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Global Existence and Stability for Neutral Functional Evolution Equations with State-Dependent Delay

Abstract: In this paper we prove the global existence and attractivity of mild solutions for neutral semilinear evolution equations with state-dependent delay in a Banach space.

Keywords: Semilinear functional differential equations, mild solution, attractivity, evolution system, fixed-point, infinite delay, infinite interval

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1 Introduction

In this paper, we consider the existence and attractivity of mild solutions of the following neutral evolution equation with state-dependent delay

$$\begin{cases} \frac{d}{dt}[y(t) - g(t, y_{\rho(t, y_t)})] - A(t)y(t) = f(t, y_{\rho(t, y_t)}), & t \in J := [0, \infty) \\ y(t) = \phi(t) & t \in (-\infty, 0], \end{cases} \quad (1)$$

where $\mathcal{B}$ is an abstract *phase space* to be specified later, f and g are given functions from $J \times \mathcal{B}$ into E , the function $\rho : J \times \mathcal{B} \rightarrow \mathbb{R}$ is an appropriate function, $\phi \in \mathcal{B}$ and $\{A(t)\}_{0 \leq t < +\infty}$ is a family of linear closed (not necessarily bounded) operators from E into E that generate an evolution system of operators $\{U(t, s)\}_{(t,s) \in J \times J}$ for $0 \leq s \leq t < +\infty$.

For any continuous function y and any $t \geq 0$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state up to the present time t . We assume that the histories y_t belong to $\mathcal{B}$.

In many fields of science we can describe various evolutionary process by differential equations with delay and for this reason the study of this type of equations has received great attention during the last years. The literature on differential equations with delay is very extensive, we refer, for instance to the books by Hale [13], Hale and Verduyn Lunel [15], Lakshmikantham *et al.* [20], Kolmanovskii and Myshkis [19] and Wu [23]. Some recent results on various classes of semilinear functional differential equations are reported in [6, 22]. In this paper we are interested in the existence and attractivity of mild solutions of neutral state-dependent delay functional differential equations. In the literature there are many papers studying these equations using

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different methods. Using the fixed point method combined with the semigroup theory in Fréchet space, Baghli and Benchohra [3–5] considered some classes of neutral evolution equations on unbounded intervals. On other hand, for the local existence, we refer, for instance, to the papers by Aiello *et al.* [2], dos Santos [10], Hernández *et al.* [16, 17], and the references therein.

In this paper, sufficient conditions are provided to get the global existence and attractivity of mild solutions of the problem (1) using Burton-Kirk's fixed point theorem (see [7]). Let us mention that under a Lipschitz condition on the nonlinearities, one can get a uniqueness result. As far as we know, there are few papers dealing with global existence results for the problem (1). Most of these results are stated in the Fréchet space setting. The present paper provides sufficient conditions for the existence and attractivity mild solutions to problem (1) in the Banach space setting.

2 Preliminary

Let E a Banach space with norm $|\cdot|$ and $BC(J, E)$ the Banach space of all bounded and continuous functions y mapping J into E with the usual supremum norm

$$\|y\| = \sup_{t \in J} |y(t)|.$$

Let $\mathcal{X}$ is the space defined by

$$\mathcal{X} = \{y : \mathbb{R} \rightarrow E \text{ such that } y|_J \in BC(J, E) \text{ and } y_0 \in \mathcal{B}\}.$$

We denote by $y|_J$ the restriction of y to J .

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [14] and follow the terminology used in [18]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E , and satisfying the following axioms :

(A₁) If $y : (-\infty, b) \rightarrow E$, $b > 0$, is continuous on $[0, b]$ and $y_0 \in \mathcal{B}$, then for every $t \in [0, b)$ the following conditions hold :

- (i) $y_t \in \mathcal{B}$;
- (ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
- (iii) There exist two functions $K(\cdot), M(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ independent of y with K continuous and M locally bounded such that :

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{|y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A₂) For the function y in (A₁), y_t is a $\mathcal{B}$ -valued continuous function on $[0, b]$.

(A₃) The space $\mathcal{B}$ is complete.

Remark 2.1. In the sequel we assume that K and M are bounded on J and

$$y := \max \left\{ \sup_{t \in \mathbb{R}_+} \{K(t)\}, \sup_{t \in \mathbb{R}_+} \{M(t)\} \right\}.$$

For other details we refer, for instance to the book by Hino *et al* [18].

In what follows, we assume that $\{A(t), t \geq 0\}$ is a family of closed densely defined linear unbounded operators on the Banach space E and with domain $D(A(t))$ independent of t .

Definition 2.2. A family of bounded linear operators

$$\{U(t, s)\}_{(t,s) \in \Delta} : U(t, s) : E \rightarrow E \quad (t, s) \in \Delta := \{(t, s) \in J \times J : 0 \leq s \leq t < +\infty\}$$

is called an evolution system if the following properties are satisfied:

1. $U(t, t) = I$ where I is the identity operator in E ,
2. $U(t, s)U(s, \tau) = U(t, \tau)$ for $0 \leq \tau \leq s \leq t < +\infty$,
3. $U(t, s) \in B(E)$ the space of bounded linear operators on E , where for every $(s, t) \in \Delta$ and for each $y \in E$, the mapping $(t, s) \rightarrow U(t, s)y$ is continuous.

More details on evolution systems and their properties could be found on the books of Ahmed [1], Engel and Nagel [11] and Pazy [21].

Lemma 2.3. [8][Corduneanu]

Let $C \subset BC(J, E)$ be a set satisfying the following conditions:

- (i) C is bounded in $BC(J, E)$;
- (ii) the functions belonging to C are equicontinuous on any compact interval of J ;
- (iii) the set $C(t) := \{y(t) : y \in C\}$ is relatively compact on any compact interval of J ;
- (iv) the functions from C are equiconvergent, i.e., given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|y(t) - y(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $y \in C$.

Then C is relatively compact in $BC(J, E)$.

Theorem 2.4. [7][Burton-Kirk's fixed point theorem]

Let X Banach space, and $A, B : X \rightarrow X$ two operators. Suppose that B is a contraction and A a compact operator. Then either

- (i) $x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax$ has a solution for $\lambda = 1$, or
- (ii) the set $\{x \in X : x = \lambda B\left(\frac{x}{\lambda}\right) + \lambda Ax, \lambda \in (0, 1)\}$ is unbounded.

3 Main result

Definition 3.1. A function $y \in \mathcal{X}$ is said to be a mild solution of the problem (1), if

$$y(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ U(t, 0)(\phi(0) - g(0, \phi)) + g(t, y_{\rho(t, y_t)}) \\ + \int_0^t U(t, s)A(s)g(s, y_{\rho(s, y_s)})ds + \int_0^t U(t, s)f(s, y_{\rho(s, y_s)})ds, & \text{if } t \in J. \end{cases} \quad (2)$$

To prove our results we introduce the following conditions:

(H₁) There exists a constant $\widehat{M} \geq 1$ and $\omega > 0$ such that

$$\|U(t, s)\|_{B(E)} \leq \widehat{M}e^{-\omega(t-s)} \quad \text{for every } (s, t) \in \Delta.$$

(H₂) There exists a function $p \in L^1(J, \mathbb{R}_+)$ such that:

$$|f(t, u)| \leq p(t)(\|u\|_{\mathcal{B}} + 1) \quad \text{for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

(H₃) For each $(t, s) \in \Delta$ we have

$$\lim_{t \rightarrow +\infty} \int_0^t e^{-w(t-s)} p(s) ds = 0.$$

(H₄) There exists a constant $\tilde{M} > 0$ such that:

$$\|A^{-1}(t)\|_{B(E)} \leq \tilde{M} \quad \text{for all } t \in J.$$

(H₅) There exists a constant $\ell > 0$ such that

$$|A(t)g(t, \phi) - A(s)g(s, \varphi)| \leq \ell(|t - s| + \|\phi - \varphi\|_{\mathcal{B}})$$

for all $t, s \in J$ and $\phi, \varphi \in \mathcal{B}$.

(H₆) There exists a bounded continuous function $\zeta : J \rightarrow \mathbb{R}_+$ such that:

$$|A(t)g(t, \phi)| \leq \zeta(t)\|\phi\|_{\mathcal{B}} \quad \text{for all } t \in J, \phi \in \mathcal{B}.$$

(H₇) Let $\mathcal{R}_\rho^- = \{\rho(t, y_t) : \rho(t, y_t) \leq 0, (t, y_t) \in \mathcal{B}\}$, and $\phi_t : \mathcal{R}_\rho^- \rightarrow \mathcal{B}$ continuous function and there exists a continuous and bounded function $L^\phi : \mathcal{R}_\rho^- \rightarrow (0, \infty)$ such that $\|\phi_t\| \leq L^\phi(t)\|\phi\|_{\mathcal{B}}$ for all $t \in \mathcal{R}_\rho^-$.

Remark 3.2. For more detailed information concerning (H₇) see ([18], Lemma 3.4 and Proposition 7.1.1).

For the proof of our results, we are going to need the following lemma.

Lemma 3.3 ([16], Lemma 3.1). Let $y : (-\infty, \infty) \rightarrow E$ continuous and bounded and $y_0 = \phi$. If (H₇) holds, then

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $L^\phi = \sup_{t \in \mathcal{R}(\rho^-)} L^\phi(t)$.

Theorem 3.4. If assumptions (H₁) – (H₇) are satisfied, and if

$$y \left(\ell \tilde{M} + \frac{\tilde{M}}{\omega} \right) < 1,$$

and

$$\tilde{M}\zeta^* y + \frac{\tilde{M}\zeta^*}{\omega} + \tilde{M}y < 1,$$

then the problem (1) admits at least one mild solution.

Proof. It is clear that we will obtain the results if we show that the operator $T : \mathcal{X} \rightarrow \mathcal{X}$ defined by:

$$Ty(t) = \begin{cases} \phi(t), & \text{if } t \leq 0 \\ U(t, 0)(\phi(0) - g(0, \phi)) + g(t, y_{\rho(t, y_t)}) \\ + \int_0^t U(t, s)A(s)g(s, y_{\rho(s, y_s)})ds + \int_0^t U(t, s)f(s, y_{\rho(s, y_s)})ds, & \text{if } t \in J, \end{cases} \quad (3)$$

has a fixed point.

For $\phi \in \mathcal{B}$, Let $x : (-\infty, +\infty) \rightarrow E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty, 0] \\ U(t, 0)\phi(0) & \text{if } t \in J. \end{cases}$$

Then $x_0 = \phi$. For each function $z \in \mathcal{X}$, we denote

$$y(t) = x(t) + z(t).$$

It is obvious that y satisfies (3) if and only if z satisfies $z_0 = 0$ and for all $t \in J$

$$\begin{aligned} z(t) = & U(t, 0)g(0, \phi) + g(t, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)}) + \int_0^t U(t, s)A(s)g(s, x_{\rho(s, z_s + x_s)} + z_{\rho(s, z_s + x_s)})ds \\ & + \int_0^t U(t, s)f(s, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)})ds. \end{aligned} \quad (4)$$

In the sequel, we always denote $\mathcal{X}_0$ is Banach space defined by

$$\mathcal{X}_0 = \{z \in \mathcal{X} : z_0 = 0\},$$

endowed with the norm

$$\|z\|_{\mathcal{X}_0} = \sup_{t \in J} |z(t)| + \|z_0\|_{\mathcal{B}} = \sup_{t \in J} |z(t)|.$$

Now, we can decompose (4) as $z(t) = Fz(t) + Lz(t)$ which $F, L : \mathcal{X}_0 \rightarrow \mathcal{X}_0$ given by

$$Fz(t) = \int_0^t U(t, s)f(s, x_{\rho(s, z_s + x_s)} + z_{\rho(s, z_s + x_s)})ds, \text{ for } t \in J,$$

and

$$Lz(t) = U(t, 0)g(0, \phi) + g(t, x_{\rho(s, z_s + x_s)} + z_{\rho(s, z_s + x_s)}) + \int_0^t U(t, s)A(s)g(s, x_{\rho(s, z_s + x_s)} + z_{\rho(s, z_s + x_s)})ds.$$

Obviously the problem (1) has a solution is equivalent to $F + L$ has a fixed point. To prove this end, we start with the following estimate.

For each $z \in \mathcal{X}_0$ and $t \in J \cup \mathcal{R}(\rho^-)$, we have

$$\begin{aligned} \|z_t + x_t\|_{\mathcal{B}} &\leq \|z_t\|_{\mathcal{B}} + \|x_t\|_{\mathcal{B}} \\ &\leq K(t)|z(t)| + K(t)\|U(t, 0)\|_{B(E)}\|\phi\|_{\mathcal{B}} + (M(t) + L^\phi)\|\phi\|_{\mathcal{B}} \\ &\leq y\|z\|_{\mathcal{X}_0} + y\widehat{M}e^{-\omega t}\|\phi\|_{\mathcal{B}} + (y + L^\phi)\|\phi\|_{\mathcal{B}} \\ &\leq y\|z\|_{\mathcal{X}_0} + [y(\widehat{M} + 1) + L^\phi]\|\phi\|_{\mathcal{B}}. \end{aligned} \quad (5)$$

Now, we will show that the operators F, L satisfy the conditions of Theorem 2.4.

Step 1.

F is continuous and compact.

• **F is continuous.** Let $(z^k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{X}_0$ such that $z^k \rightarrow z$ in $\mathcal{X}_0$, then For every $t \in J$ we obtain

$$\begin{aligned} |F(z^k)(t) - F(z)(t)| &\leq \int_0^t \|U(t, s)\|_{B(E)} \left| f(t, x_{\rho(t, x_s + z_s^k)} + z_{\rho(t, x_s + z_s^k)}^k) - f(t, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)}) \right| ds \\ &\leq \widehat{M} \int_0^t e^{-\omega(t-s)} \left| f(s, z_{\rho(t, x_s + z_s^k)}^k + x_{\rho(t, x_s + z_s^k)}) - f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) \right| ds. \end{aligned}$$

Hence, from the continuity of the function f and the Lebesgue dominated convergence theorem we obtain

$$\|Fz_k - Fz\|_{\mathcal{X}_0} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

So F is continuous.

• $F(D)$ **relatively compact**.

Let D be a bounded subset of $\mathcal{X}_0$. To show that $F(D)$ is relatively compact we will use Lemma 2.3.

Let $\eta \geq 0$ such that $D = \{z \in \mathcal{X}_0 : \|z\|_{\mathcal{X}_0} \leq \eta\}$. Let $z \in D$, then

$$\begin{aligned} |F(z)(t)| &\leq \int_0^t \|U(t, s)\|_{B(E)} \left| f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) \right| ds \\ &\leq \widehat{M} \int_0^t e^{-\omega(t-s)} p(s) \left(\|z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + 1 \right) ds. \\ &\leq \widehat{M} \left(\gamma \|z\|_{\mathcal{X}_0} + (\gamma(\widehat{M} + 1) + L^\phi) \|\phi\|_{\mathcal{B}} + 1 \right) \int_0^t e^{-\omega(t-s)} p(s) ds \\ &\leq \widehat{M} \xi \|p\|_{L^1}, \end{aligned}$$

with

$$\xi := \gamma \eta + (\gamma(\widehat{M} + 1) + L^\phi) \|\phi\|_{\mathcal{B}} + 1.$$

Thus $F(D)$ is bounded.

• $F(D)$ is **equicontinuous**.

Let $s, t \in [0, b]$ with $t > s$ and $z \in D$. Then, we have

$$\begin{aligned} |(Fz)(t) - (Fz)(s)| &= \left| \int_0^s (U(t, \tau) - U(s, \tau)) f(\tau, z_{\rho(\tau, x_\tau + z_\tau)} + x_{\rho(\tau, x_\tau + z_\tau)}) d\tau \right. \\ &\quad \left. + \int_s^t U(t, \tau) f(\tau, z_{\rho(\tau, x_\tau + z_\tau)} + x_{\rho(\tau, x_\tau + z_\tau)}) d\tau \right| \\ &\leq \int_0^s \|U(t, \tau) - U(s, \tau)\|_{B(E)} p(\tau) \left(\|z_{\rho(\tau, x_\tau + z_\tau)} + x_{\rho(\tau, x_\tau + z_\tau)}\|_{\mathcal{B}} + 1 \right) d\tau \\ &\quad + \widehat{M} \int_s^t e^{-\omega(t-\tau)} p(\tau) \left(\|z_{\rho(\tau, x_\tau + z_\tau)} + x_{\rho(\tau, x_\tau + z_\tau)}\|_{\mathcal{B}} + 1 \right) d\tau. \end{aligned}$$

Now, by the inequality (5) we get

$$|(Fz)(t) - (Fz)(s)| \leq \xi \int_0^s \|U(t, \tau) - U(s, \tau)\|_{B(E)} p(\tau) d\tau + \widehat{M} \xi \int_s^t p(\tau) d\tau.$$

The right-hand side of the above inequality tends to zero as $t - s \rightarrow 0$, which implies that $F(D)$ is equicontinuous.

Now, we will prove that $\Lambda := \{(Fz)(t) : z \in D\}$ is relatively compact in E . Let $t \in J$ be a fixed and let $0 < \varepsilon < t \leq b$. For $z \in D$ we define

$$F_\varepsilon(z)(t) = U(t, t - \varepsilon) \int_0^{t-\varepsilon} U(t - \varepsilon, s) f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) ds.$$

Since $U(t, s)$ is a compact operator, and the set $\Lambda_\varepsilon := \{(F_\varepsilon z)(t) : z \in D\}$ is the image of bounded set of E by $U(t, s)$ then Λ_ε is precompact in E . Furthermore, for $z \in D$, we have

$$|F(z)(t) - F_\varepsilon(z)(t)| \leq \int_{t-\varepsilon}^t \|U(t, s)\|_{B(E)} \left| f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) \right| ds$$

$$\begin{aligned}
&\leq \int_{t-\varepsilon}^t \|U(t, s)\|_{B(E)} p(s) \left(\|z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + 1 \right) ds \\
&\leq \xi \widehat{M} \int_{t-\varepsilon}^t e^{-\omega(t-s)} p(s) ds.
\end{aligned}$$

The right-hand side tends to zero as $\varepsilon \rightarrow 0$, then $F_\varepsilon(z)$ converge uniformly to $F(z)$ which implies that $D(t)$ is precompact in E .

Finally, it rest to show that F is **equiconvergent**.

Let $z \in D$, then from the assumptions (H_1) , (H_2) and (5) we have

$$|(Fz)(t)| \leq \widehat{M} \xi \int_0^t e^{-\omega(t-s)} p(s) ds,$$

it follows immediately by (5) that $|(Fz)(t)| \rightarrow 0$ as $t \rightarrow +\infty$.

Then

$$\lim_{t \rightarrow +\infty} |(Fz)(t) - (Fz)(+\infty)| = 0$$

which implies that F is equiconvergent.

Step 2.

L is a contraction.

Let $z, \bar{z} \in \mathcal{X}_0$, then by (H_1) , $(H_4) - (H_6)$ and (5) and for each $t \in J$ we have

$$\begin{aligned}
|(Lz)(t) - (L\bar{z})(t)| &\leq \left| g(t, x_{\rho(t, x_t + z_t)} + z_{\rho(t, x_t + z_t)}) - g(t, x_{\rho(t, x_t + \bar{z}_t)} + \bar{z}_{\rho(t, x_t + \bar{z}_t)}) \right| \\
&\quad + \int_0^t \|U(t, s)\|_{B(E)} \left| A(s)g(s, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)}) - A(s)g(s, x_{\rho(s, x_s + \bar{z}_s)} + \bar{z}_{\rho(s, x_s + \bar{z}_s)}) \right| ds \\
&\leq \|A^{-1}(t)\|_{B(E)} \left| A(t)g(t, x_{\rho(t, x_t + z_t)} + z_{\rho(t, x_t + z_t)}) - A(t)g(t, x_{\rho(t, x_t + \bar{z}_t)} + \bar{z}_{\rho(t, x_t + \bar{z}_t)}) \right| \\
&\quad + \widehat{M} \int_0^t e^{-\omega(t-s)} \left| A(s)g(s, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)}) - A(s)g(s, x_{\rho(s, x_s + \bar{z}_s)} + \bar{z}_{\rho(s, x_s + \bar{z}_s)}) \right| ds \\
&\leq \ell \widehat{M} \|z_{\rho(t, x_t + z_t)} - \bar{z}_{\rho(t, x_t + \bar{z}_t)}\|_{\mathcal{B}} + \widehat{M} \int_0^t e^{-\omega(t-s)} \|z_{\rho(s, x_s + z_s)} - \bar{z}_{\rho(s, x_s + \bar{z}_s)}\|_{\mathcal{B}} ds.
\end{aligned}$$

So, by Lemma 3.3 we obtain

$$\begin{aligned}
|(Lz)(t) - (L\bar{z})(t)| &\leq y \left(\ell \widehat{M} + \widehat{M} \int_0^t e^{-\omega(t-s)} ds \right) \|z - \bar{z}\|_{\mathcal{X}_0} \\
&\leq y \left(\ell \widehat{M} + \frac{\widehat{M}}{\omega} \right) \|z - \bar{z}\|_{\mathcal{X}_0}.
\end{aligned}$$

Therefore

$$\|Lz - L\bar{z}\|_{\mathcal{X}_0} \leq y \left(\ell \widehat{M} + \frac{\widehat{M}}{\omega} \right) \|z - \bar{z}\|_{\mathcal{X}_0}.$$

Thus, the operator L is a contraction.

In order to apply Theorem 2.4, we must check hypothesis (ii) does not hold, i.e., prove that the set

$$D_\lambda = \left\{ z \in \mathcal{X}_0 : z = \lambda L \left(\frac{z}{\lambda} \right) + \lambda F(z) \text{ for } \lambda \in (0, 1) \right\},$$

is bounded. Let $z \in D_\lambda$ then for each $t \in J$, we have

$$z(t) = \lambda L \left(\frac{z}{\lambda} \right) (t) + \lambda F(z)(t).$$

Then we get

$$\begin{aligned} |z(t)| &\leq \lambda \|U(t, 0)\|_{B(E)} \|A^{-1}(t)\|_{B(E)} |A(t)g(0, \phi)| + \|A^{-1}(t)\|_{B(E)} \left| A(t)g(t, x_{\rho(t, x_t + z_t)} + \frac{z_{\rho(t, x_t + z_t)}}{\lambda}) \right| \\ &\quad + \lambda \int_0^t \|U(t, s)\|_{B(E)} \left| A(s)g(s, x_{\rho(s, x_s + z_s)} + \frac{z_{\rho(s, x_s + z_s)}}{\lambda}) \right| ds \\ &\quad + \lambda \int_0^t \|U(t, s)\|_{B(E)} |f(s, x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)})| ds. \\ &\leq \lambda \widehat{M} \widetilde{M} \zeta(t) \|\phi\|_{\mathcal{B}} + \lambda \widetilde{M} \zeta(t) \left\| x_{\rho(t, x_t + z_t)} + \frac{z_{\rho(t, x_t + z_t)}}{\lambda} \right\|_{\mathcal{B}} + \lambda \widehat{M} \int_0^t e^{-\omega(t-s)} \zeta(s) \left\| x_{\rho(s, x_s + z_s)} + \frac{z_{\rho(s, x_s + z_s)}}{\lambda} \right\|_{\mathcal{B}} ds \\ &\quad + \lambda \widehat{M} \int_0^t e^{-\omega(t-s)} p(s) (\|x_{\rho(s, x_s + z_s)} + z_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + 1) ds. \end{aligned}$$

Thus, by Lemma 3.3 we have

$$|z(t)| \leq \widehat{M} \widetilde{M} \zeta^* \|\phi\|_{\mathcal{B}} + \widetilde{M} \zeta^* \gamma (\|z\|_{\mathcal{X}_0} + \mu) + \frac{\widehat{M} \zeta^*}{\omega} (\|z\|_{\mathcal{X}_0} + \mu) + \widehat{M} \gamma (\|z\|_{\mathcal{X}_0} + \mu + 1) \|p\|_{L^1},$$

with $\mu := (\widehat{M} + 1) \|\phi\|_{\mathcal{B}}$ and $\zeta^* := \sup_{t \in J} |\zeta(t)|$.

Therefore,

$$\|z\|_{\mathcal{X}_0} \leq \frac{\widehat{M} \widetilde{M} \zeta^* \|\phi\|_{\mathcal{B}} + \widetilde{M} \zeta^* \gamma \mu + \frac{\widehat{M} \zeta^*}{\omega} \mu + \widehat{M} (\mu + 1) \|p\|_{L^1}}{(1 - \widetilde{M} \zeta^* \gamma - \frac{\widehat{M} \zeta^*}{\omega} - \widehat{M} \gamma)} := c,$$

which implies that D_λ is bounded.

Thus by Theorem 3.4 the operator T has at least one fixed point which is a mild solution of problem (1).

4 Attractivity of solutions

In this section we study the local attractivity of solutions the problem (1)

Definition 4.1. [9] We say that solutions of (1) are locally attractive if there exists a closed ball $\overline{B}(z^*, \sigma)$ in the space $\mathcal{X}_0$ for some $z^* \in \mathcal{X}$ such that for arbitrary solutions z and $\tilde{z}$ of (1) belonging to $\overline{B}(z^*, \sigma)$ we have

$$\lim_{t \rightarrow +\infty} (z(t) - \tilde{z}(t)) = 0.$$

Under the assumptions of Section 3, let z^* be a solution of (1) and $\overline{B}(z^*, \sigma)$ the closed ball in $\mathcal{X}_0$ with σ satisfying the following inequality

$$\sigma \geq \frac{2\widehat{M}\|p\|_{L^1}}{1 - \widetilde{M}\ell\gamma - \frac{\widehat{M}\ell\gamma}{\omega} - 2\widehat{M}\gamma\|p\|_{L^1}}.$$

Moreover, we assume that

$$\lim_{t \rightarrow \infty} \zeta(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \int_0^t e^{-\omega(t-s)} \zeta(s) ds = 0, \quad (6)$$

where ζ is the function in (H_6) .

Then, for $z \in \overline{B}(z^*, \sigma)$ by (H_1) – (H_2) and (5) we have

$$\begin{aligned} |(Tz)(t) - z^*(t)| &= |(Tz)(t) - (Tz^*)(t)| \\ &\leq \|A^{-1}(t)\|_{B(E)} \left| A(t)g(t, z_{\rho(t, x_t + z_t)} + x_{\rho(t, x_t + z_t)}) - A(t)g(t, z_{\rho(t, x_t + z_t)}^* + x_{\rho(t, x_t + z_t)}) \right| \\ &\quad + \int_0^t \|U(t, s)\|_{B(E)} \left| f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) - f(s, z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}) \right| ds \\ &\quad + \int_0^t \|U(t, s)\|_{B(E)} \left| A(s)g(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) - A(s)g(s, z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}) \right| ds \\ &\leq \tilde{M}\ell \|z_{\rho(t, x_t + z_t)} - z_{\rho(t, x_t + z_t)}^*\|_{\mathcal{B}} + \widehat{M}\ell \int_0^t e^{-\omega(t-s)} \|z_{\rho(s, x_s + z_s)} - z_{\rho(s, x_s + z_s)}^*\|_{\mathcal{B}} ds \\ &\quad + \widehat{M} \int_0^t e^{-\omega(t-s)} p(t) \left(\|z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + \|z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + 2 \right) ds \\ &\leq \tilde{M}\ell y\rho + \frac{\widehat{M}\ell y\rho}{\omega} + 2\widehat{M}(y\rho + 1)\|p\|_{L^1} \\ &\leq \tilde{M}\ell y\rho + \frac{\widehat{M}\ell y\rho}{\omega} + 2\widehat{M}(y\rho + 1)\|p\|_{L^1} \\ &\leq \sigma. \end{aligned}$$

Therefore, we get $T(\overline{B}(z^*, \sigma)) \subset \overline{B}(z^*, \sigma)$.

So, for each $z \in \overline{B}(z^*, \rho)$ solution of problem (1) and $t \in J$, we have

$$\begin{aligned} |z(t) - z^*(t)| &= |(Tz)(t) - (Tz^*)(t)| \\ &\leq \|A^{-1}(t)\|_{B(E)} \left| A(t)g(t, z_{\rho(t, x_t + z_t)} + x_{\rho(t, x_t + z_t)}) - A(t)g(t, z_{\rho(t, x_t + z_t)}^* + x_{\rho(t, x_t + z_t)}) \right| \\ &\quad + \int_0^t \|U(t, s)\|_{B(E)} \left| f(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) - f(s, z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}) \right| ds \\ &\quad + \int_0^t \|U(t, s)\|_{B(E)} \left| A(s)g(s, z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}) - A(s)g(s, z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}) \right| ds \\ &\leq \tilde{M}\zeta(t) (\|z_{\rho(t, x_t + z_t)} + x_{\rho(t, x_t + z_t)}\|_{\mathcal{B}} + \|z_{\rho(t, x_t + z_t)}^* + x_{\rho(t, x_t + z_t)}\|_{\mathcal{B}}) \\ &\quad + \widehat{M} \int_0^t e^{-\omega(t-s)} p(s) \left(\psi(\|z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}}) + \psi(\|z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}}) \right) ds \\ &\quad + \widehat{M} \int_0^t e^{-\omega(t-s)} \zeta(s) \left(\|z_{\rho(s, x_s + z_s)} + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} + \|z_{\rho(s, x_s + z_s)}^* + x_{\rho(s, x_s + z_s)}\|_{\mathcal{B}} \right) ds \\ &\leq 2y\tilde{M}(\sigma + \mu)\zeta(t) + 2\widehat{M}\psi(y\sigma + y\mu) \int_0^t e^{-\omega(t-s)} p(s) ds + 2y\widehat{M}(\sigma + \mu) \int_0^t e^{-\omega(t-s)} \zeta(s) ds \end{aligned}$$

Hence, from (H_3) and (6), we conclude that

$$\lim_{t \rightarrow \infty} |z(t) - \tilde{z}(t)| = 0.$$

Consequently, the solutions of problem (1) are locally attractive.

5 An example

To apply the results of sections 2 and 3, we consider the problem

$$\begin{aligned} \frac{\partial}{\partial t} \left(v(t, \xi) + \int_{-\infty}^t a(t-s)v(s - \rho_1(t)\rho_2(|v(t, \xi)|), \xi) ds \right) = \\ \frac{\partial^2}{\partial \xi^2} v(t, \xi) + b(t, \xi)v(t, \xi) + \int_{-\infty}^t c(s-t)v(t - \rho_1(t)\rho_2(|v(t, \xi)|), \xi) ds, \end{aligned} \quad (7)$$

$$\begin{cases} v(t, 0) = v(t, \pi) = 0 & t \geq 0, \\ v(s, \xi) = v_0(s, \xi), & s \in (-\infty, 0], \xi \in [0, \pi], \end{cases} \quad (8)$$

where $\rho_1, \rho_2 : J \rightarrow \mathbb{R}_+$ and $a, c : \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and $b(t, \xi)$ is continuous with respect to t .

Let $X = L^2([0, \pi])$ and $A(t) : D(A) \subset X \rightarrow X$ be the operator defined as follows

$$A(t)x(\xi) = Ax(\xi) + b(t, \xi)x(\xi),$$

where $Ax = x''$ be the operator with domain $D(A) = \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$ which generates an analytic semigroup on X .

Now, we note $\mathcal{B} = C_0 \times L^p(h, X)$ the phase space introduced in Hino *et al.* [18], where $h : (-\infty, 0] \rightarrow \mathbb{R}$ be non-negative measurable function which satisfies the appropriate conditions cited in [18].

For $t \in J, \xi \in [0, \pi]$, and by defining the functions $f, g : J \times \mathcal{B} \rightarrow X$ by

$$y(t)(\xi) = v(t, \xi),$$

$$\phi(\tau)(\xi) = v_0(\tau, \xi), \text{ for } -\infty < \tau \leq 0,$$

$$g(t, y_t)(\xi) = \int_{-\infty}^t a(t-s)y(s - \rho(t, y_t))(\xi) ds,$$

and

$$f(t, y_t)(\xi) = \int_{-\infty}^t c(t-s)y(s - \rho(t, y_t))(\xi) ds,$$

where $\rho : J \times \mathcal{B} \rightarrow \mathbb{R}$ is defined by

$$\rho(t, y_t) := \rho_1(t)\rho_2(|y(t)(\xi)|),$$

We can transform the problem (7) – (8) into the abstract form (1).

Thus, under appropriate conditions on the functions a, b , and c , problem (7)-(8) has a solution which is locally attractive.

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