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Coherently enhanced measurements in classical mechanics

Abstract: In all quantitative sciences, it is common practice to increase the signal-to-noise ratio of noisy measurements by measuring identically prepared systems N times and averaging the measurement results. This leads to a scaling of the sensitivity as $1/\sqrt{N}$, known in quantum measurement theory as the "standard quantum limit" (SQL). It is known that if one puts the N systems into an entangled state, a scaling as $1/N$ can be achieved, the so-called "Heisenberg limit" (HL), but decoherence problems have so far prevented implementation of such protocols for large N. Here we show that a method of coherent averaging inspired by a recent entanglement-free quantum enhanced measurement protocol is capable of achieving a sensitivity that scales as $1/N$ in a purely classical setup. This may substantially improve the measurement of very weak interactions in the classical realm, and, in particular, open a novel route to measuring the gravitational constant with enhanced precision.

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Quantum enhanced measurements deal very generally with the increase of the sensitivity of measurement by exploiting quantum mechanical effects. The fact that the classical $1/\sqrt{N}$ behavior might be improved upon has received large attention [1–22]. Examples include the use of NOON states in a Mach-Zehnder interferometer [23, 24], or squeezed spin states for magnetometers based on atomic vapors [25]. Quantum-enhanced measurements were investigated for many different physical observables and contexts, including the measurements of time [23], position [26–29], temperature, and chemical potential [30–32]. Quantum-enhanced measurement of arbitrary parameters encoded in multi-mode Gaussian light was examined in [33], and in [5], very generally, a parameter multiplying an arbitrary hermitian generator was treated. Unfortunately, the entangled states typically required in these schemes are very unstable and prone to decoherence. Experiments with NOON states showing a slight improvement over the SQL have not surpassed yet the stage of more than a few entangled photons [4, 34]. On the other hand, classical experiments such as LIGO have already sensitivities of the order $10^{-22}/\sqrt{\text{Hz}}$ [35, 36]. To compete with such performance by changing the scaling law with the number of photons using entanglement, one would have to entangle a macroscopic number of particles (or create a NOON state with a macroscopic number of photons), which seems out of reach considering the experimental difficulties of creating a NOON state with just 4 photons. Also, from theoretical grounds, it has become clear that for NOON states the slightest amount of Markovian decoherence leads back to the SQL scaling for sufficiently large N [37–39]. Nevertheless, quantum enhanced metrology was recently succesfully implemented by the LIGO collaboration by using squeezed vacuum [40, 41]. This goes back to one of the earliest theoretical proposals of quantum enhanced measurements [1] and improves the prefactor of the scaling law. While less ambitious, this constitutes an important milestone for the practical use of entanglement. It was also shown that given the photon losses this approach is essentially optimal [42]. Non-classical states of light with small photon numbers may also be interesting for niche applications, such as in biological systems that require low intensities [10], and there is some hope for improving over classical schemes in the case of non-Markovian decoherence at time scales shorter than the correlation time scale of the heat bath [43].

Recently, an alternative quantum enhanced method for reaching Heisenberg-limited sensitivity was introduced, in which N distinguishable systems interact with a $N + 1$st system and one reads out the latter [44, 45]. Furthermore, the scaling with N is stable under local decoherence, and even decoherence itself can be used as a signal, if the $N + 1$st system is an environment. The effect can be understood as "coherent averaging": a phase

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accumulates in the state of the $N + 1$st system from the interaction with the N other systems. No entanglement is needed. These properties make one wonder whether a classical analogue of this mechanism exists. Here we show that this indeed true. More specifically, we demonstrate a phase accumulation mechanism in the classical motion of a central harmonic oscillator interacting with N other harmonic oscillators that is completely analogous to the quantum mechanical scenario, and that allows one to achieve a sensitivity that scales as $1/N$, just as in the quantum case with Heisenberg-limited sensitivity. This shows that even in the classical realm there are situations where one can improve upon the venerated averaging of N independent measurement results by coupling the same resources "coherently" to a $N+1$st system, and measuring the latter.

Consider a classical harmonic oscillator with frequency $\omega_0 \equiv \Omega$ harmonically coupled to N other harmonic oscillators with frequency ω_i, $i = 1, \dots, N$. The Hamilton function (with masses $m_i = 1$) reads

$$H = \frac{1}{2}\sum_{i=0}^{N}\left(p_i^2 + \omega_i^2 q_i^2\right) + \frac{1}{2}\xi^2\sum_{i=1}^{N}(q_i - q_0)^2 \,, \qquad (1)$$

where p_i and q_i are the canonical momenta and coordinates, respectively, and ξ^2 denotes the coupling strength, such that ξ has the dimension of a frequency. This is the parameter we want to determine. The oscillators could be mechanical, LC circuits, e.m. fields in cavities and many more. We are interested in very small couplings, $\xi^2 \ll \omega_i^2$, $i = 0, \dots, N$, and restrict ourselves to the situation where the ω_i are narrowly distributed about a central frequency $\overline{\omega}$, and off-resonant from Ω. We consider two sources of uncertainty: *i.)* an uncertainty in the frequencies ω_i, $i = 1, \dots, N$, described by a distribution $P(\omega)$, and *ii.)* time dependent noise. For a given signal $s(t)$ that depends on the parameter ξ^2, we follow common experimental practice and define the sensitivity with which ξ^2 can be measured as

$$\delta\xi^2_{min} = \frac{\sigma(s(t))}{\sqrt{M}\left|\langle\frac{\partial s(t)}{\partial \xi^2}\rangle\right|} \,, \qquad (2)$$

where $\langle\dots\rangle$ means average over the noise process, $\sigma(s(t))$ is the standard deviation of $s(t)$ with respect to this distribution, and M is the number of measurements. It has the meaning of the smallest variation in ξ^2 that moves the average of the signal at least a distance given by the width of the distribution of the signal.

1 Methods

The total potential energy in the problem can be rewritten as a quadratic form,

$$V(\mathbf{q}) \equiv H - \sum_{i=0}^{N} p_i^2/2 = \frac{1}{2}\mathbf{q}^t\mathbf{C}\mathbf{q} \,, \qquad (3)$$

with $\mathbf{q}^t = (q_0, \dots, q_N)$, and

$$\mathbf{C} = \begin{pmatrix} \Omega^2 + N\xi^2 & -\xi^2 & \dots & -\xi^2 \\ -\xi^2 & \omega_1^2 + \xi^2 & 0\dots & 0 \\ \vdots & 0 & \ddots & \vdots \\ -\xi^2 & 0 & \dots & \omega_N^2 + \xi^2 \end{pmatrix} . \qquad (4)$$

The dynamics of the model is easily found by diagonalizing the interaction matrix C. Under the conditions outlined non-degenerate perturbation theory (PT) will suffice to obtain the correction to the frequency Ω of the central oscillator. To order $\mathcal{O}(\xi^2)$ we have $\lambda_0 = \Omega^2 + N\xi^2 + \mathcal{O}(\xi^4)$ and $\lambda_l = \omega_l^2 + \xi^2 + \mathcal{O}(\xi^4)$, $\quad l = 1, \dots, N$. The perturbed eigenmodes $\mathbf{u}_l$ are summarized in the orthogonal transformation matrix $\mathbf{U} = (\mathbf{u}_0, \mathbf{u}_1 \dots, \mathbf{u}_N)$ to order $\mathcal{O}(\xi^2)$ as

$$\mathbf{U} = \begin{pmatrix} 1 & -\frac{\xi^2}{\omega_1^2-\Omega^2} & \dots & -\frac{\xi^2}{\omega_N^2-\Omega^2} \\ \frac{\xi^2}{\omega_1^2-\Omega^2} & 1 & 0 & \dots \\ \vdots & 0 & \ddots & 0 \\ \frac{\xi^2}{\omega_N^2-\Omega^2} & 0 & \dots & 1 \end{pmatrix} . \qquad (5)$$

The equations of motion without external driving, $\ddot{q}_i + \sum_j C_{ij}q_j = 0$ are decoupled into $N + 1$ independent harmonic oscillators by the transformation $\mathbf{q} \to \tilde{\mathbf{q}} = \mathbf{U}^t\mathbf{q}$. Solving them and transforming back leads to the response of the central oscillator. In order to simplify expressions, we specialize to vanishing initial speeds for all oscillators, $\dot{q}_j(0) = 0$, $j = 0, \dots, N$, but we emphasize that this is by no means necessary for the method to work. We will furthermore assume $N\xi^2 \ll \Omega^2$, such that $\sqrt{\lambda_0} = \Omega(1 + N\xi^2/(2\Omega^2)) + \mathcal{O}(\xi^4))$. This limits N, but for very small ξ^2, N can become very large (see also the comments in the *Discussion* for the validity beyond PT).

In the case of time-dependent noise, we get after transformation to the eigenmodes $\tilde{q}_i$ defined through $q_j = \sum_l U_{jl}\tilde{q}_l$ from eq.(13)

$$\ddot{\tilde{q}}_k + \lambda_k\tilde{q}_k = \sum_i U^{\dagger}_{ki}f_i(t) \equiv \tilde{f}_k(t) \,. \qquad (6)$$

A special solution of this equation can be found with the help of the Greens-function of the harmonic oscillator. The back transformation to the original coordinates gives for the central oscillator

$$q_0(t) = q_0(0)\cos(\sqrt{\lambda_0}t) + \frac{\dot{q}_0(0)}{\sqrt{\lambda_0}}\sin(\sqrt{\lambda_0}t) + \int_0^t \frac{\sin\sqrt{\lambda_0}(t-t')}{\sqrt{\lambda_0}} f_0(t')dt' + \mathcal{O}(\xi^2). \quad (7)$$

The noise enters here alread at order ξ^0, i.e. perturbs even the uncoupled central oscillator. Nevertheless, we will now see that the phase accumulation of the central oscillator due to the coupling to the N other oscillators still leads to a $1/N$ scaling of the sensitivity.

We restrict ourselves again to $\dot{q}_j(0) = 0$ for all $j = 0,\ldots,N$ and $N\xi^2 \ll \Omega^2$ such that $\sqrt{\lambda_0} = \Omega(1 + N\xi^2/(2\Omega^2)) + \mathcal{O}(\xi^4))$, and calculate the direct response to the noise. Once more we emphasize that this is for the ease of presentation only. For simplicity we consider noise with zero average, $\langle f_0(t)\rangle = 0\ \forall t$, where $\langle\ldots\rangle$ means now average over the noise-process. We then have $\langle q_0(t)\rangle = q_0(0)\cos\sqrt{\lambda_0}t$ and $\sigma^2(q_0(t)) = \sigma^2(n(t))$, where

$$n(t) = \int_0^t \frac{\sin\sqrt{\lambda_0}(t-t')}{\sqrt{\lambda_0}} f_0(t')dt' \quad (8)$$

is the noise response.

2 Results

i.) Uncertainty in frequencies. Assuming for simplicity vanishing initial speeds for all oscillators, the solution of Newton's equations of motion leads for the central oscillator to

$$q_0(t) = q_0(0)\cos((\Omega + \frac{N\xi^2}{2\Omega})t) + \xi^2\sum_{j=1}^N \frac{q_j(0)}{\omega_j^2 - \Omega^2} \times\left(\cos((\Omega + \frac{N\xi^2}{2\Omega})t) - \cos((\omega_j + \frac{\xi^2}{2\omega_j})t)\right), \quad (9)$$

which is correct to $\mathcal{O}(\xi^2)$. The appearance of a phase shift that scales proportional to N and the parameter to be measured is reminiscent of phase superresolution [34]. This signal can be isolated by mixing the response of the central oscillator with a $\cos(\Omega t)$ signal corresponding to the unperturbed oscillator, followed by a low-pass filter. One can do this either by having the central oscillator physically interact with an oscillator of frequency Ω, or by high-frequency sampling of $q_0(t)$ and subsequent data analysis. The mixing with the unperturbed signal is not necessary for the method to work, but the frequency down-conversion may be advantageous experimentally and also leads to simplified theoretical expressions. If the spectral width of the low-pass filter is much smaller than 2Ω and $\omega_j - \Omega$, the remaining signal $s(t)$ reads

$$s(t) = \left(q_0(0) + \xi^2 r(\{\omega_i\})\right)\cos\left(\frac{N\xi^2}{2\Omega}t\right),$$

$$\text{where } r(\{\omega_i\}) = \sum_{j=1}^N \frac{q_j(0)}{\omega_j^2 - \Omega^2} \quad (10)$$

is a random variable whose distribution is given by the distribution $P(\{\omega_j\})$ of the ω_j. Using eq.(2), we find

$$\delta\xi^2_{min} = \frac{\xi^2\sigma(r)|\cos(\frac{N\xi^2}{2\Omega}t)|}{\sqrt{M}\left|\frac{N}{2\Omega}t\left(q_0(0) + \xi^2\langle r\rangle\right)\sin(\frac{N\xi^2}{2\Omega}t) - \langle r\rangle\cos\left(\frac{N\xi^2}{2\Omega}t\right)\right|}. \quad (11)$$

If one waits long enough ($N\xi^2 t/(2\Omega) \gg 1$), the first term in the denominator dominates. If we set in addition $q_0(0) = 0$, we obtain the final result

$$\delta\xi^2_{min} = \frac{1}{N}\frac{2\Omega}{\sqrt{M}t}\left|\cot\left(\frac{N\xi^2 t}{2\Omega}\right)\right|\frac{\sigma(r)}{\langle r\rangle}. \quad (12)$$

ii.) Time dependent noise. In the presence of time-dependent noise forces $f_i(t)$ acting on oscillator i, the equations of motion in the original oscillator coordinates q_i read

$$\ddot{q}_i + \sum_j C_{ij}q_j = f_i(t). \quad (13)$$

Consider first white noise, defined through $\langle f_i(t_1)f_i(t_2)\rangle = f_0^2 T\delta(t_1 - t_2)$, where we have introduced a unit of time T for dimensional grounds, in addition to the force amplitudes f_0. One then immediately gets (see *Methods*)

$$\sigma^2(q_0(t)) = f_0^2 T\int_0^t \frac{\sin^2\sqrt{\lambda_0}(t-t')}{\lambda_0}dt' \le \frac{f_0^2 Tt}{\lambda_0}. \quad (14)$$

In fact, for large times, $t \gg 1/\sqrt{\lambda_0}$, one has $\sigma^2(q_0(t)) \simeq \frac{f_0^2 Tt}{2\lambda_0}$. All the dependence on N of the sensitivity arises again from the derivative of $\langle q_0(t)\rangle$ and thus λ_0 with respect to ξ^2. Inserting everything in eq.(2), we are led to

$$\delta\xi^2_{\min} \le \frac{2f_0\sqrt{T/t}}{\sqrt{M}N|q_0(0)\sin(\sqrt{\lambda_0}t)|}, \quad (15)$$

where for large times still a factor $1/\sqrt{2}$ can be gained on the rhs.

The above considerations are easily generalized to colored noise. In fact, unless the noise f_0 on oscillator 0 depends already at order ξ^0 on N (which appears to be a highly artificial situation, since without interaction the central oscillator should not "know" about the number of additional oscilltors), $\sigma^2(n(t))$ is independent of N, and the same scaling analysis concerning N therefore applies and always leads to a $1/N$ scaling of the sensitivity. Only the time dependence will differ. As a more general example, consider stationary colored noise with a correlation function $\langle f_0(t_1)f_0(t_2)\rangle = f_0^2 C(t_1 - t_2)$ where we take $C(0) = 1$, and $C(-t) = C(t)$. One then easily finds the upper bound

$$\sigma^2(q_0(t)) \leq \frac{2f_0^2}{\lambda_0} \int_0^t dt_- \, |C(t_-)|(t - t_-) \,. \tag{16}$$

If the correlation function vanishes for $t > t_c$, one has $\sigma^2(q_0(t)) \leq \frac{2f_0^2}{\lambda_0} b(t)$ with

$$b(t) = \begin{cases} tt_c - \frac{1}{2}t_c^2 & t_c < t \\ \frac{1}{2}t^2 & t_c \geq t \,, \end{cases} \tag{17}$$

where we have used $|C(t)| \leq |C(0)|$. Correspondingly, we have for the sensitivity

$$\delta\xi^2_{\min} \leq \frac{2\sqrt{2}f_0}{\sqrt{M}} \frac{\sqrt{b(t)}}{Nt|q_0(0)\sin(\sqrt{\lambda_0}t)|} \,. \tag{18}$$

3 Discussion

The prefactor $1/N$ in eqs.(12,15,18) identifies the sensivitities for all noise process considered as analogous to "Heisenberg-limited" in quantum enhanced measurements. Of course, this has nothing to do with Heisenberg's uncertainty relation. Rather, we have considered purely classical noise processes (uncertainty in the original frequencies ω_i, or classical time dependent noise forces), but have found a way to reduce the resulting smallest classical uncertainty with which the parameter ξ^2 can be measured from a $1/\sqrt{N}$ scaling that would be obtained by measuring it separately for each system and then averaging, to a $1/N$ scaling. The process how this happens is completely analogous to the quantum mechanical collective phase accumulation described in [44, 45]: N systems interact with a common central system, and lead to an accumulated phase proportional to N and the parameter to be measured. This manifests itself in an oscillation with a frequency proportional to $N\xi^2$ in a corotating frame — the equivalent of homodyne detection in the quantum optical setting. In the case of uncertain frequencies, it also leads to the same scaling with t, namely as $1/t$, and not the usual $1/\sqrt{t}$. It means that the sensitivity per square root of Hertz, $\delta\xi^2_{min}\sqrt{t}$, still decreases as $1/\sqrt{t}$, just as in the Heisenberg limited quantum case. For time dependent noise, the time dependence of the minimal uncertainty decays only as $1/\sqrt{t}$, just as in the standard quantum limit.

Note that some authors, when counting the resources, count the total number of times that the parameter to be measured (or even more generally: some observable that may not be the same as the parameter) is sampled in a measurement setup. This is particularly important in multi-round protocols (see e.g.[9]) for comparing different protocols on the same footing. In our case both ways of counting lead to the same definition of N: each of the N harmonic oscillators coupled to the central one can be thought of sampling the interaction strength ξ once, as they are coupled to the latter permanently, such that the number of samplings is identical to the number N of these oscillators.

Different from the quantum-mechanical case is the restriction to N such that $N\xi^2 \ll \Omega^2$. This is true beyond the validity of PT, as is seen by analysing the very strong coupling limit, where **C** can again be diagonalized analytically. There is no such limitation on the interaction strength in [44], and quantum mechanics thus still provides an advantage. Nevertheless, the method proposed here may be advatageous compared to traditional classical data analysis for very small interactions, where N can be very large before the $1/N$ scaling breaks down, and dominating classical noise.

We emphasize that we are not concerned here with reaching the "true" Heisenberg-limit due to quantum fluctuations. This would require the calculation of the quantum Cramèr-Rao bound for the corresponding quantum model and is not the topic of the present work (the general framework for doing so can be found in [44]). Rather, we accept that in many experimental situations classical sources of noise and uncertainty dominate by far. This is in particular true for the measurement of the gravitational constant examined below. In such a situation, the so far only available approach to reducing the uncertainty of the observable in question, after all efforts have been made to reduce technical noise and classical uncertainties as much as possible, is to average the measured signals from N repetitions of the experiment, which leads to the well known $1/\sqrt{N}$ scaling of the uncertainty of the average. Our fundamental new insight is that this approach is not optimal and can be improved upon by "coherent averaging", which allows one to reduce the uncertainty by a factor $1/N$. This may open the road to a new kind of classical precision measurements.

It is also interesting to compare our method to sequential procedures of quantum enhanced measurements. In such an approach one goes back to the original classical way of counting resources, namely one has one system that one samples N times, rather than having N copies of it which then can be entangled in the quantum case. In such a scenario obviously the notion of entanglement between N subsystems does not even arise. Nevertheless, it was shown that a $1/N$ scaling of the sensitivity of a phase shift measurement can be achieved. This was demonstrated by passing a single photon N times through a phase shifter, which increases the phase shift by a factor N [9]. As long as photon loss can be neglected, the uncertainty in the measured signal is essentially independent of the number of passes through the phase shifter, the signal-to-noise ratio increases by factor N, and the sensitivity scales as $1/N$. On a formal level, the phase accumulation in the one-photon state of the two modes representing the two arms of the Mach-Zehnder interferometer used in [9], $|\psi\rangle = (e^{iN\theta}|1\rangle|0\rangle + e^{iN\varphi}|0\rangle|1\rangle)/\sqrt{2}$, where φ is the phase to be measured, and θ a reference phase that is adapted to match φ by an adaptive phase estimation algorithm, looks quite similar to the phase accumulation in the quantum version [44] of our "coherent averaging" protocol. However, $|\psi\rangle$ is still a one-photon mode-entangled state, whereas with coherent averaging there is no need for entanglemend at all, allowing the classical analog in the first place. Another difference is that we measure here an interaction between N probes and a central system, whereas in the sequential procedure there is only the probe itself, subjected to N transformations parametrized by the parameter one wants to measure. Nevertheless, loosely speaking, there is some parallel between the two mechanisms in the sense that our coherent averaging approach leads to a phase accumulation that does in parallel (and in a single system rather than in two modes) what in [9] was done sequentially.

A possible application of our "coherent averaging" technique might be a novel way of measuring the gravitational constant G, which is one of the least well-determined natural constants with a relative uncertainty of order 10^{-4}, orders of magnitude above the uncertainty due to quantum noise, and the 1986 CODATA recommended value is based on conflicting experimental results [46]. One of the reasons for this dire situation is the extremely weak strength of the gravitational interaction. This, and the impossibility to shield the gravitational field from other disturbing bodies, render the determination of the absolute value of G very difficult, in spite of continued strong interest, driven in part by attempts to detect a variation of G as function of distance, time, or other physical quantities. Since Cavendish's pioneering work in 1798, essentially all lab-experiments attempting to measure G were based on a beam balance or a torsion pendulum, and measured either a static response (some by counterbalancing deflections of the small test masses), or a dynamic one (allowing frequency-specific analysis synchronized with a periodic excitation, see e.g. the recent attempt to measure deviations of the $1/r^2$ behavior below the dark energy length scale of about 85μm [47]). Our "coherent averaging" method suggests a new, massively parallel way of attempting to measure G more precisely: Couple $N \gg 1$ torsion balances gravitationally to a central one. The central oscillator consists of a central rigid axis (say, placed vertically) and N horizontal beams with masses fixed at the ends; all beams are attached rigidly, at equal intervals, to the common axis. Each of the N torsion balances is placed close to one of the beams of the central oscillator. These torsion balances are assumed sufficiently far away from each other that their interaction can be neglected compared to their interaction with the central one. This can always be achieved by having the distance between the central beams large enough. For sufficiently small amplitudes of the oscillations, this setup maps to model (3) up to numerical factors and with $\xi^2 \propto G$. Then measure the shift in frequency of that central oscillator as function of N and the positions of the test masses. This will enable a reduction of the uncertainty in the measured value of G with a scaling $1/N$ and should become competitive with traditional measurements for large N.

In summary, the above analysis shows that even in the classical realm there are situations where it is possible to beat the venerated procedure of averaging N measurement results of a physical quantity that leads to a $1/\sqrt{N}$ scaling of the sensitivity. It can be improved upon by coupling the N samples "coherently" to a central oscillator that will pick up a collective phase proportional to N and the parameter to be measured, and yield in the end a $1/N$ scaling with the number of samples available. The $1/N$ scaling is largely independent of the kind of noise considered. This was demonstrated by considering uncertainties in the frequencies of the N oscillators, and time dependent noise forces for both white and colored noise. Applications might be found in the measurement of very weak interactions, such as the gravitational interaction between lab-scale test masses, suggesting a new, massively parallel way of determining the gravitational constant.

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