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Neighbourhoods of independence and associated geometry in manifolds of bivariate Gaussian and Freund distributions

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Abstract: We provide explicit information geometric tubular neighbourhoods containing all bivariate distributions sufficiently close to the cases of independent Poisson or Gaussian processes. This is achieved via affine immersions of the 4-manifold of Freund bivariate distributions and of the 5-manifold of bivariate Gaussians. We provide also the α-geometry for both manifolds. The Central Limit Theorem makes our neighbourhoods of independence limiting cases for a wide range of bivariate distributions; the topological character of the results makes them stable under small perturbations, which is important for applications in models of stochastic processes.

Keywords: Information geometry, statistical manifold, bivariate distribution, neighbourhoods of independence, exponential distribution, Freund distribution, Gaussian distribution
MSC (2000): 53B20, 60G05

1 Introduction

In general, a probability density function depends on a set of parameters, $\theta^1, \theta^2, \ldots, \theta^n$ and we say that we have an n-dimensional family of probability density functions. Let Θ be the parameter space of an n-dimensional smooth such family defined on some fixed event space Ω

$$\{p_\theta | \theta \in \Theta\} \quad \text{with} \quad \int_\Omega p_\theta = 1 \quad \text{for all } \theta \in \Theta.$$

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Then, the derivatives of the log-likelihood function, $l = \log p_\theta$, yield a matrix with entries

$$g_{ij} = \int_\Omega p_\theta \left(\frac{\partial l}{\partial \theta^i} \frac{\partial l}{\partial \theta^j} \right) = - \int_\Omega p_\theta \left(\frac{\partial^2 l}{\partial \theta^i \partial \theta^j} \right) , \tag{1}$$

for coordinates (θ^i) about $\theta \in \Theta \subseteq \mathbb{R}^n$.

This gives rise to a positive definite matrix, so inducing a Riemannian metric g, the Fisher metric on Θ using for coordinates the parameters (θ^i); this metric is called the information metric for the family of probability density functions—the second equality here is subject to certain regularity conditions. Amari [1] and Amari and Nagaoka [2] provide modern accounts of the differential geometry that arises from the Fisher information metric.

An n-dimensional set of probability density functions $S = \{p_\theta | \theta \in \Theta \subset \mathbb{R}^n\}$ for random variable $x \in \Omega \subseteq \mathbb{R}$ is said to be an exponential family [2] when the density functions can be expressed in terms of functions $\{C, F_1, ..., F_n\}$ on $\mathbb{R}$ and a function φ on Θ as:

$$p_\theta(x) = e^{\{C(x) + \sum_i (\theta^i F_i(x)) - \varphi(\theta)\}} . \tag{2}$$

Then we say that (θ^i) are its natural coordinates, and φ is its potential function. From the normalization condition $\int_\Omega p_\theta(x)\, dx = 1$ we obtain:

$$\varphi(\theta) = \log \int_\Omega e^{\{C(x) + \sum_i (\theta^i F_i(x))\}} \, dx . \tag{3}$$

With $\partial_i = \frac{\partial}{\partial \theta^i}$, we use the log-likelihood function $l(\theta, x) = \log(p_\theta(x))$ to obtain

$$\partial_i l(\theta, x) = F_i(x) - \partial_i \varphi(\theta)$$

and

$$\partial_i \partial_j l(\theta, x) = -\partial_i \partial_j \varphi(\theta) .$$

The Fisher information metric g on the n-dimensional space of parameters $\Theta \subset \mathbb{R}^n$, equivalently on the set $S = \{p_\theta | \theta \in \Theta \subset \mathbb{R}^n\}$, has coordinates:

$$[g_{ij}] = - \int_\Omega [\partial_i \partial_j l(\theta, x)] \; p_\theta(x) \; dx = \partial_i \partial_j \varphi(\theta) = \varphi_{ij}(\theta) \; . \tag{4}$$

Then, (S, g) is a Riemannian n-manifold with Levi-Civita connection given by:

$$\begin{aligned}
\Gamma^k_{ij}(\theta) &= \sum_{h=1}^n \frac{1}{2} \, g^{kh} \, (\partial_i g_{jh} + \partial_j g_{ih} - \partial_h g_{ij}) \\
&= \sum_{h=1}^n \frac{1}{2} \, g^{kh} \, \partial_i \partial_j \partial_h \varphi(\theta) = \sum_{h=1}^n \frac{1}{2} \, \varphi^{kh}(\theta) \, \varphi_{ijh}(\theta)
\end{aligned}$$

where $[\varphi^{hk}(\theta)]$ represents the inverse to $[\varphi_{hk}(x)]$.

There is a family of symmetric connections which includes the Levi-Civita case and has significance in mathematical statistics. Consider for $\alpha \in \mathbb{R}$ the function $\Gamma^{(\alpha)}_{ij,k}$ which maps each point $\theta \in \Theta$ to the following value:

$$\begin{aligned}\Gamma^{(\alpha)}_{ij,k}(\theta) &= \int_\Omega \left(\partial_i\partial_j l + \frac{1-\alpha}{2}\,\partial_i l\,\partial_j l\right)\,\partial_k l\; p_\theta \\ &= \frac{1-\alpha}{2}\,\partial_i\partial_j\partial_k\varphi(\theta) = \frac{1-\alpha}{2}\,\varphi_{ijk}(\theta)\,. \end{aligned} \tag{5}$$

So we have an affine connection $\nabla^{(\alpha)}$ on the statistical manifold (S, g) defined by

$$g(\nabla^{(\alpha)}_{\partial_i}\partial_j, \partial_k) = \Gamma^{(\alpha)}_{ij,k}\,,$$

where g is the Fisher information metric. We call this $\nabla^{(\alpha)}$ the α-connection and it is clearly a symmetric connection and defines an α-curvature. We have also

$$\begin{aligned}\nabla^{(\alpha)} &= (1-\alpha)\;\nabla^{(0)} + \alpha\,\nabla^{(1)}\,, \\ &= \frac{1+\alpha}{2}\,\nabla^{(1)} + \frac{1-\alpha}{2}\,\nabla^{(-1)}\,.\end{aligned}$$

For a submanifold $M \subset S$, the α-connection on M is simply the restriction with respect to g of the α-connection on S. Note that the 0-connection is the Riemannian or Levi-Civita connection with respect to the Fisher metric and its uniqueness implies that an α-connection is a metric connection if and only if $\alpha = 0$.

In [4] we proved that every neighbourhood of an exponential distribution contains a neighbourhood of gamma distributions, in the subspace topology of $\mathbb{R}^3$ using an information geometric affine immersion of Dodson and Matsuzoe [10]. As part of a study of the information geometry and topology of near random and bivariate stochastic processes cf. [3, 4, 6–9], we calculated the geometry of the Riemannian 4-manifold of Freund bivariate (mixture) exponential density functions. This family is important because exponential distributions represent intervals between events for Poisson processes and Freund distributions can model bivariate processes with positive and negative covariance. We derive the induced α-geometry, i.e., the α-Ricci curvature, the α-scalar curvature etc. The case $\alpha = 0$ recovers the Levi-Civita connection and it has a positive constant 0-scalar curvature.

Sato et al [16] provided the bivariate Gaussian distributions as a Riemannian 5-manifold; it has a negative constant 0-scalar curvature and if the covariance is zero, the space becomes an Einstein space. We calculate the α-geometry. In each of the Freund and bivariate Gaussian cases we provide explicitly an affine immersion and examples of neighbourhoods of independence.

Thus, including the results we reported in [4], we now have explicit representations in $\mathbb{R}^3$ of information geometric tubular neighbourhoods containing by continuity each of the following:

- All distributions sufficiently close to a Poisson distribution
- All distributions sufficiently close to a uniform distribution

- All bivariate distributions sufficiently close to the independent bivariate Poisson distribution
- All bivariate distributions sufficiently close to the independent bivariate Gaussian distribution.

These results have wide application in the theory of stochastic processes because Poisson distributions model the random state, of independent haphazard events, and provide good limiting models for some binomial distributions. Moreover, the Central Limit Theorem makes our neighbourhoods of Gaussian independence limiting cases for a wide range of bivariate distributions other than Gaussian.

There are practical applications because the topological character of the results makes them stable under small perturbations. The authors used *Mathematica* to perform analytic calculations and the interactive notebooks are available for others to use [5].

2 Freund bivariate exponential 4-manifold F

Freund [11] introduced a bivariate exponential mixture distribution arising from the following reliability considerations. Suppose that an instrument has two components A and B with lifetimes X and Y respectively having probability density functions (when both components are in operation)
$f_X(x) = \alpha_1\, e^{-\alpha_1 x}$; $f_Y(y) = \alpha_2\, e^{-\alpha_2 y}$ for $(\alpha_1, \alpha_2 > 0; x, y > 0)$.

Then X and Y are dependent in that a failure of either component changes the parameter of the life distribution of the other component. Thus when A fails, the parameter for Y becomes β_2; when B fails, the parameter for X becomes β_1. There is no other dependence. Hence the joint probability density function of X and Y is:

$$f(x,y) = \begin{cases} \alpha_1 \beta_2 e^{-\beta_2 y - (\alpha_1+\alpha_2-\beta_2)x} \text{ for } 0 < x < y \\ \alpha_2 \beta_1 e^{-\beta_1 x - (\alpha_1+\alpha_2-\beta_1)y} \text{ for } 0 < y < x \end{cases} \tag{6}$$

where $\alpha_i, \beta_i > 0 \quad (i = 1, 2)$.

The marginal probability density function of $X \geq 0$ is (provided that $\alpha_1 + \alpha_2 \neq \beta_1$)

$$f_X(x) = \left(\frac{\alpha_2}{\alpha_1 + \alpha_2 - \beta_1}\right) \beta_1\, e^{-\beta_1 x} + \left(\frac{\alpha_1 - \beta_1}{\alpha_1 + \alpha_2 - \beta_1}\right) (\alpha_1 + \alpha_2)\, e^{-(\alpha_1+\alpha_2)x}. \tag{7}$$

The marginal probability density function of $Y \geq 0$ is (provided that $\alpha_1 + \alpha_2 \neq \beta_2$)

$$f_Y(y) = \left(\frac{\alpha_1}{\alpha_1 + \alpha_2 - \beta_2}\right) \beta_2\, e^{-\beta_2 y} + \left(\frac{\alpha_2 - \beta_2}{\alpha_1 + \alpha_2 - \beta_2}\right) (\alpha_1 + \alpha_2)\, e^{-(\alpha_1+\alpha_2)y}. \tag{8}$$

We can see that the marginal functions are not exponential but rather mixtures of exponential densities if $\alpha_i > \beta_i$; otherwise, they are weighted averages. This family should be termed bivariate mixture exponential densities rather than simply bivariate exponential densities. The marginal density functions $f_X(x)$ and $f_Y(y)$ are exponential distributions

only in the special case $\alpha_i = \beta_i \quad (i = 1, 2)$.
The covariance and correlation coefficient of X and Y are given by:

$$Cov(X, Y) = \frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\beta_1\,\beta_2\,(\alpha_1 + \alpha_2)^2}\,, \tag{9}$$

$$\rho(X, Y) = \frac{\beta_1\,\beta_2 - \alpha_1\,\alpha_2}{\sqrt{{\alpha_2}^2 + 2\,\alpha_1\,\alpha_2 + {\beta_1}^2}\,\sqrt{{\alpha_1}^2 + 2\,\alpha_1\,\alpha_2 + {\beta_2}^2}} \tag{10}$$

Note that $-\frac{1}{3} < \rho(X, Y) < 1$. The correlation coefficient $\rho(X, Y) \to 1$ when $\alpha_1,\ \beta_2 \to \infty$, and $\rho(X, Y) \to -\frac{1}{3}$ when $\alpha_1 = \alpha_2$ and $\beta_1,\ \beta_2 \to 0$. In many applications, $\beta_i > \alpha_i \quad (i = 1, 2)$ (i.e., lifetime tends to be shorter when the other component is out of action); in such cases the correlation is positive.

2.1 Freund Fisher metric

The Freund family F in coordinates $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ has Fisher information metric components

$$[g_{ij}] = \frac{1}{\alpha_1 + \alpha_2}\begin{bmatrix} \frac{1}{\alpha_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\alpha_2} & 0 & 0 \\ 0 & 0 & \frac{\alpha_2}{{\beta_1}^2} & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{{\beta_2}^2} \end{bmatrix} \tag{11}$$

with inverse

$$[g^{ij}] = (\alpha_1 + \alpha_2)\begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \frac{\beta_1^2}{\alpha_2} & 0 \\ 0 & 0 & 0 & \frac{\beta_2^2}{\alpha_1} \end{bmatrix}. \tag{12}$$

2.2 Natural coordinates and potential function

It was noted by Leurgans, Tsai, and Crowley [14] that the family of Freund distributions forms an exponential family, with natural parameters

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\alpha_1 + \beta_1, \alpha_2, \log\left(\frac{\alpha_1\,\beta_2}{\alpha_2\,\beta_1}\right), \beta_2) \tag{13}$$

and potential function

$$\varphi(\theta) = -\log\left(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4}\right) = -\log(\alpha_2\,\beta_1). \tag{14}$$

So by solving the equations

$$\theta_1 = \alpha_1+\beta_1,\ \theta_2=\alpha_2,\ \theta_3=\log\left(\frac{\alpha_1\,\beta_2}{\alpha_2\,\beta_1}\right),\ \theta_4=\beta_2.,$$

we obtain that:

$$\alpha_1 = \frac{\theta_1\,\theta_2}{e^{\theta_3}\,\theta_2+\theta_4}\,e^{\theta_3},\ \beta_1=\frac{\theta_1\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4},\ \alpha_2=\theta_2,\ \beta_2=\theta_4.,$$

so (6) can be written in term of the natural coordinate system as:

$$f(x,y) = \begin{cases} e^{\theta_1(-x)+\theta_3+\theta_4(x-y)+\log\left(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4}\right)} & \text{for } 0<x<y \\ e^{\theta_1(y)+\theta_2(y-x)+\log\left(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4}\right)} & \text{for } 0<y<x \end{cases}. \tag{15}$$

The Fisher metric with respect to the natural coordinates (θ_i) (13) is given by

$$\left[\frac{\partial^2\varphi(\theta)}{\partial\theta_i\partial\theta_j}\right] = \begin{bmatrix} \frac{1}{\theta_1{}^2} & 0 & 0 & 0 \\ 0 & \frac{\theta_4\left(2\,e^{\theta_3}\,\theta_2+\theta_4\right)}{\theta_2{}^2\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & \frac{e^{\theta_3}\,\theta_4}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & -\frac{e^{\theta_3}}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} \\ 0 & \frac{e^{\theta_3}\,\theta_4}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & \frac{e^{\theta_3}\,\theta_2\,\theta_4}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & -\frac{e^{\theta_3}\,\theta_2}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2}, \\ 0 & -\frac{e^{\theta_3}}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & -\frac{e^{\theta_3}\,\theta_2}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} & \frac{1}{\theta_4{}^2}-\frac{1}{\left(e^{\theta_3}\,\theta_2+\theta_4\right)^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{(\alpha_1+\beta_1)^2} & 0 & 0 & 0 \\ 0 & \frac{\beta_1\,(2\,\alpha_1+\beta_1)}{\alpha_2{}^2\,(\alpha_1+\beta_1)^2} & \frac{\alpha_1\,\beta_1}{\alpha_2\,(\alpha_1+\beta_1)^2} & -\frac{\alpha_1\,\beta_1}{\alpha_2\,(\alpha_1+\beta_1)^2\,\beta_2} \\ 0 & \frac{\alpha_1\,\beta_1}{\alpha_2\,(\alpha_1+\beta_1)^2} & \frac{\alpha_1\,\beta_1}{(\alpha_1+\beta_1)^2} & -\frac{\alpha_1\,\beta_1}{(\alpha_1+\beta_1)^2\,\beta_2} \\ 0 & -\frac{\alpha_1\,\beta_1}{\alpha_2\,(\alpha_1+\beta_1)^2\,\beta_2} & -\frac{\alpha_1\,\beta_1}{(\alpha_1+\beta_1)^2\,\beta_2} & \frac{\alpha_1\,(\alpha_1+2\,\beta_1)}{(\alpha_1+\beta_1)^2\,\beta_2{}^2} \end{bmatrix}. \tag{16}$$

2.3 Freund α-geometry

We report the analytic expressions for the α-connections and the α-curvature objects with respect to coordinates $(\alpha_1,\alpha_2,\beta_1,\beta_2)$; this is simpler than using the natural coordinates (13). Detailed expressions are given are given in Appendix A for the components of the α-connection components (5).

Proposition 2.1. *The nonzero independent components $R^{(\alpha)}_{ijkl}$ of the α-curvature tensor are given by:*

$$\begin{aligned}
R^{(\alpha)}_{1313} &= \frac{\left(\alpha^2-1\right)\alpha_2{}^2}{4\,\alpha_1\,(\alpha_1+\alpha_2)^3\,\beta_1{}^2}\,,\\
R^{(\alpha)}_{1332} &= \frac{\left(\alpha^2-1\right)\alpha_2}{4\,(\alpha_1+\alpha_2)^3\,\beta_1{}^2}\,,\\
R^{(\alpha)}_{1414} &= \frac{\left(\alpha^2-1\right)\alpha_2}{4\,(\alpha_1+\alpha_2)^3\,\beta_2{}^2}\,,\\
R^{(\alpha)}_{1424} &= \frac{-\left(\alpha^2-1\right)\alpha_1}{4\,(\alpha_1+\alpha_2)^3\,\beta_2{}^2}\,,\\
R^{(\alpha)}_{3232} &= \frac{\left(\alpha^2-1\right)\alpha_1}{4\,(\alpha_1+\alpha_2)^3\,\beta_1{}^2}\,,\\
R^{(\alpha)}_{3434} &= \frac{\left(\alpha^2-1\right)\alpha_1\,\alpha_2}{4\,(\alpha_1+\alpha_2)^2\,\beta_1{}^2\,\beta_2{}^2}\,,\\
R^{(\alpha)}_{2424} &= \frac{\left(\alpha^2-1\right)\alpha_1{}^2}{4\,\alpha_2\,(\alpha_1+\alpha_2)^3\,\beta_2{}^2}. \qquad \square
\end{aligned} \tag{17}$$

Contracting $R^{(\alpha)}_{ijkl}$ with g^{il} we obtain the components $R^{(\alpha)}_{jk}$ of the α-Ricci tensor.

Proposition 2.2. *The α-Ricci tensor $R^{(\alpha)} = [R^{(\alpha)}_{jk}]$ is given by:*

$$R^{(\alpha)} = [R^{(\alpha)}_{jk}] = \frac{(1-\alpha^2)}{(\alpha_1+\alpha_2)} \begin{bmatrix} \frac{\alpha_2}{2\,\alpha_1\,(\alpha_1+\alpha_2)} & \frac{-1}{2\,(\alpha_1+\alpha_2)} & 0 & 0 \\ \frac{-1}{2\,(\alpha_1+\alpha_2)} & \frac{\alpha_1}{2\,\alpha_2\,(\alpha_1+\alpha_2)} & 0 & 0 \\ 0 & 0 & \frac{\alpha_2}{2\,\beta_1{}^2} & 0 \\ 0 & 0 & 0 & \frac{\alpha_1}{2\,\beta_2{}^2} \end{bmatrix} \tag{18}$$

The α-eigenvalues and the α-eigenvectors of the α-Ricci tensor are given by:

$$\left(1-\alpha^2\right) \begin{pmatrix} 0 \\ \frac{1}{2\,\alpha_1\,\alpha_2} - \frac{1}{(\alpha_1+\alpha_2)^2} \\ \frac{\alpha_2}{2\,(\alpha_1+\alpha_2)\,\beta_1{}^2} \\ \frac{\alpha_1}{2\,(\alpha_1+\alpha_2)\,\beta_2{}^2} \end{pmatrix} \tag{19}$$

$$\begin{pmatrix} \frac{\alpha_1}{\alpha_2} & 1 & 0 & 0 \\ -\frac{\alpha_2}{\alpha_1} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \square \tag{20}$$

Proposition 2.3. *The manifold F has constant α-scalar curvature*

$$R^{(\alpha)} = \frac{3\,(1-\alpha^2)}{2} \tag{21}$$

□

Proposition 2.4. *The α-sectional curvatures $\varrho^{(\alpha)}(\lambda,\mu)$ $(\lambda,\mu = 1,2,3,4)$ are given by:*

$$\begin{aligned}
\varrho^{(\alpha)}(1,3) &= \varrho^{(\alpha)}(1,4) = \frac{(1-\alpha^2)\,\alpha_2}{4\,(\alpha_1+\alpha_2)}\,,\\
\varrho^{(\alpha)}(1,2) &= 0,\\
\varrho^{(\alpha)}(2,3) &= \varrho^{(\alpha)}(2,4) = \frac{(1-\alpha^2)\,\alpha_1}{4\,(\alpha_1+\alpha_2)}\,,\\
\varrho^{(\alpha)}(3,4) &= \frac{1-\alpha^2}{4}\,.\square
\end{aligned} \tag{22}$$

Proposition 2.5. *The α-mean curvatures $\varrho^{(\alpha)}(\lambda)$ $(\lambda = 1,2,3,4)$ are given by:*

$$\begin{aligned}
\varrho^{(\alpha)}(1) &= \frac{(1-\alpha^2)\,\alpha_2}{6\,(\alpha_1+\alpha_2)}\,,\\
\varrho^{(\alpha)}(2) &= \frac{(1-\alpha^2)\,\alpha_1}{6\,(\alpha_1+\alpha_2)}\,,\\
\varrho^{(\alpha)}(3) &= \varrho(4) = \frac{1-\alpha^2}{6}\,.\square
\end{aligned} \tag{23}$$

2.4 Dual coordinates

Since F is an exponential family, a mixture coordinate system is given by the potential function (14), that is,

$$\begin{aligned}
\eta_1 &= \frac{\partial\varphi(\theta)}{\partial\theta_1} = -\frac{1}{\theta_1} = -\frac{1}{\alpha_1+\beta_1}\,,\\
\eta_2 &= \frac{\partial\varphi(\theta)}{\partial\theta_2} = -\frac{\theta_4}{\theta_2\,(e^{\theta_3}\,\theta_2+\theta_4)} = -\frac{\beta_1}{\alpha_2\,(\alpha_1+\beta_1)}\,,\\
\eta_3 &= \frac{\partial\varphi(\theta)}{\partial\theta_3} = 1 - \frac{\theta_4}{e^{\theta_3}\,\theta_2+\theta_4} = \frac{\alpha_1}{\alpha_1+\beta_1}\,,\\
\eta_4 &= \frac{\partial\varphi(\theta)}{\partial\theta_4} = -\frac{1}{\theta_4} + \frac{1}{e^{\theta_3}\,\theta_2+\theta_4} = -\frac{\alpha_1}{(\alpha_1+\beta_1)\,\beta_2}\,.
\end{aligned} \tag{24}$$

Next $(\theta_1,\theta_2,\theta_3,\theta_4)$ is a 1-affine coordinate system, $(\eta_1,\eta_2,\eta_3,\eta_4)$ is a (-1)-affine coordinate system, and they are dual with respect to the Fisher information metric. The coordinates (η_i) (24) have a potential function given by:

$$\lambda = \log\left(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4}\right) + \frac{e^{\theta_3}\,\theta_2\,\theta_3}{e^{\theta_3}\,\theta_2+\theta_4} - 2 = \log(\alpha_2\,\beta_1) + \frac{\alpha_1}{\alpha_1+\beta_1}\log\left(\frac{\alpha_1\,\beta_2}{\alpha_2\,\beta_1}\right) - 2. \tag{25}$$

The coordinates (θ_i) and (η_i) form a dual coordinate system. Therefore the Freund manifold has dually orthogonal foliations (See Section 3.7 in [1]) for example.

Take

$$(\eta_1, \theta_2, \theta_3, \theta_4) = (-\frac{1}{\theta_1}, \theta_2, \theta_3, \theta_4)$$

as a coordinate system for F; then the Freund distributions take the form:

$$f(x, y; \eta_1, \theta_2, \theta_3, \theta_4) = \begin{cases} -\frac{\theta_2\,\theta_4\,e^{\theta_3}}{\left(\theta_2\,e^{\theta_3}+\theta_4\right)\eta_1}\,e^{\theta_4\,(x-y)+\frac{x}{\eta_1}} & \text{for } 0 < x < y \\ -\frac{\theta_2\,\theta_4}{\left(\theta_2\,e^{\theta_3}+\theta_4\right)\eta_1}\,e^{\theta_2\,(y-x)+\frac{y}{\eta_1}} & \text{for } 0 < y < x \end{cases} \tag{26}$$

where $\eta_1 < 0$ and $\theta_i > 0 \quad (i = 2, 3, 4)$.

The Fisher metric is

$$[g_{ij}] = \frac{1}{\left(\theta_2\,e^{\theta_3}+\theta_4\right)^2}\begin{bmatrix} \frac{\left(\theta_2\,e^{\theta_3}+\theta_4\right)^2}{{\eta_1}^2} & 0 & 0 & 0 \\ 0 & \frac{\theta_4\left(2\,\theta_2\,e^{\theta_3}+\theta_4\right)}{{\theta_2}^2} & \theta_4\,e^{\theta_3} & -\,e^{\theta_3} \\ 0 & \theta_4\,e^{\theta_3} & \theta_2\,\theta_4\,e^{\theta_3} & -\theta_2\,e^{\theta_3} \\ 0 & -\,e^{\theta_3} & -\theta_2\,e^{\theta_3} & \frac{\left(\theta_2\,e^{\theta_3}+\theta_4\right)^2}{{\theta_4}^2}-1 \end{bmatrix}$$

$$= \frac{1}{\left(\alpha_1+\beta_1\right)^2}\begin{bmatrix} \left(\alpha_1+\beta_1\right)^4 & 0 & 0 & 0 \\ 0 & \frac{\beta_1\,(2\,\alpha_1+\beta_1)}{{\alpha_2}^2} & \frac{\alpha_1\,\beta_1}{\alpha_2} & -\frac{\alpha_1\,\beta_1}{\alpha_2\,\beta_2} \\ 0 & \frac{\alpha_1\,\beta_1}{\alpha_2} & \alpha_1\,\beta_1 & -\frac{\alpha_1\,\beta_1}{\beta_2} \\ 0 & -\frac{\alpha_1\,\beta_1}{\alpha_2\,\beta_2} & -\frac{\alpha_1\,\beta_1}{\beta_2} & \frac{\alpha_1\,(\alpha_1+2\,\beta_1)}{{\beta_2}^2} \end{bmatrix}. \tag{27}$$

We remark that (θ_i) is a geodesic coordinate system of $\nabla^{(1)}$, and (η_i) is a geodesic coordinate system of $\nabla^{(-1)}$.

2.5 Submanifolds of F

We consider four submanifolds $F_i\,(i = 1, 2, 3, 4)$ of the 4-manifold F of Freund bivariate exponential densities $f(x, y; \alpha_1, \alpha_2, \beta_1, \beta_2)$ (6), which includes the case of independent random variables. It includes also the special case of an Absolutely Continuous Bivariate Exponential Distribution called ACBED (or ACBVE) by Block and Basu (cf. Hutchinson and Lai [12]). We use the coordinate system $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ for the submanifolds $F_i\,(i \neq 4)$, and the coordinate system $(\lambda_1, \lambda_{12}, \lambda_2)$ for ACBED of the Block and Basu case.

2.5.1 Independence submanifold: $F_1 \subset F$: $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$

The densities are of form:

$$f(x, y; \alpha_1, \alpha_2) = f_1(x; \alpha_1) f_2(y; \alpha_2) \tag{28}$$

where the f_i are the univariate exponential densities with parameters $\alpha_i > 0\,(i = 1, 2)$. This is the case for the independence of X and Y, so F_1 is the direct product of the Riemannian spaces $\{f_1(x; \alpha_1) = \alpha_1 e^{-\alpha_1 x},\ \alpha_1 > 0\}$ and $\{f_2(y; \alpha_2) = -\alpha_2 e^{-\alpha_2 y},\ \alpha_2 > 0\}$.

Proposition 2.6. *The metric tensor $[g_{ij}]$ is as follows:*

$$[g_{ij}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\alpha_2^2} \end{bmatrix} \square \tag{29}$$

Proposition 2.7. *The α-curvature tensor, α-Ricci tensor, and α-scalar curvature of F_1 are zero.* □

2.5.2 Submanifold: $F_2 \subset F$: $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$

The probability density functions are of form:

$$f(x, y; \alpha_1, \beta_1) = \begin{cases} \alpha_1\beta_1\, e^{-\beta_1 y-(2\,\alpha_1-\beta_1)x} \text{ for } 0 < x < y \\ \alpha_1\beta_1\, e^{-\beta_1 x-(2\,\alpha_1-\beta_1)y} \text{ for } 0 < y < x \end{cases} \tag{30}$$

with parameters $\alpha_1, \beta_1 > 0$. The covariance, correlation coefficient and marginal density functions, of X and Y are given by:

$$Cov(X, Y) = \frac{1}{4}\left(\frac{1}{{\alpha_1}^2} - \frac{1}{{\beta_1}^2}\right), \tag{31}$$

$$\rho(X, Y) = 1 - \frac{4\,{\alpha_1}^2}{3\,{\alpha_1}^2 + {\beta_1}^2}, \tag{32}$$

$$f_X(x) = \left(\frac{\alpha_1}{2\,\alpha_1 - \beta_1}\right)\beta_1\, e^{-\beta_1 x} + \left(\frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1}\right)(2\,\alpha_1)\, e^{-2\,\alpha_1 x}\,,\ x \geq 0\,, \tag{33}$$

$$f_Y(y) = \left(\frac{\alpha_1}{2\,\alpha_1 - \beta_1}\right)\beta_1\, e^{-\beta_1 y} + \left(\frac{\alpha_1 - \beta_1}{2\,\alpha_1 - \beta_1}\right)(2\,\alpha_1)\, e^{-2\,\alpha_1 y}\,,\ y \geq 0\,. \tag{34}$$

We see that $\rho(X, Y) = 0$ if and only if $\alpha_1 = \beta_1$. Also, F_2 forms an exponential family, with natural parameters (α_1, β_1) and potential function $\varphi = -\log(\alpha_1\,\beta_1)$.

Proposition 2.8. *The submanifold F_2 is an isometric isomorph of the manifold F_1.*

Proof. Since $\varphi = -\log(\alpha_1\,\beta_1)$ is a potential function, the Fisher metric is the Hessian of φ, that is,

$$[g_{ij}] = [\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j}] = \begin{bmatrix} \frac{1}{\alpha_1^2} & 0 \\ 0 & \frac{1}{\beta_1^2} \end{bmatrix} \tag{35}$$

where $(\theta_1, \theta_2) = (\alpha_1, \beta_1)$. □

2.5.3 Submanifold: $F_3 \subset F$: $\beta_1 = \beta_2 = \alpha_1 + \alpha_2$

The probability density functions are of form:

$$f(x,y;\alpha_1,\alpha_2,\beta_2) = \begin{cases} \alpha_1\ (\alpha_1+\alpha_2)\ e^{-(\alpha_1+\alpha_2)y} \text{ for } 0<x<y \\ \alpha_2\ (\alpha_1+\alpha_2)\ e^{-(\alpha_1+\alpha_2)x} \text{ for } 0<y<x \end{cases} \tag{36}$$

with parameters $\alpha_1, \alpha_2 > 0$. The covariance, correlation coefficient and marginal functions, of X and Y are given by:

$$Cov(X,Y) = \frac{{\alpha_1}^2 + \alpha_1\,\alpha_2 + {\alpha_2}^2}{(\alpha_1+\alpha_2)^4}, \tag{37}$$

$$\rho(X,Y) = \frac{{\alpha_1}^2 + \alpha_1\,\alpha_2 + {\alpha_2}^2}{\sqrt{2\,(\alpha_1+\alpha_2)^2 - {\alpha_1}^2}\,\sqrt{2\,{\alpha_1}^2 + 4\,\alpha_1\alpha_2 + {\alpha_2}^2}}, \tag{38}$$

$$f_X(x) = (\alpha_2\,(\alpha_1+\alpha_2)x + \alpha_1)\ e^{-(\alpha_1+\alpha_2)x},\ x \geq 0 \tag{39}$$

$$f_Y(y) = (\alpha_1\,(\alpha_1+\alpha_2)y + \alpha_2)\ e^{-(\alpha_1+\alpha_2)y},\ y \geq 0 \tag{40}$$

Note that the correlation coefficient is positive.

Proposition 2.9. *The metric tensor on F_3 is*

$$[g_{ij}] = \begin{bmatrix} \frac{\alpha_2+2\,\alpha_1}{\alpha_1\,(\alpha_1+\alpha_2)^2} & \frac{1}{(\alpha_1+\alpha_2)^2} \\ \frac{1}{(\alpha_1+\alpha_2)^2} & \frac{\alpha_1+2\,\alpha_2}{\alpha_2\,(\alpha_1+\alpha_2)^2} \end{bmatrix}. \square \tag{41}$$

Proposition 2.10. *The α-curvature tensor, α-Ricci curvature, and α-scalar curvature of F_3 are zero.* □

2.5.4 Submanifold: $F_4 \subset F$, ACBED of Block and Basu

The probability density functions are

$$f(x,y;\lambda_1,\lambda_{12},\lambda_2) = \begin{cases} \frac{\lambda_1\,\lambda\,(\lambda_2+\lambda_{12})}{\lambda_1+\lambda_2}\, e^{-\lambda_1\,x-(\lambda_2+\lambda_{12})\,y} & \text{for } 0<x<y \\ \frac{\lambda_2\,\lambda\,(\lambda_1+\lambda_{12})}{\lambda_1+\lambda_2}\, e^{-(\lambda_1+\lambda_{12})\,x-\lambda_2\,y} & \text{for } 0<y<x \end{cases} \tag{42}$$

where the parameters $\lambda_1, \lambda_{12}, \lambda_2$ are positive, and $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$.
This distribution was derived originally by omitting the singular part of the Marshall and Olkin distribution (cf. [13], page [139]); Block and Basu called it the ACBED to emphasize that these are the absolutely continuous bivariate exponential distributions. Alternatively, these distributions can be obtained from (6), by taking

$$\begin{aligned} \alpha_1 &= \lambda_1 + \frac{\lambda_1\,\lambda_{12}}{(\lambda_1+\lambda_2)}, \\ \beta_1 &= \lambda_1 + \lambda_{12}, \\ \alpha_2 &= \lambda_2 + \frac{\lambda_2\,\lambda_{12}}{(\lambda_1+\lambda_2)}, \\ \beta_2 &= \lambda_2 + \lambda_{12}. \end{aligned}$$

By substitution we obtain the covariance, correlation coefficient and marginal probability density functions:

$$Cov(X,Y) = \frac{(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12}) - \lambda^2\,\lambda_1\,\lambda_2}{\lambda^2\,(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12})}, \tag{43}$$

$$\rho(X,Y) = \frac{(\lambda_1+\lambda_2)^2\,(\lambda_1+\lambda_{12})\,(\lambda_2+\lambda_{12}) - \lambda^2\,\lambda_1\,\lambda_2}{\sqrt{\prod_{i=1,\,j\neq i}^{2}\left((\lambda_1+\lambda_2)^2(\lambda_i+\lambda_{12})^2 + \lambda_j\lambda^2\,(\lambda_j+2\lambda_i)\right)}}, \tag{44}$$

$$f_X(x) = \left(\frac{-\lambda_{12}}{\lambda_1+\lambda_2}\right)\lambda\,e^{-\lambda\,x} + \left(\frac{\lambda}{\lambda_1+\lambda_2}\right)(\lambda_1+\lambda_{12})\,e^{-(\lambda_1+\lambda_{12})\,x},\ x \geq 0 \tag{45}$$

$$f_Y(y) = \left(\frac{-\lambda_{12}}{\lambda_1+\lambda_2}\right)\lambda\,e^{-\lambda\,y} + \left(\frac{\lambda}{\lambda_1+\lambda_2}\right)(\lambda_2+\lambda_{12})\,e^{-(\lambda_2+\lambda_{12})\,y},\ y \geq 0 \tag{46}$$

The correlation coefficient is positive, and the marginal density functions are a mixture of two exponentials.

Proposition 2.11. *The metric tensor* $[g_{ij}]$ *using the coordinate system* $(\lambda_1, \lambda_{12}, \lambda_2)$ *is*

$$[g_{ij}] = \begin{bmatrix} \frac{\lambda_2\left(\frac{1}{\lambda_1}+\frac{\lambda_1+\lambda_2}{(\lambda_1+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \frac{\lambda_2}{(\lambda_1+\lambda_2)\,(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \\ \frac{\lambda_2}{(\lambda_1+\lambda_2)\,(\lambda_1+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{\frac{\lambda_2}{(\lambda_1+\lambda_{12})^2}+\frac{\lambda_1}{(\lambda_2+\lambda_{12})^2}}{\lambda_1+\lambda_2} + \frac{1}{\lambda^2} & \frac{\lambda_1}{(\lambda_1+\lambda_2)\,(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} \\ \frac{-1}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} & \frac{\lambda_1}{(\lambda_1+\lambda_2)\,(\lambda_2+\lambda_{12})^2} + \frac{1}{\lambda^2} & \frac{\lambda_1\left(\frac{1}{\lambda_2}+\frac{\lambda_1+\lambda_2}{(\lambda_2+\lambda_{12})^2}\right)}{(\lambda_1+\lambda_2)^2} + \frac{1}{\lambda^2} \end{bmatrix}. \square \tag{47}$$

The Christoffel symbols, curvature tensor, Ricci tensor, scalar curvature, sectional curvatures and the mean curvatures were computed [3] but these are not listed here because the expressions are somewhat cumbersome.

In the case when $\lambda_1 = \lambda_2$, this family of distributions becomes

$$f(x, y; \lambda_1, \lambda_{12}) = \begin{cases} \frac{(2\,\lambda_1+\lambda_{12})\,(\lambda_1+\lambda_{12})}{2}\,e^{-\lambda_1\,x-(\lambda_1+\lambda_{12})\,y} & \text{for } 0 < x < y \\ \frac{(2\,\lambda_1+\lambda_{12})\,(\lambda_1+\lambda_{12})}{2}\,e^{-\lambda_1\,y-(\lambda_1+\lambda_{12})\,x} & \text{for } 0 < y < x \end{cases} \tag{48}$$

which is an exponential family with natural parameters $(\theta_1, \theta_2) = (\lambda_1, \lambda_{12})$ and potential function $\varphi(\theta) = \log(2) - \log(\lambda_1+\lambda_{12}) - \log(2\,\lambda_1+\lambda_{12})$, note that from equations (45, 46) this family of bivariate distributions has two equal marginal density functions.

So it is easy to derive the α-geometry; the metric tensor is:

$$[g_{ij}] = \left[\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j}\right] = \begin{bmatrix} \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{4}{(2\,\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{2}{(2\,\lambda_1+\lambda_{12})^2} \\ \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{2}{(2\,\lambda_1+\lambda_{12})^2} & \frac{1}{(\lambda_1+\lambda_{12})^2} + \frac{1}{(2\,\lambda_1+\lambda_{12})^2} \end{bmatrix} \tag{49}$$

In this case, the α-curvature tensor, α-Ricci curvature, and α-scalar curvature are zero.

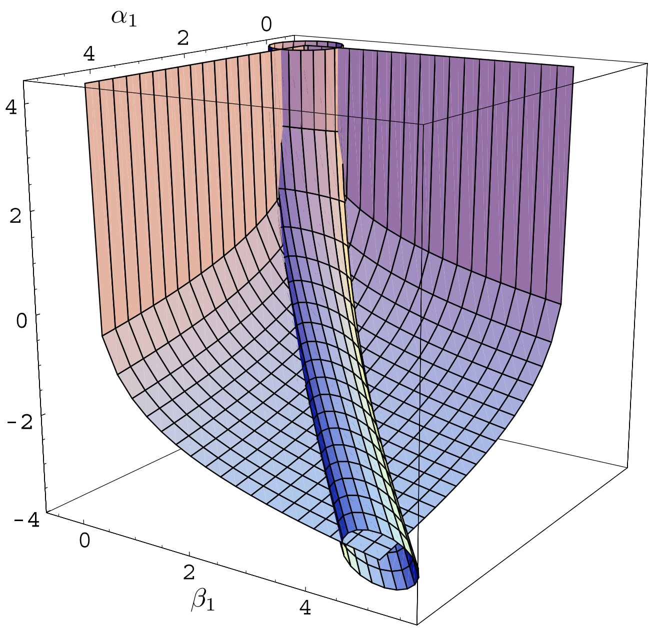

Fig. 1 Affine immersion in natural coordinates (α_1, β_1) as a surface in $\mathbb{R}^3$ for the Freund submanifold F_2. The curve $\alpha_1 = \beta_1$ in the surface consists of all bivariate distributions having common exponential marginals and zero covariance; its tubular neighbourhoods contain by continuity all immersions of bivariate exponential processes sufficiently close to the case of independence.

Additionally, since $(\lambda_1, \lambda_{12})$ is a 1-affine coordinate system, a (-1)-affine coordinate system is

$$(\eta_1, \eta_2) = (-\frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{\lambda_1 + 2\,\lambda_{12}}, -\frac{1}{\lambda_1 + \lambda_{12}} - \frac{1}{2\,\lambda_1 + \lambda_{12}})$$

with potential function

$$\lambda = -2 - \log(2) + \log(2\,\lambda_1 + \lambda_{12}) + \log(\lambda_1 + \lambda_{12}).$$

2.6 Affine immersion and neighbourhoods of independence

An important practical application of the Freund submanifold F_2 is the representation of a bivariate stochastic process with common marginal exponentials. The next results are important because it provides topological neighbourhoods of that subspace W in F_2 consisting of the bivariate processes that have zero covariance: we obtain neighbourhoods of independence for random (ie exponentially distributed) processes.

Proposition 2.12. *Let F be the Freund 4-manifold with the Fisher metric g and the exponential connection $\nabla^{(1)}$. Denote by (θ_i) the natural coordinate system (13). Then F can be realized in $\mathbb{R}^5$ by the graph of a potential function, the affine immersion f:*

$$f : F \to \mathbb{R}^5 : \begin{bmatrix} \theta_i \end{bmatrix} \mapsto \begin{bmatrix} \theta_i \\ \varphi(\theta) \end{bmatrix}, \tag{50}$$

where $\varphi(\theta)$ *is the potential function* $\varphi(\theta) = -\log(\frac{\theta_1\,\theta_2\,\theta_4}{e^{\theta_3}\,\theta_2+\theta_4}) = -\log(\alpha_2\,\beta_1)$. □

The case of Freund distributions with $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$ is represented by the surface:

$$\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^3 : (\alpha_1, \beta_1) \mapsto (\alpha_1, \beta_1, \varphi)\,.$$

where $\varphi = -\log(\alpha_1\,\beta_1)$.

The submanifold $W \subset F_2$ consisting of the independent case ($\alpha_1 = \beta_1$) is represented by the curve:

$$(0, \infty) \to \mathbb{R}^3 : (\alpha_1) \mapsto (\alpha_1, \alpha_1, -2\log\alpha_1)\,.$$

This is illustrated in Figure 1 which shows an affine embedding of F_2 as a surface in $\mathbb{R}^3$, and an $\mathbb{R}^3$-tubular neighbourhood of W, the curve $\alpha_1 = \beta_1$ in the surface. This curve represents all bivariate distributions having common exponential marginals and zero covariance; by continuity its tubular neighourhoods contain all small enough departures from independence.

Proposition 2.13. *In the affine embedding of the Freund submanifold* F_2 *in* $\mathbb{R}^3$*, a tubular neighbourhood of the curve* $\alpha_1 = \beta_1$ *will contain all affine immersions of bivariate exponential distributions sufficiently close to the case of independence.* □

3 Bivariate Gaussian 5-manifold N

The bivariate Gaussian distribution has the form:

$$f(x, y) = \frac{1}{2\pi\sqrt{\sigma_1\,\sigma_2 - {\sigma_{12}}^2}}\, e^{-\frac{1}{2(\sigma_1\,\sigma_2 - {\sigma_{12}}^2)}\left(\sigma_2(x-\mu_1)^2 - 2\sigma_{12}(x-\mu_1)(y-\mu_2) + \sigma_1(y-\mu_2)^2\right)}, \tag{51}$$

defined on $-\infty < x\,, y < \infty$ with parameters $(\mu_1,\, \mu_2,\, \sigma_1,\, \sigma_{12},\, \sigma_2)$; where $-\infty < \mu_1\,, \mu_2 < \infty$, $\quad 0 < \sigma_1\,, \sigma_2 < \infty$ and σ_{12} is the covariance of X and Y.

The marginal density functions of X and Y are univariate Gaussian:

$$f_X(x, \mu_1, \sigma_1) = \frac{1}{\sqrt{2\,\pi\,\sigma_1}}\, e^{-\frac{(x-\mu_1)^2}{2\,\sigma_1}}\,, \tag{52}$$

$$f_Y(y, \mu_2, \sigma_2) = \frac{1}{\sqrt{2\,\pi\,\sigma_2}}\, e^{-\frac{(y-\mu_2)^2}{2\,\sigma_2}}\,. \tag{53}$$

The correlation coefficient is:

$$\rho(X,Y) = \frac{\sigma_{12}}{\sqrt{\sigma_1\,\sigma_2}}$$

Since ${\sigma_{12}}^2 < \sigma_1\,\sigma_2$ then $-1 < \rho(X,Y) < 1$; so we do not have the case when Y is a linear function of X.

3.1 Fisher metric

The family N of bivariate Gaussian distributions with $(\mu_1, \mu_2, \sigma_1, \sigma_{12}, \sigma_2)$ as coordinate system, becomes a 5-manifold with Fisher information metric components

$$[g_{ij}] = \begin{bmatrix} \frac{\sigma_2}{\Delta} & -\frac{\sigma_{12}}{\Delta} & 0 & 0 & 0 \\ -\frac{\sigma_{12}}{\Delta} & \frac{\sigma_1}{\Delta} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_2^2}{2\,\Delta^2} & -\frac{\sigma_{12}\,\sigma_2}{\Delta^2} & \frac{\sigma_{12}^2}{2\,\Delta^2} \\ 0 & 0 & -\frac{\sigma_{12}\,\sigma_2}{\Delta^2} & \frac{\sigma_1\,\sigma_2+\sigma_{12}^2}{\Delta^2} & -\frac{\sigma_1\,\sigma_{12}}{\Delta^2} \\ 0 & 0 & \frac{\sigma_{12}^2}{2\,\Delta^2} & -\frac{\sigma_1\,\sigma_{12}}{\Delta^2} & \frac{\sigma_1^2}{2\,\Delta^2} \end{bmatrix}, \tag{54}$$

The inverse is

$$[g^{ij}] = \begin{bmatrix} \sigma_1 & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{12} & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & 2\,\sigma_1^2 & 2\,\sigma_1\,\sigma_{12} & 2\,\sigma_{12}^2 \\ 0 & 0 & 2\,\sigma_1\,\sigma_{12} & \sigma_1\,\sigma_2+\sigma_{12}^2 & 2\,\sigma_{12}\,\sigma_2 \\ 0 & 0 & 2\,\sigma_{12}^2 & 2\,\sigma_{12}\,\sigma_2 & 2\,\sigma_2^2 \end{bmatrix} \tag{55}$$

where $\Delta = \sigma_1\sigma_2 - \sigma_{12}^2$.

See Skovgaard [17] for the metric in the case of general multivariate Gaussians, which also form an exponential family.

3.2 Natural coordinates and potential function

Proposition 3.1. *The set of all bivariate Gaussian distributions forms an exponential family, with natural coordinate system*

$$(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\frac{\mu_1\,\sigma_2 - \mu_2\,\sigma_{12}}{\Delta}, \frac{\mu_2\,\sigma_1 - \mu_1\,\sigma_{12}}{\Delta}, \frac{-\sigma_2}{2\,\Delta}, \frac{\sigma_{12}}{\Delta}, \frac{-\sigma_1}{2\,\Delta}) \tag{56}$$

and corresponding potential function

$$\begin{aligned} \varphi(\theta) &= \log(2\,\pi\,\sqrt{\Delta}) + \frac{{\mu_2}^2\,\sigma_1 + {\mu_1}^2\,\sigma_2 - 2\,\mu_1\,\mu_2\,\sigma_{12}}{2\,\Delta} \\ &= \log(2\,\pi\,\sqrt{\Delta}) - \Delta\,\left({\theta_2}^2\,\theta_3 - \theta_1\,\theta_2\,\theta_4 + {\theta_1}^2\,\theta_5\right)\,. \end{aligned} \tag{57}$$

where

$$\Delta = \frac{1}{4\,\theta_3\,\theta_5 - {\theta_4}^2}\,.$$

Proof.

$$\log f(x,y) = \log\left(\frac{1}{2\,\pi\,\sqrt{\Delta}}\,e^{-\frac{1}{2\,\Delta}\left(\sigma_2 (x-\mu_1)^2 - 2\,\sigma_{12}\,(x-\mu_1)\,(y-\mu_2) + \sigma_1 (y-\mu_2)^2\right)}\right)$$

$$= \frac{\mu_1\,\sigma_2 - \mu_2\,\sigma_{12}}{\Delta}\,x + \frac{\mu_2\,\sigma_1 - \mu_1\,\sigma_{12}}{\Delta}\,y + \frac{-\sigma_2}{2\,\Delta}\,x^2 + \frac{\sigma_{12}}{\Delta}\,x\,y + \frac{-\sigma_1}{2\,\Delta}\,y^2 \quad (58)$$

$$- \left(\log(2\,\pi\,\sqrt{\Delta}) + \frac{{\mu_2}^2\,\sigma_1 + {\mu_1}^2\,\sigma_2 - 2\,\mu_1\,\mu_2\,\sigma_{12}}{2\,\Delta}\right). \quad (59)$$

Hence the set of all bivariate Gaussian distributions is an exponential family. The line (58) implies that $(\frac{\mu_1\,\sigma_2 - \mu_2\,\sigma_{12}}{\Delta}, \frac{\mu_2\,\sigma_1 - \mu_1\,\sigma_{12}}{\Delta}, \frac{-\sigma_2}{2\,\Delta}, \frac{\sigma_{12}}{\Delta}, \frac{-\sigma_1}{2\,\Delta})$ is a natural coordinate system and

$$(x_1, x_2, x_3, x_4, x_5) = (F_1(x), F_2(x), F_3(x), F_4(x), F_5(x)) = (x, y, x^2, x\,y, y^2)$$

is a random variable, and (59) implies that

$$\varphi(\theta) = \log(2\,\pi\,\sqrt{\Delta}) + \frac{{\mu_2}^2\,\sigma_1 + {\mu_1}^2\,\sigma_2 - 2\,\mu_1\,\mu_2\,\sigma_{12}}{2\,\Delta}$$

is its potential function.

We can write the potential function in terms of natural coordinates by solving the set of equations:

$$\left\{\theta_1 = \frac{\mu_1\sigma_2 - \mu_2\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2}, \theta_2 = \frac{\mu_2\sigma_1 - \mu_1\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2}, \theta_3 = \frac{-\sigma_2}{2\,(\sigma_1\sigma_2 - {\sigma_{12}}^2)},\right.$$

$$\left.\theta_4 = \frac{\sigma_{12}}{\sigma_1\sigma_2 - {\sigma_{12}}^2}, \theta_5 = \frac{-\sigma_1}{2\,(\sigma_1\sigma_2 - {\sigma_{12}}^2)}\right\}$$

we obtain:

$$\left\{\mu_1 = \frac{2\theta_1\,\theta_5 - \theta_2\,\theta_4}{{\theta_4}^2 - 4\,\theta_3\,\theta_5}, \mu_2 = \frac{2\theta_2\,\theta_3 - \theta_1\,\theta_4}{{\theta_4}^2 - 4\,\theta_3\,\theta_5},\right.$$

$$\left.\sigma_1 = \frac{2\theta_5}{{\theta_4}^2 - 4\,\theta_3\,\theta_5}, \sigma_{12} = \frac{\theta_4}{4\,\theta_3\,\theta_5 - {\theta_4}^2}, \sigma_2 = \frac{2\theta_3}{{\theta_4}^2 - 4\,\theta_3\,\theta_5}\right\}$$

Then

$$\varphi = \log(2\,\pi\,\sqrt{\Delta}) - \Delta\,\left({\theta_2}^2\,\theta_3 - \theta_1\,\theta_2\,\theta_4 + {\theta_1}^2\,\theta_5\right),\ \text{where}\ \ \Delta = \sigma_1\,\sigma_2 - {\sigma_{12}}^2 = \frac{1}{4\,\theta_3\,\theta_5 - {\theta_4}^2}.$$

□

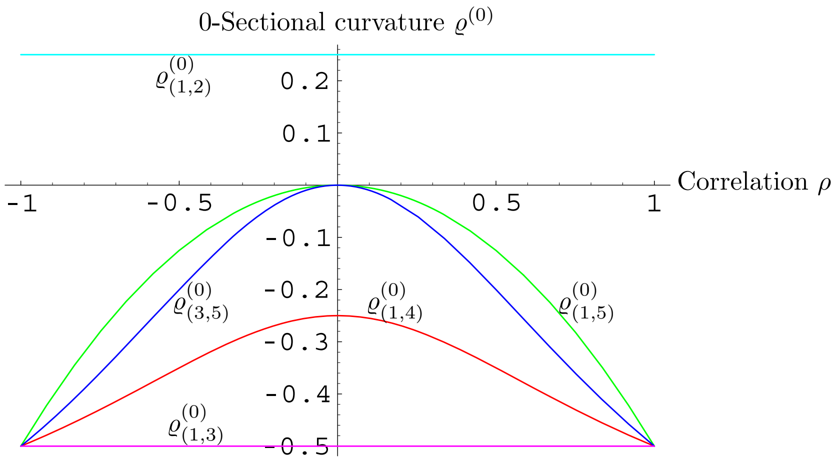

Fig. 2 The 0-sectional curvatures $\varrho^{(0)}$ as a function of correlation ρ for bivariate Gaussian manifold N where: $\varrho^{(0)}(1,3) = \varrho^{(0)}(2,5) = \varrho^{(0)}(3,4) = \varrho^{(0)}(4,5) = -\frac{1}{2}$, $\varrho^{(0)}(1,2) = \frac{1}{4}$, $\varrho^{(0)}(1,4) = \varrho^{(0)}(2,4)$ and $\varrho^{(0)}(1,5) = \varrho^{(0)}(2,3)$. Note that $\varrho^{(0)}(1,4)$, $\varrho^{(0)}(1,5)$ and $\varrho^{(0)}(3,5)$ have limiting value $-\frac{1}{2}$ as $\rho \to \pm 1$.

3.3 α-geometry

Since the analytic expressions for the α-connections and the α-curvature objects are very large in the natural coordinate system, we report these components in terms of the coordinate system $(\mu_1, \mu_2, \sigma_1, \sigma_{12}, \sigma_2)$. Details are given in Appendix B for the components of the α-connection from equation (5). Skovgaard [17] has given the formula for the 0-connection for multivariate Gaussians. These results were extended by Mitchell [15] who gave the metric and α-connections for multivariate elliptic distributions.

Proposition 3.2. *The components of the α-Ricci tensor are given by the symmetric matrix $R^{(\alpha)} = [R^{(\alpha)}_{ij}]$:*

$$R^{(\alpha)} = \left(\alpha^2 - 1\right) \begin{bmatrix} \frac{\sigma_2}{2\Delta} & -\frac{\sigma_{12}}{2\Delta} & 0 & 0 & 0 \\ -\frac{\sigma_{12}}{2\Delta} & \frac{\sigma_1}{2\Delta} & 0 & 0 & 0 \\ 0 & 0 & \frac{{\sigma_2}^2}{2\Delta^2} & -\frac{\sigma_2\sigma_{12}}{\Delta^2} & \frac{3{\sigma_{12}}^2-\sigma_1\sigma_2}{4\Delta^2} \\ 0 & 0 & -\frac{\sigma_2\sigma_{12}}{\Delta^2} & \frac{3\sigma_1\sigma_2+{\sigma_{12}}^2}{2\Delta^2} & -\frac{\sigma_1\sigma_{12}}{\Delta^2} \\ 0 & 0 & \frac{3{\sigma_{12}}^2-\sigma_1\sigma_2}{4\Delta^2} & -\frac{\sigma_1\sigma_{12}}{\Delta^2} & \frac{{\sigma_1}^2}{2\Delta^2} \end{bmatrix} \tag{60}$$

□

Proposition 3.3. *The bivariate Gaussian manifold N has a constant α-scalar curvature*

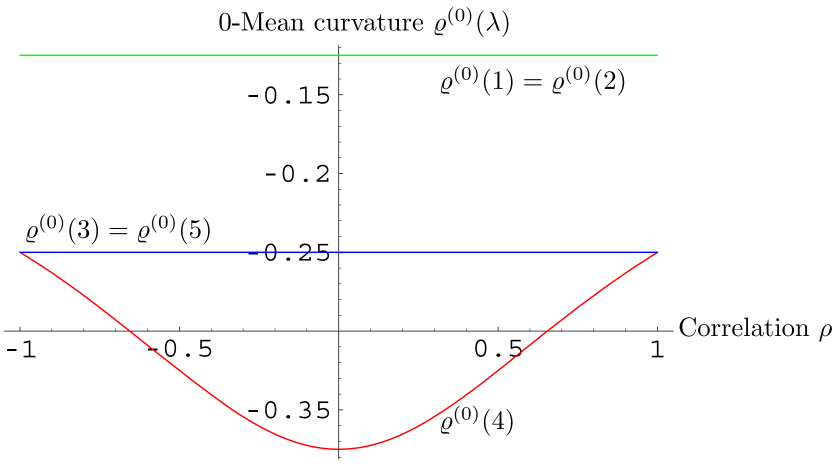

Fig. 3 The 0-mean curvatures $\varrho^{(0)}(\lambda)$ as a function of correlation ρ for bivariate Gaussian manifold N where; $\varrho^{(0)}(1) = \varrho^{(0)}(2) = -\frac{1}{8}$, $\varrho^{(0)}(3) = \varrho^{(0)}(5) = -\frac{1}{4}$, and $\varrho^{(0)}(4) \to -\frac{1}{4}$ as $\rho \to \pm 1$, and $\varrho^{(0)}(4) \to -\frac{3}{8}$ as $\rho \to 0$.

$R^{(\alpha)}$:

$$R^{(\alpha)} = \frac{9\,(\alpha^2 - 1)}{2} \tag{61}$$

This recovers the known result for the 0*-scalar curvature* $R^{(0)} = -\frac{9}{2}$. □

Hence N is ± 1-flat, as in fact also are the multivariate Gaussians, by Theorems 2.5, 3.5 in Amari and Nagaoki [2].

Proposition 3.4. *The* α*-sectional curvatures of* N *can be written as a function of correlation* $\rho(X,Y)$ *only, as follows:*

$$\varrho^{(\alpha)} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{\rho^2}{2} \\ -\frac{1}{4} & 0 & \frac{\rho^2}{2} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1}{2} \\ \frac{1}{2} & \frac{\rho^2}{2} & 0 & \frac{1}{2} & \frac{\rho^2}{1+\rho^2} \\ \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1+3\rho^2}{4(1+\rho^2)} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\rho^2}{2} & \frac{1}{2} & \frac{\rho^2}{1+\rho^2} & \frac{1}{2} & 0 \end{bmatrix} . \square \tag{62}$$

Figure 2 shows a plot of the 0*-sectional curvatures* $\varrho^{(0)}$ *as a function of correlation* ρ *for bivariate Gaussian manifold* N. □

Proposition 3.5. *The α-mean curvatures $\varrho^{(0)}(\lambda,\mu)$ $(\lambda = 1,2,3,4,5)$ are given by:*

$$\begin{aligned}
\varrho^{(\alpha)}(1) &= \varrho^{(\alpha)}(2) = \frac{\alpha^2-1}{8}\,,\\
\varrho^{(\alpha)}(3) &= \varrho^{(\alpha)}(5) = \frac{\alpha^2-1}{4}\,,\\
\varrho^{(\alpha)}(4) &= \frac{(\alpha^2-1)\,(3\,\sigma_1\,\sigma_2+{\sigma_{12}}^2)}{8\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2)} = \frac{(\alpha^2-1)\,(3+\rho^2)}{8\,(1+\rho^2)}\,.
\end{aligned} \tag{63}$$

Figure 3 shows a plot of the 0-mean curvatures $\varrho^{(0)}(\lambda)$ as a function of correlation ρ for bivariate Gaussian manifold N.

□

3.4 Dual coordinates

Since N is an exponential family, a mixture coordinate system is given by the potential function (59), that is,

$$\begin{aligned}
\eta_1 &= \frac{\partial\varphi}{\partial\theta_1} = \frac{2\,\theta_1\,\theta_5-\theta_2\,\theta_4}{{\theta_4}^2-4\,\theta_3\,\theta_5} = \mu_1\,,\\
\eta_2 &= \frac{\partial\varphi}{\partial\theta_2} = \frac{2\,\theta_2\,\theta_3-\theta_1\,\theta_4}{{\theta_4}^2-4\,\theta_3\,\theta_5} = \mu_2\,,\\
\eta_3 &= \frac{\partial\varphi}{\partial\theta_3} = \frac{{\theta_2}^2\,{\theta_4}^2+2\,\theta_4\,(-2\,\theta_1\,\theta_2+\theta_4)\,\theta_5+4\,\left({\theta_1}^2-2\,\theta_3\right)\,{\theta_5}^2}{\left({\theta_4}^2-4\,\theta_3\,\theta_5\right)^2} = {\mu_1}^2+\sigma_1\,,\\
\eta_4 &= \frac{\partial\varphi}{\partial\theta_4} = -\frac{2{\theta_2}^2\,\theta_3\theta_4+{\theta_4}^3+2\,\left({\theta_1}^2-2\theta_3\right)\theta_4\theta_5-\theta_1\theta_2\left({\theta_4}^2+4\theta_3\theta_5\right)}{\left({\theta_4}^2-4\theta_3\theta_5\right)^2} = \mu_1\mu_2+\sigma_{12},\\
\eta_5 &= \frac{\partial\varphi}{\partial\theta_5} = \frac{4{\theta_2}^2\,{\theta_3}^2-4\theta_1\,\theta_2\,\theta_3\,\theta_4+\left({\theta_1}^2+2\,\theta_3\right)\,{\theta_4}^2-8{\theta_3}^2\,\theta_5}{\left({\theta_4}^2-4\theta_3\,\theta_5\right)^2} = {\mu_2}^2+\sigma_2\,.
\end{aligned} \tag{64}$$

We have $(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5)$ as a 1-affine coordinate system, so $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5)$ is a (-1)-affine coordinate system, and they are dual with respect to the Fisher information metric. The coordinates (η_i) have a potential function given by:

$$\lambda = -\left(1+\log(2\,\pi\,\sqrt{\Delta})\right). \tag{65}$$

The coordinates (θ_i) and (η_i) form a dual coordinate system. Therefore the bivariate Gaussian manifold has dually orthogonal foliations (See Section 3.7 in [1]) for example.

Take

$$(\eta_1,\eta_2,\theta_3,\theta_4,\theta_5) = \left(\mu_1,\mu_2,\frac{-\sigma_2}{2\,(\sigma_1\sigma_2-{\sigma_{12}}^2)},\frac{\sigma_{12}}{\sigma_1\sigma_2-{\sigma_{12}}^2},\frac{-\sigma_1}{2\,(\sigma_1\sigma_2-{\sigma_{12}}^2)}\right)$$

as a coordinate system for N; then the bivariate Gaussian distributions take the form:

$$f(x,y;\eta_1,\eta_2,\theta_3,\theta_4,\theta_5) = \frac{1}{2\pi}\,\sqrt{4\theta_3\theta_5-\theta_4^2}\;e^{\theta_3\,(x-\mu_1)^2+\theta_4\,(x-\mu_1)\,(y-\mu_2)+\theta_5\,(y-\mu_2)^2} \tag{66}$$

and the Fisher metric is

$$\begin{bmatrix} \sigma_1 & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{12} & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \frac{{\sigma_2}^2}{2\,\Delta^2} & -\frac{\sigma_2\,\sigma_{12}}{\Delta^2} & \frac{{\sigma_{12}}^2}{2\,\Delta^2} \\ 0 & 0 & -\frac{\sigma_2\,\sigma_{12}}{\Delta^2} & \frac{\sigma_1\,\sigma_2+{\sigma_{12}}^2}{\Delta^2} & -\frac{\sigma_1\,\sigma_{12}}{\Delta^2} \\ 0 & 0 & \frac{{\sigma_{12}}^2}{2\,\Delta^2} & -\frac{\sigma_1\,\sigma_{12}}{\Delta^2} & \frac{{\sigma_1}^2}{2\,\Delta^2} \end{bmatrix}. \tag{67}$$

We remark that (θ_i) is a geodesic coordinate system of $\nabla^{(1)}$, and (η_i) is a geodesic coordinate system of $\nabla^{(-1)}$.

3.5 Bivariate Gaussian submanifolds

3.5.1 Independence submanifold: $N_1 \subset N$: $\sigma_{12} = 0$

The distributions are of form:

$$f(x, y; \mu_1, \mu_2, \sigma_1, \sigma_2) = f_X(x, \mu_1, \sigma_1).f_Y(y, \mu_2, \sigma_2) \tag{68}$$

This is the case for statistical independence of X and Y, so the space N_1 is the direct product of two Riemannian spaces

$$\{f_X(x, \mu_1, \sigma_1),\ \mu_1 \in \mathbb{R},\ \sigma_1 \in \mathbb{R}^+\} \ \text{ and } \ \{f_Y(y, \mu_2, \sigma_2),\ \mu_2 \in \mathbb{R},\ \sigma_2 \in \mathbb{R}^+\}.$$

We report expressions for the metric, the α-connections and the α-curvature objects using the natural coordinate system

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (\frac{\mu_1}{\sigma_1}, \frac{\mu_2}{\sigma_2}, -\frac{1}{2\sigma_1}, -\frac{1}{2\,\sigma_2})$$

and potential function $\varphi = \log(2\,\pi\,\sqrt{\Delta}) - \Delta\,\left({\theta_2}^2\,\theta_3 + {\theta_1}^2\,\theta_4\right); \quad \Delta = \frac{1}{4\,\theta_3\,\theta_4}.$

Proposition 3.6. *The metric tensor is:*

$$[g_{ij}] = \begin{bmatrix} \sigma_1 & 0 & 2\,\mu_1\,\sigma_1 & 0 \\ 0 & \sigma_2 & 0 & 2\,\mu_2\,\sigma_2 \\ 2\,\mu_1\,\sigma_1 & 0 & 2\,\sigma_1\,\left(2\,{\mu_1}^2 + \sigma_1\right) & 0 \\ 0 & 2\,\mu_2\,\sigma_2 & 0 & 2\,\sigma_2\,\left(2\,{\mu_2}^2 + \sigma_2\right) \end{bmatrix}. \square \tag{69}$$

Proposition 3.7. *By direct calculation we have the α-curvature tensor given by*

$$R^{(\alpha)}_{1313} = -\left(\alpha^2 - 1\right)\,{\sigma_1}^3, \quad R^{(\alpha)}_{2424} = -\left(\alpha^2 - 1\right)\,{\sigma_2}^3 \tag{70}$$

while the other independent components are zero.

By contraction we obtain the α- Ricci tensor:

$$R^{(\alpha)} = \left(\alpha^2 - 1\right) \begin{bmatrix} \frac{\sigma_1}{2} & 0 & \mu_1\,\sigma_1 & 0 \\ 0 & \frac{\sigma_2}{2} & 0 & \mu_2\,\sigma_2 \\ \mu_1\,\sigma_1 & 0 & \sigma_1\,\left(2\,{\mu_1}^2 + \sigma_1\right) & 0 \\ 0 & \mu_2\,\sigma_2 & 0 & \sigma_2\,\left(2\,{\mu_2}^2 + \sigma_2\right) \end{bmatrix}, \tag{71}$$

The α-eigenvalues of the α-Ricci tensor are given by:

$$\left(\alpha^2 - 1\right) \begin{pmatrix} \frac{\sigma_1}{4} + {\mu_1}^2\,\sigma_1 + \frac{\sigma_1}{4}\left(\sqrt{16\,{\mu_1}^4 + (1 - 2\,\sigma_1)^2 + 8\,{\mu_1}^2\,(1 + 2\,\sigma_1)} + \frac{2}{\sigma_1}\right) \\ \frac{\sigma_1}{4} + {\mu_1}^2\,\sigma_1 - \frac{\sigma_1}{4}\left(\sqrt{16\,{\mu_1}^4 + (1 - 2\,\sigma_1)^2 + 8\,{\mu_1}^2\,(1 + 2\,\sigma_1)} - \frac{2}{\sigma_1}\right) \\ \frac{\sigma_2}{4} + {\mu_2}^2\,\sigma_2 + \frac{\sigma_2}{4}\left(\sqrt{16\,{\mu_2}^4 + (1 - 2\,\sigma_2)^2 + 8\,{\mu_2}^2\,(1 + 2\,\sigma_2)} + \frac{2}{\sigma_2}\right) \\ \frac{\sigma_2}{4} + {\mu_2}^2\,\sigma_2 - \frac{{\sigma_2}^2}{4}\left(\sqrt{16\,{\mu_2}^4 + (1 - 2\,\sigma_2)^2 + 8\,{\mu_2}^2\,(1 + 2\,\sigma_2)} - \frac{2}{\sigma_2}\right) \end{pmatrix}$$

The α-scalar curvature of N_1 is constant:

$$R^{(\alpha)} = 2\left(\alpha^2 - 1\right) \tag{72}$$

The α-sectional curvatures:

$$\varrho^{(\alpha)} = \frac{(\alpha^2 - 1)}{2} \begin{bmatrix} 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \\ 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \end{bmatrix} \tag{73}$$

The α-mean curvatures:

$$\varrho^{(\alpha)}(1) = \varrho^{(\alpha)}(2) = \varrho^{(\alpha)}(3) = \varrho^{(\alpha)}(4) = \frac{\alpha^2 - 1}{6}\,. \tag{74}$$

□

Proposition 3.8. *The submanifold N_1 is an Einstein space.*

Proof. By comparison of the metric tensor (69) with the Ricci tensor (71), we see that

$$R^{(0)}_{ij} = \frac{R^{(0)}}{k} g_{ij}, \quad k = dim(N_1).$$

So the submanifold N_1 with statistically independent random variables is an Einstein space. □

3.5.2 Identical marginal Gaussian submanifold: $N_2 \subset N$: $\sigma_1 = \sigma_2 = \sigma$ and $\mu_1 = \mu_2 = \mu$

The distributions are of form:

$$f(x, y; \mu, \sigma, \sigma_{12}) = \frac{1}{2\pi\sqrt{\sigma^2 - {\sigma_{12}}^2}}\, e^{-\frac{1}{2(\sigma^2 - {\sigma_{12}}^2)}\left(\sigma(x-\mu)^2 - 2\sigma_{12}(x-\mu)(y-\mu) + \sigma(y-\mu)^2\right)} \tag{75}$$

The marginal functions are $f_X = f_Y \equiv N(\mu, \sigma)$, with correlation coefficient $\rho(X, Y) = \frac{\sigma_{12}}{\sigma}$.

We report the expressions for the metric, the α-connections and the α-curvature objects using the natural coordinate system

$$(\theta_1, \theta_2, \theta_3) = (\frac{\mu}{\sigma + \sigma_{12}}, \frac{-\sigma}{2\,(\sigma^2 - {\sigma_{12}}^2)}, \frac{\sigma_{12}}{(\sigma^2 - {\sigma_{12}}^2)})$$

and the potential function

$$\varphi = -\frac{{\theta_1}^2}{2\,\theta_2 + \theta_3} + \log(2\,\pi) - \frac{1}{2}\,\log(4\,{\theta_2}^2 - {\theta_3}^2)\,.$$

Proposition 3.9. *The metric tensor $[g_{ij}]$ is as follows:*

$$\begin{bmatrix} 2\,(\sigma + \sigma_{12}) & 4\,\mu\,(\sigma + \sigma_{12}) & 2\,\mu\,(\sigma + \sigma_{12}) \\ 4\,\mu\,(\sigma + \sigma_{12}) & 4\,(\sigma\,(2\,\mu^2 + \sigma) + 2\,\mu^2\,\sigma_{12} + {\sigma_{12}}^2) & 4\,(\mu^2\,\sigma + (\mu^2 + \sigma)\,\sigma_{12}) \\ 2\,\mu\,(\sigma + \sigma_{12}) & 4\,(\mu^2\,\sigma + (\mu^2 + \sigma)\,\sigma_{12}) & \sigma\,(2\,\mu^2 + \sigma) + 2\,\mu^2\,\sigma_{12} + {\sigma_{12}}^2 \end{bmatrix} \tag{76}$$

Proposition 3.10. *By direct calculation we have the α-curvature tensor of N_2*

$$R^{(\alpha)}_{12kl} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -2\,(\sigma + \sigma_{12})^3 & -(\sigma + \sigma_{12})^3 \\ 2\,(\sigma + \sigma_{12})^3 & 0 & 0 \\ (\sigma + \sigma_{12})^3 & 0 & 0 \end{bmatrix}$$

$$R^{(\alpha)}_{13kl} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -(\sigma + \sigma_{12})^3 & \frac{-(\sigma+\sigma_{12})^3}{2} \\ (\sigma + \sigma_{12})^3 & 0 & 0 \\ \frac{(\sigma+\sigma_{12})^3}{2} & 0 & 0 \end{bmatrix} \tag{77}$$

while the other independent components are zero.

By contraction we obtain:
The α- Ricci tensor:

$$\left(\alpha^2-1\right)\begin{bmatrix} (\sigma+\sigma_{12}) & 2\,\mu\,(\sigma+\sigma_{12}) & \mu\,(\sigma+\sigma_{12}) \\ 2\,\mu\,(\sigma+\sigma_{12}) & (\sigma+\sigma_{12})\,(4\,\mu^2+\sigma+\sigma_{12}) & \frac{(\sigma+\sigma_{12})\left(4\,\mu^2+\sigma+\sigma_{12}\right)}{2} \\ \mu\,(\sigma+\sigma_{12}) & \frac{(\sigma+\sigma_{12})\left(4\,\mu^2+\sigma+\sigma_{12}\right)}{2} & \frac{(\sigma+\sigma_{12})\left(4\,\mu^2+\sigma+\sigma_{12}\right)}{4} \end{bmatrix}, \tag{78}$$

The α-eigenvalues of the α-Ricci tensor are given by:

$$\left(\alpha^2-1\right)\begin{pmatrix} 0 \\ \frac{10(\sigma+\sigma_{12})^2}{4(1+5\mu^2)+5-\sqrt{400\mu^4+(4-5\sigma)^2+40\mu^2(4+5\sigma)+5\sigma_{12}\,(-8+40\,\mu^2+10\sigma+5\,\sigma_{12})}} \\ \frac{10(\sigma+\sigma_{12})^2}{4(1+5\mu^2)+5+\sqrt{400\,\mu^4+(4-5\,\sigma)^2+40\mu^2\,(4+5\sigma)+5\sigma_{12}(-8+40\,\mu^2+10\sigma+5\,\sigma_{12})}} \end{pmatrix} \tag{79}$$

The α-scalar curvature:

$$R^{(\alpha)}=\left(\alpha^2-1\right) \tag{80}$$

The α-sectional curvatures:

$$\varrho^{(\alpha)}=\left(\alpha^2-1\right)\begin{bmatrix} 0 & \frac{(\sigma+\sigma_{12})^2}{4\,(\sigma^2+\sigma_{12}{}^2)} & \frac{(\sigma+\sigma_{12})^2}{4\,(\sigma^2+\sigma_{12}{}^2)} \\ \frac{(\sigma+\sigma_{12})^2}{4\,(\sigma^2+\sigma_{12}{}^2)} & 0 & 0 \\ \frac{(\sigma+\sigma_{12})^2}{4\,(\sigma^2+\sigma_{12}{}^2)} & 0 & 0 \end{bmatrix} \tag{81}$$

The α-mean curvatures:

$$\begin{aligned} \varrho^{(\alpha)}(1) &= \frac{1}{4}\left(\alpha^2-1\right)\,, \\ \varrho^{(\alpha)}(2) = \varrho^{(\alpha)}(3) &= \frac{(\alpha^2-1)\,(\sigma+\sigma_{12})\,(4\,\mu^2+\sigma+\sigma_{12})}{8\,(\sigma\,(2\,\mu^2+\sigma)+2\,\mu^2\,\sigma_{12}+\sigma_{12}{}^2)}\,.\ \square \end{aligned} \tag{82}$$

3.5.3 Central mean submanifold: $N_3 \subset N$: $\mu_1=\mu_2=0$

The distributions are of form:

$$f(x,y;\sigma_1,\sigma_2,\sigma_{12})=\frac{1}{2\,\pi\,\sqrt{\sigma_1\,\sigma_2-\sigma_{12}{}^2}}\,e^{-\frac{1}{2\,(\sigma_1\,\sigma_2-\sigma_{12}{}^2)}\left(\sigma_2 x^2-2\,\sigma_{12}\,x\,y+\sigma_1 y^2\right)} \tag{83}$$

The marginal functions are $f_X(x,0,\sigma_1)$ and $f_Y(y,0,\sigma_2)$, with correlation coefficient $\rho(X,Y)=\frac{\sigma_{12}}{\sqrt{\sigma_1\,\sigma_2}}$.

We report the metric, and the α-curvature objects using the natural coordinate system

$$(\theta_1,\theta_2,\theta_3)=(-\frac{\sigma_2}{2\,(\sigma_1\,\sigma_2-\sigma_{12}{}^2)},\frac{\sigma_{12}}{\sigma_1\,\sigma_2-\sigma_{12}{}^2},-\frac{\sigma_1}{2\,(\sigma_1\,\sigma_2-\sigma_{12}{}^2)})$$

and the potential function

$$\varphi = \log(2\,\pi) - \frac{1}{2}\,\log(\sqrt{4\,\theta_1\,\theta_3 - {\theta_4}^2})\,.$$

Proposition 3.11. *The metric tensor is as follows:*

$$[g_{ij}] = \begin{bmatrix} 2\,{\sigma_1}^2 & 2\,\sigma_1\,\sigma_{12} & 2\,{\sigma_{12}}^2 \\ 2\,\sigma_1\,\sigma_{12} & \sigma_1\,\sigma_2 + {\sigma_{12}}^2 & 2\,\sigma_2\,\sigma_{12} \\ 2\,{\sigma_{12}}^2 & 2\,\sigma_2\,\sigma_{12} & 2\,{\sigma_2}^2 \end{bmatrix},$$

□

Proposition 3.12. *By direct calculation we have the nonzero independent components of the α-curvature tensor of N_3*

$$R^{(\alpha)}_{12kl} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -{\sigma_1}^2\,\Delta & -2\,\sigma_1\,\sigma_{12}\,\Delta \\ {\sigma_1}^2\,\Delta & 0 & -\sigma_1\,\sigma_2\,\Delta \\ 2\,\sigma_1\,\sigma_{12}\,\Delta & \sigma_1\,\sigma_2\,\Delta & 0 \end{bmatrix}$$

$$R^{(\alpha)}_{13kl} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -2\,\sigma_1\,\sigma_{12}\,\Delta & -4\,{\sigma_{12}}^2\,\Delta \\ 2\,\sigma_1\,\sigma_{12}\,\Delta & 0 & -2\,\sigma_2\,\sigma_{12}\,\Delta \\ 4\,{\sigma_{12}}^2\,\Delta & 2\,\sigma_2\,\sigma_{12}\,\Delta & 0 \end{bmatrix}$$

$$R^{(\alpha)}_{23kl} = \left(\alpha^2 - 1\right) \begin{bmatrix} 0 & -\sigma_1\,\sigma_2\,\Delta & -2\,\sigma_2\,\sigma_{12}\,\Delta \\ \sigma_1\,\sigma_2\,\Delta & 0 & -{\sigma_2}^2\,\Delta \\ 2\,\sigma_2\,\sigma_{12}\,\Delta & {\sigma_2}^2\,\Delta & 0 \end{bmatrix}. \qquad (84)$$

By contraction we obtain:
The α- Ricci tensor:

$$R^{(\alpha)} = \left(\alpha^2 - 1\right) \begin{bmatrix} {\sigma_1}^2 & \sigma_1\,\sigma_{12} & 2\,{\sigma_{12}}^2 - \sigma_1\,\sigma_2 \\ \sigma_1\,\sigma_{12} & \sigma_1\,\sigma_2 & \sigma_2\,\sigma_{12} \\ 2\,{\sigma_{12}}^2 - \sigma_1\,\sigma_2 & \sigma_2\,\sigma_{12} & {\sigma_2}^2 \end{bmatrix}, \qquad (85)$$

The α-eigenvalues of the α-Ricci tensor are given by:

$$\frac{(\alpha^2-1)}{2}\begin{pmatrix} 0 \\ {\sigma_1}^2+\sigma_1\,\sigma_2+{\sigma_2}^2-\sqrt{({\sigma_1}^2-\sigma_1\,\sigma_2+{\sigma_2}^2)^2+4\,({\sigma_1}^2-4\,\sigma_1\,\sigma_2+{\sigma_2}^2)\,{\sigma_{12}}^2+16\,{\sigma_{12}}^4} \\ {\sigma_1}^2+\sigma_1\,\sigma_2+{\sigma_2}^2+\sqrt{({\sigma_1}^2-\sigma_1\,\sigma_2+{\sigma_2}^2)^2+4\,({\sigma_1}^2-4\,\sigma_1\,\sigma_2+{\sigma_2}^2)\,{\sigma_{12}}^2+16\,{\sigma_{12}}^4} \end{pmatrix}$$

The α-scalar curvature:

$$R^{(\alpha)}=2\,\left(\alpha^2-1\right) \tag{86}$$

The α-sectional curvatures:

$$\varrho^{(\alpha)}=\left(\alpha^2-1\right)\begin{bmatrix} 0 & \frac{1}{2} & \frac{\rho^2}{1+\rho^2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{\rho^2}{1+\rho^2} & \frac{1}{2} & 0 \end{bmatrix} \tag{87}$$

The α-mean curvatures:

$$\begin{aligned} \varrho^{(\alpha)}(1)&=\varrho^{(\alpha)}(3)=\frac{1}{4}\left(\alpha^2-1\right)\,, \\ \varrho^{(\alpha)}(2)&=\frac{(\alpha^2-1)\,\sigma_1\,\sigma_2}{2\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2)}=\frac{(\alpha^2-1)}{2\,(1+\rho^2)}\,. \end{aligned} \tag{88}$$

For N_3 the α-mean curvatures have limiting value $\frac{(\alpha^2-1)}{4}$ as $\rho^2\to 1$.

□

3.6 Affine immersion

Proposition 3.13. *Let N be the bivariate Gaussian manifold with the Fisher metric g and the exponential connection $\nabla^{(1)}$. Denote by (θ_i) the natural coordinate system (58). Then N can be realized in $\mathbb{R}^6$ by the graph of a potential function, via the affine immersion $\{f,\xi\}$:*

$$f:\Theta\to\mathbb{R}^6:\begin{bmatrix}\theta_i\end{bmatrix}\mapsto\begin{bmatrix}\theta_i \\ \varphi(\theta)\end{bmatrix}, \tag{89}$$

where $\varphi(\theta)$ is the potential function $\varphi(\theta)=\log(2\,\pi\,\sqrt{\Delta})-\Delta\,\left({\theta_2}^2\,\theta_3-\theta_1\,\theta_2\,\theta_4+{\theta_1}^2\,\theta_5\right)$.

□

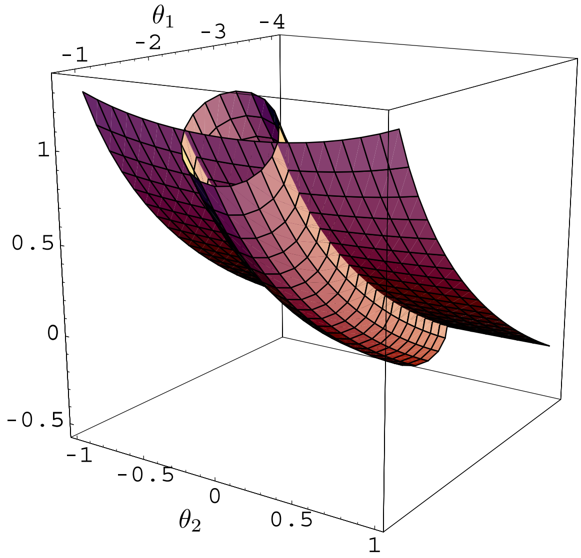

Fig. 4 Affine immersion in natural coordinates $(\theta_1, \theta_2) = (\frac{-\sigma}{2\Delta}, \frac{\sigma_{12}}{\Delta})$ as a surface in $\mathbb{R}^3$ for the bivariate Gaussian distributions with zero means and common standard deviation σ. The tubular neighbourhood surrounds the curve $\sigma_{12} = 0$ in the surface; this curve represents bivariate distributions having common Gaussian marginals and zero covariance; its tubular neighourhoods contain by continuity all sufficient small departures from independence.

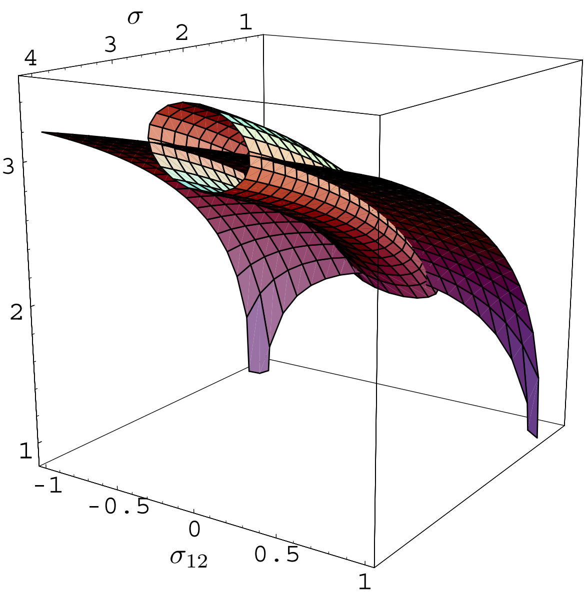

Fig. 5 Continuous image of the affine immersion in Figure 4 as a surface in $\mathbb{R}^3$ using standard coordinates for the bivariate Gaussian distributions with zero means and common standard deviation σ; the tubular neighbourhood surrounds the curve $\sigma_{12} = 0$ in the surface.

3.7 Neighbourhoods of independence

The case of bivariate Gaussian distributions with zero means ($\mu_1 = \mu_2 = 0$) and common standard deviation $\sigma_1 = \sigma_2 = \sigma$ is represented by the surface in $\mathbb{R}^3$:

$$\mathbb{R}^- \times \mathbb{R} \to \mathbb{R}^3 : (\theta_1, \theta_2) \mapsto (\theta_1, \theta_2, \varphi(\theta)) ,$$

where $(\theta_1, \theta_2) = (\frac{-\sigma}{2\,\Delta}, \frac{\sigma_{12}}{\Delta})$; $\Delta = \sigma^2 - {\sigma_{12}}^2$ and $\varphi(\theta) = \log(2\,\pi\,\sigma)$.

So the submanifold consisting of the independent case with zero means and common standard deviations is represented by the curve:

$$\begin{aligned}(-\infty, 0) \to \mathbb{R}^3 : (\theta_1) &\mapsto (\theta_1, 0, \log(-4\,\pi\,\Delta\,\theta_1)) \\ : (-\frac{1}{2\,\sigma}) &\mapsto (-\frac{1}{2\,\sigma}, 0, \log(2\,\pi\,\sigma)).\end{aligned}$$

Proposition 3.14. *In the affine immersion as a surface in $\mathbb{R}^3$ for the bivariate Gaussian distributions with zero means and common standard deviation σ, tubular neighbourhoods of the curve of zero covariance will contain by continuity all immersions of bivariate Gaussian processes sufficiently close to the independence case.* □

Corollary 3.15. *Via the Central Limit Theorem, the tubular neighbourhoods of the curve of zero covariance will contain all immersions of limiting bivariate processes sufficiently close to the independence case for all processes with marginals that converge in distribution to Gaussians.* □

The figures show an affine embedding of the bivariate Gaussian with zero means ($\mu_1 = \mu_2 = 0$) and common standard deviation σ as a surface in $\mathbb{R}^3$, and an $\mathbb{R}^3$-tubular neighbourhood of the curve $\sigma_{12} = 0$ in the surface. This curve represents bivariate distributions having common Gaussian marginals and zero covariance; its tubular neighbourhoods represent departures from independence. In Figure 4 this is depicted in natural coordinates $(\frac{-\sigma}{2\,\Delta}, \frac{\sigma_{12}}{\Delta})$ and in Figure 5 the corresponding surface and tubular neighbourhood (not here an affine immersion, just a continuous image) is shown in the usual (σ, σ_{12}) coordinates of the bivariate Gaussian family, with zero means and common standard deviation σ.

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References

[1] S.-I. Amari, O.E. Barndorff-Neilsen, R.E. Kass, S.L. Lauritzen and C.R. Rao: *Differential Geometrical Methods in Statistics*, Springer Lecture Notes in Statistics 28, Springer-Verlag, Berlin, 1985.

[2] S-I. Amari and H. Nagaoka: *Methods of Information Geometry*, American Mathematical Society, Oxford University Press, 2000.

[3] Khadiga Arwini: *Differential geometry in neighbourhoods of randomness and independence*, PhD thesis, UMIST, 2004.

[4] Khadiga Arwini and C.T.J. Dodson: "Information geometric neighbourhoods of randomness and geometry of the McKay bivariate gamma 3-manifold", *Sankhya: Indian Journal of Statistics*, Vol. 66(2), (2004), pp. 211–231.

[5] Khadiga Arwini and C.T.J. Dodson: "Neighbourhoods of independence for random processes via information geometry", *Math. J.*, Vol. 9(4), (2005).

[6] Khadiga Arwini, L. Del Riego and C.T.J. Dodson: "Universal connection and curvature for statistical manifold geometry", *Houston J. Math.*, in press, (2006).

[7] Y. Cai, C.T.J. Dodson, O. Wolkenhauer and A.J. Doig: "Gamma Distribution Analysis of Protein Sequences shows that Amino Acids Self Cluster", *J. Theoretical Biology*, Vol. 218(4), (2002), pp. 409–418.

[8] C.T.J. Dodson: "Spatial statistics and information geometry for parametric statistical models of galaxy clustering", *Int. J. Theor. Phys.*, Vol. 38(10), (1999), pp. 2585–2597.

[9] C.T.J. Dodson: "Geometry for stochastically inhomogeneous spacetimes", *Nonlinear Analysis*, Vol. 47, (2001), pp. 2951–2958.

[10] C.T.J. Dodson and Hiroshi Matsuzoe: "An affine embedding of the gamma manifold", *Appl. Sci.*, Vol. 5(1), (2003), pp. 1–6.

[11] R.J. Freund: "A bivariate extension of the exponential distribution", J. Am. Stat. Assoc., Vol. 56, (1961), pp. 971–977.

[12] T.P. Hutchinson and C.D. Lai: *Continuous Multivariate Distributions, Emphasising Applications*, Rumsby Scientific Publishing, Adelaide 1990.

[13] S. Kotz, N. Balakrishnan and N. Johnson: *Continuous Multivariate Distributions*, Volume 1, 2nd ed., John Wiley, New York, 2000.

[14] S. Leurgans, T.W.-Y. Tsai and J. Crowley: "Freund's bivariate exponential distribution and censoring", In: R.A. Johnson (Ed.): *Survival Analysis*, IMS Lecture Notes, Hayward, California, Institute of Mathematical Statistics, 1982.

[15] A.F.S. Mitchell: "The information matrix, skewness tensor and α-connections for the general multivariate elliptic distribution", *Ann. Ins. Stat. Math.*, Vol. 41, (1989), pp. 289–304.

[16] Y. Sato, K. Sugawa and M. Kawaguchi: *The geometrical structure of the parameter space of the two-dimensional normal distribution*, Division of information engineering, Hokkaido University, Sapporo, Japan, 1977.

[17] L.T. Skovgaard: "A Riemannian geometry of the multivariate normal model", *Scan-*

dinavian J. Stat., Vol. 11, (1984), pp. 211–223.

Appendix

A Freund 4-manifold F

A.1 α-connection and α-curvature

Proposition A.1. *The nonzero independent components* $\Gamma^{(\alpha)}_{ij,k}$ *in* $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ *coordinates are*

$$\begin{aligned}
\Gamma^{(\alpha)}_{11,1} &= \frac{2\,(\alpha-1)\,\alpha_1-(1+\alpha)\,\alpha_2}{2\,{\alpha_1}^2\,(\alpha_1+\alpha_2)^2}\,,\\
\Gamma^{(\alpha)}_{11,2} &= \frac{1+\alpha}{2\,\alpha_1\,(\alpha_1+\alpha_2)^2}\,,\\
\Gamma^{(\alpha)}_{13,3} &= \frac{(\alpha-1)\,\alpha_2}{2\,(\alpha_1+\alpha_2)^2\,{\beta_1}^2},\\
\Gamma^{(\alpha)}_{12,2} &= \frac{\alpha-1}{2\,\alpha_2\,(\alpha_1+\alpha_2)^2}\,,\\
\Gamma^{(\alpha)}_{14,4} &= \frac{-(\alpha-1)\,\alpha_2}{2\,(\alpha_1+\alpha_2)^2\,{\beta_2}^2},\\
\Gamma^{(\alpha)}_{33,3} &= \frac{(\alpha-1)\,\alpha_2}{(\alpha_1+\alpha_2)\,{\beta_1}^3}\,,\\
\Gamma^{(\alpha)}_{33,2} &= \frac{-(1+\alpha)\,\alpha_1}{2\,(\alpha_1+\alpha_2)^2\,{\beta_1}^2}\,,\\
\Gamma^{(\alpha)}_{22,2} &= \frac{-(1+\alpha)\,\alpha_1+2\,(\alpha-1)\,\alpha_2}{2\,{\alpha_2}^2\,(\alpha_1+\alpha_2)^2},\\
\Gamma^{(\alpha)}_{24,4} &= \frac{(\alpha-1)\,\alpha_1}{2\,(\alpha_1+\alpha_2)^2\,{\beta_2}^2},\\
\Gamma^{(\alpha)}_{44,4} &= \frac{(\alpha-1)\,\alpha_1}{(\alpha_1+\alpha_2)\,{\beta_2}^3}.
\end{aligned} \tag{A.1}$$

□

Proposition A.2. *The nonzero components* $\Gamma^{i(\alpha)}_{jk}$ *of the* $\nabla^{(\alpha)}$*-connections are given by:*

$$
\begin{aligned}
\Gamma^{(\alpha)1}_{11} &= -\frac{1+\alpha}{2\,\alpha_1} + \frac{-1+3\,\alpha}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{12} &= \Gamma^{(\alpha)3}_{13} = \Gamma^{(\alpha)2}_{12} = \Gamma^{(\alpha)4}_{24} = \frac{\alpha-1}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{33} &= -\Gamma^{(\alpha)2}_{33} = \frac{(1+\alpha)\,\alpha_1\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_1}^2}\,,\\
\Gamma^{(\alpha)1}_{22} &= \Gamma^{(\alpha)3}_{32} = \frac{(1+\alpha)\,\alpha_1}{2\,\alpha_2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)1}_{44} &= -\Gamma^{(\alpha)2}_{44} = \frac{-\,(1+\alpha)\,\alpha_1\,\alpha_2}{2\,(\alpha_1+\alpha_2)\,{\beta_2}^2}\,,\\
\Gamma^{(\alpha)2}_{11} &= \Gamma^{(\alpha)4}_{14} = \frac{(1+\alpha)\,\alpha_2}{2\,\alpha_1\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)2}_{22} &= -\frac{1+\alpha}{2\,\alpha_2} + \frac{-1+3\,\alpha}{2\,(\alpha_1+\alpha_2)}\,,\\
\Gamma^{(\alpha)4}_{44} &= \frac{\alpha-1}{\beta_2}\,. \quad \square
\end{aligned}
\tag{A.2}
$$

B Bivariate Gaussian 5-manifold N

We use coordinates $(\mu_1, \mu_2, \sigma_1, \sigma_{12}, \sigma_2)$.

B.1 α-connection and α-curvature

Proposition B.1. *The functions* $\Gamma^{(\alpha)}_{ij,k}$ *are given by:*

$$
[\Gamma^{(\alpha)}_{ij,1}] = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix} \quad \text{where } A = \frac{1}{2\Delta^2}\begin{bmatrix} -\,(1+\alpha)\,{\sigma_2}^2 & 2(1+\alpha)\,\sigma_2\,\sigma_{12} & -\,(1+\alpha)\,{\sigma_{12}}^2 \\ (1+\alpha)\,\sigma_2\,\sigma_{12} & -\,(1+\alpha)\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & (1+\alpha)\,\sigma_1\,\sigma_{12} \end{bmatrix}
$$

$$
[\Gamma^{(\alpha)}_{ij,2}] = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} \quad \text{where } B = \frac{1}{2\Delta^2}\begin{bmatrix} (1+\alpha)\,\sigma_2\,\sigma_{12} & -\,(1+\alpha)\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & (1+\alpha)\,\sigma_1\,\sigma_{12} \\ -\,(1+\alpha)\,{\sigma_{12}}^2 & 2(1+\alpha)\,\sigma_1\,\sigma_{12} & -(1+\alpha)\,{\sigma_1}^2 \end{bmatrix}
$$

$$
[\Gamma^{(\alpha)}_{ij,3}] = \begin{bmatrix}
\frac{-(\alpha-1)\,{\sigma_2}^2}{2\,\Delta^2} & \frac{(\alpha-1)\,\sigma_2\,\sigma_{12}}{2\,\Delta^2} & 0 & 0 & 0 \\
\frac{(\alpha-1)\,\sigma_2\,\sigma_{12}}{2\,\Delta^2} & \frac{-(\alpha-1)\,{\sigma_{12}}^2}{2\,\Delta^2} & 0 & 0 & 0 \\
0 & 0 & \frac{(1+\alpha)\,{\sigma_2}^3}{-2\,\Delta^3} & \frac{(1+\alpha)\,{\sigma_2}^2\,\sigma_{12}}{\Delta^3} & \frac{(1+\alpha)\,\sigma_2\,{\sigma_{12}}^2}{-2\,\Delta^3} \\
0 & 0 & \frac{(1+\alpha)\,{\sigma_2}^2\,\sigma_{12}}{\Delta^3} & \frac{-(1+\alpha)\,\sigma_2\left(\sigma_1\,\sigma_2+3\,{\sigma_{12}}^2\right)}{2\,\Delta^3} & \frac{(1+\alpha)\,\sigma_{12}\left(\sigma_1\,\sigma_2+{\sigma_{12}}^2\right)}{2\,\Delta^3} \\
0 & 0 & \frac{(1+\alpha)\,\sigma_2\,{\sigma_{12}}^2}{-2\,\Delta^3} & \frac{(1+\alpha)\,\sigma_{12}\left(\sigma_1\,\sigma_2+{\sigma_{12}}^2\right)}{2\,\Delta^3} & \frac{(1+\alpha)\,\sigma_1\,{\sigma_{12}}^2}{-2\,\Delta^3}
\end{bmatrix}
$$

$$[\Gamma^{(\alpha)}_{ij,4}] = (\alpha+1)\begin{bmatrix} \frac{(\alpha-1)\sigma_2\sigma_{12}}{(\alpha+1)\Delta^2} & \frac{(\alpha-1)\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{-2(\alpha+1)\Delta^2} & 0 & 0 & 0 \\ \frac{(\alpha-1)\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{-2(\alpha+1)\Delta^2} & \frac{(\alpha-1)\sigma_1\sigma_{12}}{(\alpha+1)\Delta^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\sigma_2{}^2\sigma_{12}}{\Delta^3} & \frac{\sigma_2\left(\sigma_1\sigma_2+3\sigma_{12}{}^2\right)}{-2\,\Delta^3} & \frac{\sigma_{12}\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{2\,\Delta^3} \\ 0 & 0 & \frac{\sigma_2\left(\sigma_1\sigma_2+3\sigma_{12}{}^2\right)}{-2\,\Delta^3} & \frac{\sigma_{12}\left(3\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{\Delta^3} & \frac{\sigma_1\left(\sigma_1\sigma_2+3\sigma_{12}{}^2\right)}{-2\,\Delta^3} \\ 0 & 0 & \frac{\sigma_{12}\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{2\,\Delta^3} & \frac{\sigma_1\left(\sigma_1\sigma_2+3\sigma_{12}{}^2\right)}{-2\,\Delta^3} & \frac{\sigma_1{}^2\sigma_{12}}{\Delta^3} \end{bmatrix}$$

$$[\Gamma^{(\alpha)}_{ij,5}] = \begin{bmatrix} \frac{-(\alpha-1)\sigma_{12}{}^2}{2\Delta^2} & \frac{(\alpha-1)\sigma_1\sigma_{12}}{2\Delta^2} & 0 & 0 & 0 \\ \frac{(\alpha-1)\,\sigma_1\,\sigma_{12}}{2\,\Delta^2} & \frac{-(\alpha-1)\sigma_1{}^2}{2\Delta^2} & 0 & 0 & 0 \\ 0 & 0 & \frac{(1+\alpha)\sigma_2\sigma_{12}{}^2}{-2\Delta^3} & \frac{(1+\alpha)\sigma_{12}\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{2\Delta^3} & \frac{(1+\alpha)\sigma_1\sigma_{12}{}^2}{-2\Delta^3} \\ 0 & 0 & \frac{(1+\alpha)\sigma_{12}\left(\sigma_1\sigma_2+\sigma_{12}{}^2\right)}{2\Delta^3} & \frac{-(1+\alpha)\sigma_1\left(\sigma_1\sigma_2+3\sigma_{12}{}^2\right)}{2\Delta^3} & \frac{(1+\alpha)\sigma_1{}^2\sigma_{12}}{\Delta^3} \\ 0 & 0 & \frac{(1+\alpha)\sigma_1\sigma_{12}{}^2}{2\Delta^3} & \frac{(1+\alpha)\sigma_1{}^2\sigma_{12}}{\Delta^3} & \frac{(1+\alpha)\sigma_1{}^3}{-2\Delta^3} \end{bmatrix}$$

□

We have an affine connection $\nabla^{(\alpha)}$ defined by:

$$\langle \nabla^{(\alpha)}_{\partial_i}\partial_j, \partial_k\rangle = \Gamma^{(\alpha)}_{ij,k}\,,$$

So by solving the equations

$$\Gamma^{(\alpha)}_{ij,k} = \sum_{h=1}^{3} g_{kh}\,\Gamma^{h(\alpha)}_{ij} \quad ,(k=1,2,3,4,5).$$

we obtain the components of $\nabla^{(\alpha)}$:

Proposition B.2. *The components $\Gamma^{(\alpha)i}_{jk}$ of the $\nabla^{(\alpha)}$-connections are given by:*

$$\Gamma^{(\alpha)1} = [\Gamma^{(\alpha)1}_{ij}] = \frac{(1+\alpha)}{2\Delta}\begin{bmatrix} 0 & 0 & -\sigma_2 & \sigma_{12} & 0 \\ 0 & 0 & \sigma_{12} & -\sigma_1 & 0 \\ -\sigma_2 & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{12} & -\sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{(\alpha)2} = [\Gamma^{(\alpha)2}_{ij}] = \frac{(1+\alpha)}{2\Delta}\begin{bmatrix} 0 & 0 & 0 & -\sigma_2 & \sigma_{12} \\ 0 & 0 & 0 & \sigma_{12} & -\sigma_1 \\ 0 & 0 & 0 & 0 & 0 \\ -\sigma_2 & \sigma_{12} & 0 & 0 & 0 \\ \sigma_{12} & -\sigma_1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{(\alpha)3} = [\Gamma^{(\alpha)3}_{ij}] = \frac{1}{\Delta}\begin{bmatrix} \Delta(1-\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -(1+\alpha)\,\sigma_2 & (1+\alpha)\,\sigma_{12} & 0 \\ 0 & 0 & (1+\alpha)\,\sigma_{12} & -(1+\alpha)\,\sigma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma^{(\alpha)4} = [\Gamma^{(\alpha)4}_{ij}] = \frac{1}{2\Delta}\begin{bmatrix} 0 & \Delta(1-\alpha) & 0 & 0 & 0 \\ \Delta(1-\alpha) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+\alpha)\,\sigma_2 & (1+\alpha)\,\sigma_{12} \\ 0 & 0 & -(1+\alpha)\,\sigma_2 & (1+\alpha)\,\sigma_{12} & -(1+\alpha)\,\sigma_1 \\ 0 & 0 & (1+\alpha)\,\sigma_{12} & -(1+\alpha)\,\sigma_1 & 0 \end{bmatrix}$$

$$\Gamma^{(\alpha)5} = [\Gamma^{(\alpha)5}_{ij}] = \frac{1}{\Delta}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \Delta(1-\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(1+\alpha)\,\sigma_2 & (1+\alpha)\,\sigma_{12} \\ 0 & 0 & 0 & (1+\alpha)\,\sigma_{12} & -(1+\alpha)\,\sigma_1 \end{bmatrix}.\square \qquad \text{(B.1)}$$

B.1.1 α-curvature

Proposition B.3. *The components $R^{(\alpha)}_{ijkl}$ of the α-curvature tensor are given by:*

$$[R^{(\alpha)}_{12kl}] = \frac{(\alpha^2-1)}{4\Delta^2}\begin{bmatrix} 0 & \Delta & 0 & 0 & 0 \\ -\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_2 & \sigma_{12} \\ 0 & 0 & \sigma_2 & 0 & -\sigma_1 \\ 0 & 0 & -\sigma_{12} & \sigma_1 & 0 \end{bmatrix}$$

$$[R^{(\alpha)}_{13kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & C \\ -C^T & 0 \end{bmatrix} \text{ where } C = \begin{bmatrix} -{\sigma_2}^3 & 2{\sigma_2}^2\,\sigma_{12} & -\sigma_2\,{\sigma_{12}}^2 \\ {\sigma_2}^2\,\sigma_{12} & -\sigma_2\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & \sigma_1\,\sigma_2\,\sigma_{12} \end{bmatrix}$$

$$[R^{(\alpha)}_{14kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & D \\ -D^T & 0 \end{bmatrix} \text{ where } D = \begin{bmatrix} 2{\sigma_2}^2\sigma_{12} & -\sigma_2\,(\sigma_1\sigma_2+3{\sigma_{12}}^2) & \sigma_{12}\,(\sigma_1\sigma_2+{\sigma_{12}}^2) \\ -2\sigma_2{\sigma_{12}}^2 & \sigma_{12}\,(3\sigma_1\sigma_2+{\sigma_{12}}^2) & -\sigma_1\,(\sigma_1\sigma_2+{\sigma_{12}}^2) \end{bmatrix}$$

$$[R^{(\alpha)}_{15kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & E \\ -E^T & 0 \end{bmatrix} \text{ where } E = \begin{bmatrix} -\sigma_2\,{\sigma_{12}}^2 & \sigma_{12}\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & -\sigma_1\,{\sigma_{12}}^2 \\ {\sigma_{12}}^3 & -2\sigma_1\,{\sigma_{12}}^2 & {\sigma_1}^2\,\sigma_{12} \end{bmatrix}$$

$$[R^{(\alpha)}_{23kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & H \\ -H^T & 0 \end{bmatrix} \text{ where } H = \begin{bmatrix} {\sigma_2}^2\,\sigma_{12} & -2\sigma_2\,{\sigma_{12}}^2 & {\sigma_{12}}^3 \\ -\sigma_2\,{\sigma_{12}}^2 & \sigma_{12}\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & -\sigma_1\,{\sigma_{12}}^2 \end{bmatrix}$$

$$[R^{(\alpha)}_{24kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & J \\ -J^T & 0 \end{bmatrix} \text{ where } J = \begin{bmatrix} -\sigma_2\,(\sigma_1\sigma_2+{\sigma_{12}}^2) & \sigma_{12}\,(3\sigma_1\sigma_2+{\sigma_{12}}^2) & -2\sigma_1{\sigma_{12}}^2 \\ \sigma_{12}\,(\sigma_1\sigma_2+{\sigma_{12}}^2) & -\sigma_1\,(\sigma_1\sigma_2+3{\sigma_{12}}^2) & 2{\sigma_1}^2\sigma_{12} \end{bmatrix}$$

$$[R^{(\alpha)}_{25kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & K \\ -K^T & 0 \end{bmatrix} \text{ where } K = \begin{bmatrix} \sigma_1\,\sigma_2\,\sigma_{12} & -\sigma_1\,(\sigma_1\,\sigma_2+{\sigma_{12}}^2) & {\sigma_1}^2\,\sigma_{12} \\ -\sigma_1\,{\sigma_{12}}^2 & 2{\sigma_1}^2\,\sigma_{12} & -{\sigma_1}^3 \end{bmatrix}$$

$$[R^{(\alpha)}_{34kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & -\sigma_2\Delta & 0 & 0 & 0 \\ \sigma_2\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_2^2 & \sigma_2\,\sigma_{12} \\ 0 & 0 & \sigma_2^2 & 0 & -\sigma_1\,\sigma_2 \\ 0 & 0 & -\sigma_2\,\sigma_{12} & \sigma_1\,\sigma_2 & 0 \end{bmatrix}$$

$$[R^{(\alpha)}_{35kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & \sigma_{12}\Delta & 0 & 0 & 0 \\ -\sigma_{12}\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2\,\sigma_{12} & -\sigma_{12}^2 \\ 0 & 0 & -\sigma_2\,\sigma_{12} & 0 & \sigma_1\,\sigma_{12} \\ 0 & 0 & \sigma_{12}^2 & -\sigma_1\,\sigma_{12} & 0 \end{bmatrix}$$

$$[R^{(\alpha)}_{45kl}] = \frac{(\alpha^2-1)}{4\Delta^3}\begin{bmatrix} 0 & -\sigma_1\Delta & 0 & 0 & 0 \\ \sigma_1\Delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sigma_1\,\sigma_2 & \sigma_1\,\sigma_{12} \\ 0 & 0 & \sigma_1\,\sigma_2 & 0 & -\sigma_1^2 \\ 0 & 0 & -\sigma_1\,\sigma_{12} & \sigma_1^2 & 0 \end{bmatrix}$$

□