the PLL-scheme can be demonstrated through the following numerical simulation presented in Figs.6-8 for $q = 0.98$. The initial conditions for the master and slave systems are $(0.7, 1)$ and $(-0.2, -1)$ respectively.

Fig. 6. Time series: (a) $x_1(t)$ and (b) $x_2(t)$ of (26).

Fig. 7. Time series: (a) $y_1(t)$ and (b) $y_2(t)$ of (26).

Fig. 8. Evolution of error functions: (a) $e_1(t)$, (b) $e_2(t)$ of (26).

6 Conclusion

In the present paper, the dynamics for a nonautonomous fractional-order biological system is investigated. Both of the active control and the Phase-Locked-Loop methods are applied to synchronize the fractional order biological system. The corresponding numerical simulations demonstrate the effectiveness of the proposed techniques. We underline that same methods can be used to synchronize a large variety of identical or non-identical chaotic systems such as: Duffing, Duffing-Holmes, Vander Pol, etc.

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