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Perturbation index of linear partial differential-algebraic equations with a hyperbolic part

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Abstract: This paper deals with linear partial differential-algebraic equations (PDAEs) which have a hyperbolic part. If the spatial differential operator satisfies a Gårding-type inequality in a suitable function space setting, a perturbation index can be defined. Theoretical and practical examples are considered.

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1 Introduction

Systems of differential equations are widely used to describe diverse physical phenomena in such fields as combustions, biology, chemistry, metallurgy, medicine, and fluid mechanics. The well-known Navier-Stokes system forms a representative example. Typically, these systems consist of partial differential, ordinary differential, and algebraic equations and are often called partial differential algebraic equations (PDAEs). In most cases these problems are solved numerically by the help of the vertical or the horizontal method of lines (MOL). Using the vertical method of lines, the PDAE is first semi-discretized in

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space with (conformal) finite elements. This procedure leads to a differential algebraic equation, the so-called MOL-DAE. A DAE consists of ordinary differential equations (ODEs) coupled with finite-dimensional algebraic equations. According to the survey [6], DAEs are singular implicit ODEs of the form $F(t, \mathbf{u}, \dot{\mathbf{u}}) = 0$, $t \in J$, where J is a time interval, $\dot{\mathbf{u}}$ denotes the (partial) derivative of $\mathbf{u}$ w.r.t. t, and the matrix $\frac{\partial F(t,\mathbf{u},\mathbf{v})}{\partial \mathbf{v}}$ is singular everywhere in J. Otherwise the above system leads to an implicit ODE. For a historical overview we refer to [6] and for a detailed introduction into the theory of DAEs to [7] and [3].

DAEs can be classified by means of a so-called "index" which plays a fundamental role in both theoretical and numerical investigations of such problems. It has turned out to give insight into the solution properties, as well as into the numerical difficulties to be expected when solving these problems, e.g. how to obtain consistent initial data if there are hidden constraints. To a certain extent, the DAE index is a measure of the singularity of the DAE. There are various types of indices known (see, e.g., [11, Sect. 1.2]), for example the differentiation index and the perturbation index to mention the best known indices. In [22] a comparison of both types of indices can be found which shows that the perturbation index seems to be a better measure. Of course sometimes the estimate may be too pessimistic. In [12] and, more recently, in [21] flowcharts are presented which show suggestions for the selection of numerical methods in dependence on the index of the problem. The numerical methods resulting from such a selection have good stability properties and are able to solve MOL-DAEs of index 1 and 2.

Unfortunately, a differentiation index cannot be defined for general PDAEs (see [20]). A differentiation index for special classes of PDAEs can be found in [17]. In this note we make use of the perturbation index defined in [20] which is an extension of the classical perturbation index for DAEs known from [9]. Also, in [20] a more detailed overview on related papers dealing with index concepts for PDAEs is given.

The present paper investigates linear PDAEs within the framework of weak solutions, i.e. the PDAEs are considered as abstract DAEs in suitable function spaces of Sobolev-type. The appropriate treatment of boundary conditions is obtained by the requirement that the spatial component of the differential operator has to satisfy a special inequality which is a weak form of a Gårding-type inequality. Based on this, an index concept extending the classical perturbation index is introduced and theoretical results as well as practical examples are presented.

2 The problem and its weak formulation

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$ be a domain with a Lipschitzian boundary $\partial\Omega$ and let $J := (0, \bar{t})$, $\bar{t} \in (0, \infty]$, be some time interval. In a very few examples we also will allow the case $d = 1$, where Ω reduces to an interval of the real axis. We consider the following linear system of $n \in \mathbb{N}$ partial differential, ordinary differential, and algebraic equations with respect to the unknown $\mathbf{u} = (u_1, \ldots, u_n)^\top : J \times \Omega \to \mathbb{R}^n$:

$$A\dot{\mathbf{u}} + \mathcal{L}\mathbf{u} = \mathbf{f} \qquad \text{in } J \times \Omega\,, \tag{1}$$

where $A: \Omega \to \mathbb{R}^{n,n}$, $\mathbf{f}: J \times \Omega \to \mathbb{R}^n$,

$$(\mathcal{L}\mathbf{u})_i := \sum_{j=1}^{n} \mathcal{L}_{ij} u_j, \quad i = 1, \ldots, n,$$
$$\mathcal{L}_{ij} w := -\nabla \cdot (K_{ij} \nabla w - b_{ij} w) + c_{ij} w, \qquad i, j = 1, \ldots, n.$$

Here the coefficients $K_{ij}: \Omega \to \mathbb{R}^{d,d}$, $b_{ij}: \Omega \to \mathbb{R}^d$, $c_{ij}: \Omega \to \mathbb{R}$, $\quad i, j = 1, \ldots, n$, have the properties

$$A \in L_\infty(\Omega)^{n,n}, \quad \mathbf{f} \in C(J, H^s(\Omega)^n) \quad \text{for some } s \geq 0, \tag{2}$$
$$K_{ij} = K_{ij}^\top \in W_\infty^1(\Omega)^{d,d}, \; b_{ij} \in L_\infty(\Omega)^d, \; c_{ij} \in L_\infty(\Omega), \quad i, j = 1, \ldots, n. \tag{3}$$

We do not assume that the matrix function A in (1) is regular a.e. in Ω. In such a case, the system (1) is called a partial differential-algebraic equation (PDAE).

The boundary conditions are formally formulated in a slightly different way from [20] as follows. Given piecewise continuous functions $m_{ij}, \mu_i: \partial\Omega \to \mathbb{R}$ and $u_{\Gamma i}: J \times \partial\Omega \to \mathbb{R}$, the boundary conditions read as

$$\sum_{j=1}^{n} \Big\{ \mu_i \nu \cdot (K_{ij} \nabla u_j - b_{ij} u_j) + m_{ij} u_j \Big\} + u_{\Gamma i} = 0 \qquad \text{on } J \times \partial\Omega, \quad i = 1, \ldots, n, \tag{4}$$

where ν denotes the outer unit normal. With this formulation it is possible to use the common Dirichlet and flux boundary conditions as well as the conditions described in [5].

First we set

$$\Gamma_{Ni} := \operatorname{int}(\operatorname{supp} \mu_i), \quad \Gamma_{Di} := \partial\Omega \setminus \Gamma_{Ni}, \quad \kappa_{ij} := \operatorname*{esssup}_{x \in \Omega} \|K_{ij}(x)\|_2, \quad \beta_{ij} := \operatorname*{esssup}_{x \in \Omega} \|b_{ij}(x)\|_2,$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$ or the corresponding matrix norm depending on the context, and $\operatorname{int}(\,\cdot\,)$ is the set of points which are interior as elements of a subset of the boundary $\partial\Omega$.

Furthermore, given some $\mathbf{u}_0: \Omega \to \mathbb{R}^n$, we have the implicit initial condition

$$A(\mathbf{u} - \mathbf{u}_0) = 0 \quad \text{for } x \in \Omega. \tag{5}$$

Next we define the following index sets:

$$[1, n]_{\mathbb{N}} := \{1, 2, \ldots, n\},$$
$$\mathcal{N}_E := \Big\{ i \in [1, n]_{\mathbb{N}} : \sum_{j=1}^{n} (\kappa_{ij} + \kappa_{ji}) > 0 \Big\},$$
$$\mathcal{N}_H := \Big\{ i \in [1, n]_{\mathbb{N}} \setminus \mathcal{N}_E : \sum_{j=1}^{n} \beta_{ji} > 0 \Big\},$$
$$\mathcal{N}_A := [1, n]_{\mathbb{N}} \setminus (\mathcal{N}_E \cup \mathcal{N}_H).$$

Thus we get a partition of $[1, n]_{\mathbb{N}}$ into three pairwise disjoint index sets. Without loss of generality we may assume that the indices can be arranged in such a way that

$$\max_{i \in \mathcal{N}_E} i < \min_{i \in \mathcal{N}_H} i \leq \max_{i \in \mathcal{N}_H} i < \min_{i \in \mathcal{N}_A} i. \tag{6}$$

In the following we will give a functional-analytic formulation of the PDAE (1). We set

$$l_i := \begin{cases} 1, & i \in \mathcal{N}_E \cup \mathcal{N}_H, \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

$$Y := \bigotimes_{i=1}^{n} H^{l_i}(\Omega), \qquad X := L_2(\Omega)^n.$$

The norms on Y, X are defined in the usual way, where $\mathbf{v} = (v_1, \ldots, v_n)^\top \in X$ resp. Y:

$$\|\mathbf{v}\|_X^2 := \sum_{i=1}^{n} \|v_i\|_{0,2,\Omega}^2 , \qquad \|\mathbf{v}\|_Y^2 := \sum_{i=1}^{n} \|v_i\|_{l_i,2,\Omega}^2. \tag{8}$$

The following examples illustrate these settings under the assumptions (2)–(3).

Example 2.1. Let $d := 1$, $\Omega := (0,1)$ and consider the following PDE (see [8]), where $a_{11} > 0$ and $a_{22} > 0$ a.e. in Ω and v' denotes the partial derivative of v w.r.t. x:

$$\begin{cases} a_{11}\dot{u}_1 + b_{12}u_2' + c_{11}u_1 = f_1 & \text{in} \quad J \times \Omega, \\ a_{22}\dot{u}_2 + b_{12}u_1' + c_{22}u_2 = f_2 & \text{in} \quad J \times \Omega, \\ u_1(t,0) = g_1(t) & t \in J, \\ u_2(t,1) = g_2(t) & t \in J, \\ u_1(0,x) = u_{10}(x) & x \in \Omega, \\ u_2(0,x) = u_{20}(x) & x \in \Omega. \end{cases} \tag{9}$$

Writing this problem in the form (1), (4), (5) we see that $n = 2$ and

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}, \quad (K_{ij}) = 0, \quad (b_{ij}) = \begin{pmatrix} 0 & b_{12} \\ b_{12} & 0 \end{pmatrix}, \quad (c_{ij}) = \begin{pmatrix} c_{11} & 0 \\ 0 & c_{22} \end{pmatrix},$$

$$\mu_1 = \mu_2 = 0, \quad m_{ij} = \delta_{ij} \quad \text{on } \partial\Omega = \{0,1\}, \quad u_{\Gamma i} = -g_i.$$

Then, if $\beta_{12} > 0$, we have $\mathcal{N}_E = \mathcal{N}_A = \emptyset$, $\mathcal{N}_H = \{1,2\}$, $l_1 = l_2 = 1$, $Y = H^1(\Omega)^2$, otherwise $\mathcal{N}_E = \mathcal{N}_H = \emptyset$, $\mathcal{N}_A = \{1,2\}$, $l_1 = l_2 = 0$, $Y = L_2(\Omega)^2$.

Example 2.2. Consider the following PDAE

$$\begin{cases} \dot{u}_1 + \nabla \cdot (b_{11}u_1 + b_{12}u_2) + c_{11}u_1 + c_{12}u_2 = f_1 & \text{in} \quad J \times \Omega, \\ \dot{u}_2 + \nabla \cdot (b_{12}u_1 + b_{22}u_2) + c_{21}u_1 + c_{22}u_2 = f_2 & \text{in} \quad J \times \Omega, \\ \nu \cdot (b_{11}u_1 + b_{12}u_2) = g_1 & \text{in} \quad J \times \partial\Omega, \\ \nu \cdot (b_{12}u_1 + b_{22}u_2) = g_2 & \text{in} \quad J \times \partial\Omega, \\ u_1(0,x) = u_{10}(x) & x \in \Omega, \\ u_2(0,x) = u_{20}(x) & x \in \Omega. \end{cases} \tag{10}$$

Writing this problem in the form (1), (4), (5) we see that $n = 2$ and

$$A = I, \quad (K_{ij}) = 0, \quad (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}, \quad (c_{ij}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$
$$\mu_1 = \mu_2 = 1, \quad m_{ij} = 0 \quad \text{on } \partial\Omega, \quad u_{\Gamma i} = g_i.$$

Then, if all functions b_{ij} are nontrivial, we have $\mathcal{N}_E = \mathcal{N}_A = \emptyset$, $\mathcal{N}_H = \{1,2\}$, $l_1 = l_2 = 1$, $Y = H^1(\Omega)^2$.

Example 2.3. Consider the following PDAE

$$\begin{cases} \dot{u}_1 - \nabla \cdot (K_{11}\nabla u_1 - b_{11}u_1 - b_{12}u_2) + c_{11}u_1 + c_{12}u_2 = f_1 & \text{in} \quad J \times \Omega, \\ \dot{u}_2 + \nabla \cdot (b_{12}u_1 + b_{22}u_2) + c_{21}u_1 + c_{22}u_2 = f_2 & \text{in} \quad J \times \Omega, \\ u_1 = g_1 & \text{in} \quad J \times \partial\Omega, \\ \nu \cdot (b_{12}u_1 + b_{22}u_2) = g_2 & \text{in} \quad J \times \partial\Omega, \\ u_1(0,x) = u_{10}(x) & x \in \Omega, \\ u_2(0,x) = u_{20}(x) & x \in \Omega. \end{cases} \tag{11}$$

Writing this problem in the form (1), (4), (5) we see that $n = 2$ and

$$A = I, \quad (K_{ij}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad (b_{ij}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}, \quad (c_{ij}) = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$
$$\mu_1 = 0, \quad \mu_2 = 1, \quad m_{1j} = \delta_{1j}, \quad m_{2j} = 0 \quad \text{on } \partial\Omega, \quad u_{\Gamma 1} = -g_1, \quad u_{\Gamma 2} = g_2.$$

Then, if b_{12} or b_{22} are nontrivial, we have $\mathcal{N}_E = \{1\}$, $\mathcal{N}_H = \{2\}$, $\mathcal{N}_A = \emptyset$, $l_1 = l_2 = 1$, $Y = H^1(\Omega)^2$.

More examples can be found in [20] and [23].

Now we give an abstract formulation of the problem (1), (4) and (5) under the assumptions (2) – (3) and the following symmetry condition:

$$b_{ij} = b_{ji} \qquad \forall (i,j) \in [1,n]_{\mathbb{N}}^2 \setminus \mathcal{N}_E^2 . \tag{12}$$

As usual, for $\mathbf{u}, \mathbf{v} \in Y$, we formally multiply $\mathcal{L}\mathbf{u}$ by $\mathbf{v}$, integrate the result over Ω and integrate by parts, where $(\cdot,\cdot)$ denotes the $L_2(\Omega)$- or $L_2(\Omega)^d$-inner product, and, with the subscript $\partial\Omega$ or Γ_{Ni}, the $L_2(\partial\Omega)$- or $L_2(\Gamma_{Ni})^d$-inner product:

$$\begin{aligned}
(\mathcal{L}\mathbf{u}, \mathbf{v}) &= \sum_{i,j=1}^{n} (-\nabla \cdot (K_{ij}\nabla u_j - b_{ij}u_j) + c_{ij}u_j, v_i) \\
&= \sum_{i\in\mathcal{N}_E} \sum_{j=1}^{n} \Big\{ (K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) - (\nu \cdot (K_{ij}\nabla u_j - b_{ij}u_j), v_i)_{\partial\Omega} \Big\} \\
&\quad + \sum_{i\in\mathcal{N}_H} \sum_{j=1}^{n} (\nabla \cdot (b_{ij}u_j) + c_{ij}u_j, v_i) + \sum_{i\in\mathcal{N}_A} \sum_{j=1}^{n} (c_{ij}u_j, v_i).
\end{aligned}$$

Here we have used that, by definition of the index sets, $K_{ij} = 0$ for $i \in [1,n]_{\mathbb{N}} \setminus \mathcal{N}_E$, $j \in [1,n]_{\mathbb{N}}$. Namely, a nontrivial coefficient K_{ij} for some $i \in [1,n]_{\mathbb{N}} \setminus \mathcal{N}_E$ would imply $\kappa_{ij} > 0$ and, thus, $i, j \in \mathcal{N}_E$.

A similar argument applies to the third term. A nontrivial coefficient b_{ij} for some $i \in \mathcal{N}_A$ would imply, by the symmetry assumption (12), $\beta_{ji} > 0$ and, therefore, $i \in \mathcal{N}_H$.

In the next step, we separate also with respect to the summation over j.

$$\begin{aligned}
(\mathcal{L}\mathbf{u}, \mathbf{v}) &= \sum_{i,j\in\mathcal{N}_E} \Big\{ (K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) - (\nu \cdot (K_{ij}\nabla u_j - b_{ij}u_j), v_i)_{\partial\Omega} \Big\} \\
&\quad + \sum_{i\in\mathcal{N}_E} \sum_{j\in\mathcal{N}_H} \Big\{ -(b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) + ((\nu \cdot b_{ij})u_j, v_i)_{\partial\Omega} \Big\} \\
&\quad + \sum_{i\in\mathcal{N}_E} \sum_{j\in\mathcal{N}_A} (c_{ij}u_j, v_i) \\
&\quad + \sum_{i\in\mathcal{N}_H} \sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} (\nabla \cdot (b_{ij}u_j) + c_{ij}u_j, v_i) \\
&\quad + \sum_{i\in\mathcal{N}_H} \sum_{j\in\mathcal{N}_A} (c_{ij}u_j, v_i) \\
&\quad + \sum_{i\in\mathcal{N}_A} \sum_{j=1}^{n} (c_{ij}u_j, v_i).
\end{aligned} \tag{13}$$

The coefficients K_{ij} in the second and the third lines disappear by the definition of the index sets (this is the same argument as above). The coefficients b_{ij} in the third and the fifth lines disappear by the definition of the index set $\mathcal{N}_H$ (but not as a consequence of the symmetry assumption (12)).

We introduce the following function spaces:

$$V_i := \begin{cases} \{v \in H^1(\Omega) : v|_{\Gamma_{Di}} = 0\}, & i \in \mathcal{N}_E, \\ H^1(\Omega), & i \in \mathcal{N}_H, \\ L_2(\Omega), & i \in \mathcal{N}_A, \end{cases} \qquad V := \bigotimes_{i=1}^{n} V_i(\Omega).$$

The norm on V is defined by restricting the norm of Y, cf. (8).

Using these definitions and integrating by parts in the fourth line of (13), we get

$$\begin{aligned}(\mathcal{L}\mathbf{u}, \mathbf{v}) = &\sum_{i,j\in\mathcal{N}_E} \Big\{(K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) - (\nu \cdot (K_{ij}\nabla u_j - b_{ij}u_j), v_i)_{\Gamma_{Ni}}\Big\} \\ &+ \sum_{i\in\mathcal{N}_E}\sum_{j\in\mathcal{N}_H} \Big\{ - (b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) + ((\nu \cdot b_{ij})u_j, v_i)_{\Gamma_{Ni}}\Big\} \\ &+ \sum_{i\in\mathcal{N}_H}\sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} \Big\{ - (b_{ij}u_j, \nabla v_i) + c_{ij}u_j, v_i) + ((\nu \cdot b_{ij})u_j, v_i)_{\partial\Omega}\Big\} \\ &+ \sum_{i\in\mathcal{N}_E\cup\mathcal{N}_H}\sum_{j\in\mathcal{N}_A}(c_{ij}u_j, v_i) + \sum_{i\in\mathcal{N}_A}\sum_{j=1}^{n}(c_{ij}u_j, v_i).\end{aligned}$$

If we take into consideration the definition of the index sets, the boundary conditions (4) read as follows:

$$\sum_{j\in\mathcal{N}_E} \Big\{\mu_i\nu \cdot (K_{ij}\nabla u_j - b_{ij}u_j) + m_{ij}u_j\Big\} + \sum_{j\in\mathcal{N}_H}(m_{ij} - \mu_i(\nu \cdot b_{ij}))u_j$$

$$+ \sum_{j\in\mathcal{N}_A} m_{ij}u_j + u_{\Gamma i} = 0 \quad \text{if} \quad i \in \mathcal{N}_E,$$

$$\sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H}(m_{ij} - \mu_i(\nu \cdot b_{ij}))u_j + \sum_{j\in\mathcal{N}_A} m_{ij}u_j + u_{\Gamma i} = 0 \quad \text{if} \quad i \in \mathcal{N}_H,$$

$$\sum_{j=1}^{n} m_{ij}u_j + u_{\Gamma i} = 0 \quad \text{if} \quad i \in \mathcal{N}_A.$$

We note that μ_i is piecewise continuous by assumption and, therefore, Γ_{Ni} is either empty or has a positive boundary measure $|\Gamma_{Ni}|$.

Now we use the following assumptions w.r.t. the boundary data:

$$\begin{aligned} \mu_i^{-1} &\in L_\infty(\Gamma_{Ni}), && \forall i \in \mathcal{N}_E \text{ s.t. } |\Gamma_{Ni}| > 0, \\ \mu_i &= 1, && \forall i \in \mathcal{N}_H, \\ u_{\Gamma_i} &= 0, && \forall i \in \mathcal{N}_A, \\ m_{ij} &= m_{ji}, && \forall (i,j) \in [1,n]^2_{\mathbb{N}} \setminus \mathcal{N}_H^2\,, \\ m_{ij} &= 0, && \forall j \in \mathcal{N}_E \setminus \{i\},\ \forall j \in \mathcal{N}_A. \end{aligned} \tag{14}$$

Then the boundary conditions for $i \in \mathcal{N}_A$ are satisfied identically, and the remaining conditions get the form

$$\sum_{j\in\mathcal{N}_E} \mu_i \nu \cdot (K_{ij}\nabla u_j - b_{ij}u_j) - \mu_i \sum_{j\in\mathcal{N}_H} (\nu \cdot b_{ij})u_j + m_{ii}u_i + u_{\Gamma i} = 0 \quad \text{if} \quad i \in \mathcal{N}_E,$$

$$\sum_{j\in\mathcal{N}_H} m_{ij}u_j - \sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} (\nu \cdot b_{ij})u_j = 0 \quad \text{if} \quad i \in \mathcal{N}_H.$$

This formally implies that

$$\begin{aligned}
(\mathcal{L}\mathbf{u}, \mathbf{v}) = & \sum_{i,j\in\mathcal{N}_E} \Big\{ (K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) \Big\} \\
& + \sum_{i\in\mathcal{N}_E} \sum_{j\in\mathcal{N}_H} \Big\{ -(b_{ij}u_j, \nabla v_i) + (c_{ij}u_j, v_i) \Big\} \\
& + \sum_{i\in\mathcal{N}_E} \left(\frac{m_{ii}}{\mu_i} u_i, v_i \right)_{\Gamma_{Ni}} + \sum_{i\in\mathcal{N}_E} \left(\frac{u_{\Gamma_i}}{\mu_i}, v_i \right)_{\Gamma_{Ni}} \\
& + \sum_{i\in\mathcal{N}_H} \sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} \Big\{ -(b_{ij}u_j, \nabla v_i) + c_{ij}u_j, v_i) \Big\} \\
& + \sum_{i,j\in\mathcal{N}_H} (m_{ij}u_j, v_i)_{\partial\Omega} \\
& + \sum_{i\in\mathcal{N}_E\cup\mathcal{N}_H} \sum_{j\in\mathcal{N}_A} (c_{ij}u_j, v_i) + \sum_{i\in\mathcal{N}_A} \sum_{j=1}^{n} (c_{ij}u_j, v_i) \\
= & \sum_{i,j\in\mathcal{N}_E} (K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i) \\
& - \sum_{i\in\mathcal{N}_E} \sum_{j\in\mathcal{N}_H} (b_{ij}u_j, \nabla v_i) - \sum_{i\in\mathcal{N}_H} \sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}u_j, \nabla v_i) \\
& + \sum_{i,j=1}^{n} (c_{ij}u_j, v_i) \\
& + \sum_{i\in\mathcal{N}_E} \left(\frac{m_{ii}}{\mu_i} u_i, v_i \right)_{\Gamma_{Ni}} + \sum_{i,j\in\mathcal{N}_H} (m_{ij}u_j, v_i)_{\partial\Omega} \\
& + \sum_{i\in\mathcal{N}_E} \left(\frac{u_{\Gamma_i}}{\mu_i}, v_i \right)_{\Gamma_{Ni}}.
\end{aligned}$$

Now we can introduce the following linear operators $\mathcal{A}, \mathcal{B} : Y \to V^*$ and right-hand sides

$\mathbf{f}_\Omega : J \to X$, $\mathbf{f}_N : J \to V^*$:

$$
\begin{aligned}
\langle \mathcal{A}\mathbf{u}, \mathbf{v}\rangle &:= \sum_{i,j=1}^{n} (a_{ij}u_j, v_i),\\
\langle \mathcal{B}\mathbf{u}, \mathbf{v}\rangle &:= \sum_{i,j\in\mathcal{N}_E} (K_{ij}\nabla u_j - b_{ij}u_j, \nabla v_i)\\
&\quad - \sum_{i\in\mathcal{N}_E}\sum_{j\in\mathcal{N}_H} (b_{ij}u_j, \nabla v_i) - \sum_{i\in\mathcal{N}_H}\sum_{j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}u_j, \nabla v_i)\\
&\quad + \sum_{i,j=1}^{n} (c_{ij}u_j, v_i)\\
&\quad + \sum_{i\in\mathcal{N}_E} \left(\frac{m_{ii}}{\mu_i}u_i, v_i\right)_{\Gamma_{Ni}} + \sum_{i,j\in\mathcal{N}_H} (m_{ij}u_j, v_i)_{\partial\Omega},\\
\langle \mathbf{f}_\Omega, \mathbf{v}\rangle &:= \sum_{i=1}^{n} (f_i, v_i), \qquad \langle \mathbf{f}_N, \mathbf{v}\rangle := \sum_{i\in\mathcal{N}_E} \left(\frac{u_{\Gamma_i}}{\mu_i}, v_i\right)_{\Gamma_{Ni}}.
\end{aligned}
\tag{15}
$$

With this, we get the following operator equation in V^* w.r.t. the unknown element $\mathbf{u} : J \to Y$:

$$\mathcal{A}\dot{\mathbf{u}} + \mathcal{B}\mathbf{u} = \mathbf{f}_\Omega + \mathbf{f}_N\,. \tag{16}$$

Given some $\mathbf{u}_0 \in Y$, the initial condition reads as

$$\mathcal{A}(\mathbf{u} - \mathbf{u}_0) = 0\,. \tag{17}$$

Equation (16) is called an abstract DAE (ADAE). In order to be able to include inhomogeneous Dirichlet boundary conditions, we assume that there exists some abstract function $\mathbf{u}_D : J \to Y$ with $u_{Dj} = u_{\Gamma j}$ on Γ_{Dj} for $j \in [1, n]_{\mathbb{N}}$. Using the representation $\mathbf{u} = \mathbf{u}_{hom} + \mathbf{u}_D$, where $\mathbf{u}_{hom} : J \to V$, and introducing the right-hand sides $\mathbf{f}_D, \mathbf{f} : J \to V^*$ by

$$\mathbf{f}_D := -\mathcal{A}\dot{\mathbf{u}}_D - \mathcal{B}\mathbf{u}_D \quad \text{and} \quad \mathbf{f} := \mathbf{f}_\Omega + \mathbf{f}_D + \mathbf{f}_N, \tag{18}$$

we get the following operator equation in V^* w.r.t. the unknown element $\mathbf{u}_{hom} : J \to V$:

$$\mathcal{A}\dot{\mathbf{u}}_{hom} + \mathcal{B}\mathbf{u}_{hom} = \mathbf{f}\,. \tag{19}$$

If there are no Dirichlet boundary conditions at all, then we formally set $\mathbf{u}_D = 0$.

In [20] and [19] we have seen that the perturbation index can be determined by the help of a Gårding-type inequality, i.e. there exist two constants $\lambda \geq 0$, $c > 0$ such that

$$\forall \mathbf{v} \in V : \ \langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle + \lambda\|\mathbf{v}\|_X^2 \geq c\|\mathbf{v}\|_V^2. \tag{20}$$

Sufficient conditions under which the operator $\mathcal{B}$ satisfies a Gårding-type inequality can be found in [20]. Unfortunately, (20) is not satisfied for problems with a hyperbolic part as for example (9), (10), and (11). In this case the operator $\mathcal{B}$ should satisfy an estimate of the form

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle + \lambda\|\mathbf{v}\|_X^2 \geq c\sum_{i\in\mathcal{N}_E} \|\nabla v_i\|_{0,2,\Omega}^2. \tag{21}$$

To give a short formulation of the corresponding result, we define the following matrices $B, M : \Omega \to \mathbb{R}^{n,n}$ by

$$B := \frac{1}{2}\,(\nu \cdot b_{ij})_{i,j=1}^{n}\,, \qquad M := (m_{ij})_{i,j=1}^{n}\,.$$

By assumption (6), B and M have the following block structure:

$$B = \begin{pmatrix} B_{EE} & B_{EH} & B_{EA} \\ B_{HE} & B_{HH} & B_{HA} \\ B_{AE} & B_{AH} & B_{AA} \end{pmatrix}, \qquad M = \begin{pmatrix} M_{EE} & M_{EH} & M_{EA} \\ M_{HE} & M_{HH} & M_{HA} \\ M_{AE} & M_{AH} & M_{AA} \end{pmatrix}.$$

Taking into consideration the definition of the index sets and assumption (14), this structure simplifies to

$$B = \begin{pmatrix} B_{EE} & B_{EH} & 0 \\ B_{HE} & B_{HH} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} M_{EE} & 0 & 0 \\ 0 & M_{HH} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where, thanks to (12) and (14), again,

$$B_{HE} = B_{EH}^{\top}, \quad B_{HH} = B_{HH}^{\top},$$

and M_{EE} is a diagonal matrix. Finally, we define a diagonal matrix $D_E = \operatorname{diag}(d_i)$, $i \in \mathcal{N}_E$, by

$$d_i := \begin{cases} \mu_i^{-1}, & |\Gamma_{Ni}| > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.4. *Let there exist constants $\underline{\kappa}_{ii} > 0$, $i \in \mathcal{N}_E$, such that*

$$\xi \cdot (K_{ii}(x)\xi) \geq \underline{\kappa}_{ii} \|\xi\|_2^2, \qquad \forall \xi \in \mathbb{R}^d, \quad \forall x \in \Omega.$$

Let $b_{ij} \in W_\infty^1(\Omega)^d$ ($i, j \in [1, n]_{\mathbb{N}}$) be such that the symmetry condition (12) is satisfied but for all indices, i.e.

$$b_{ij} = b_{ji} \qquad \forall i, j \in [1, n]_{\mathbb{N}}, \tag{22}$$

and let the symmetric part of the matrix

$$\begin{pmatrix} D_E M_{EE} & 0 \\ 0 & M_{HH} \end{pmatrix} - \begin{pmatrix} B_{EE} & B_{EH} \\ B_{HE} & B_{HH} \end{pmatrix}$$

be positive semidefinite. Finally, let the entries of a matrix κ be given by

$$\kappa_{ij} := \begin{cases} \underline{\kappa}_{ii}, & i = j, \\ -\operatorname*{esssup}_{x \in \Omega} \|K_{ij}(x)\|_2, & i \neq j, \end{cases} \qquad i, j \in \mathcal{N}_E,$$

and let $\kappa_{sym} := \frac{\kappa+\kappa^\top}{2}$ *be positive definite. Then the inequality* (21) *is satisfied for some positive* c *and* λ*.*

Proof. Since $\nabla(v_i v_j) = v_i \nabla v_j + v_j \nabla v_i$, we can write

$$\sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_j, \nabla v_i) = \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}, \nabla(v_i v_j)) - \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_i, \nabla v_j).$$

From the symmetry assumption (22) we see that

$$\sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_i, \nabla v_j) = \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ji}v_i, \nabla v_j) = \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_j, \nabla v_i),$$

consequently,

$$\begin{aligned}\sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_j, \nabla v_i) &= \frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}, \nabla(v_i v_j)) \\ &= -\frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} ((\nabla\cdot b_{ij})v_j, v_i)) + \frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} ((\nu\cdot b_{ij})v_j, v_i))_{\partial\Omega}.\end{aligned}$$

Thus we have, by assumption,

$$\begin{aligned}\langle\mathcal{B}\mathbf{v},\mathbf{v}\rangle &= \sum_{i,j\in\mathcal{N}_E} (K_{ij}\nabla v_j, \nabla v_i) - \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} (b_{ij}v_j, \nabla v_i) + \sum_{i,j=1}^{n} (c_{ij}v_j, v_i) \\ &\quad + \sum_{i\in\mathcal{N}_E} \left(\frac{m_{ii}}{\mu_i} v_i, v_i\right)_{\Gamma_{Ni}} + \sum_{i,j\in\mathcal{N}_H} (m_{ij}v_j, v_i)_{\partial\Omega} \\ &= \sum_{i,j\in\mathcal{N}_E} (K_{ij}\nabla v_j, \nabla v_i) + \frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} ((\nabla\cdot b_{ij})v_j, v_i)) + \sum_{i,j=1}^{n} (c_{ij}v_j, v_i) \\ &\quad + \sum_{i\in\mathcal{N}_E} \left(\frac{m_{ii}}{\mu_i} v_i, v_i\right)_{\Gamma_{Ni}} + \sum_{i,j\in\mathcal{N}_H} (m_{ij}v_j, v_i)_{\partial\Omega} - \frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} ((\nu\cdot b_{ij})v_j, v_i))_{\partial\Omega} \\ &\geq \sum_{i,j\in\mathcal{N}_E} (K_{ij}\nabla v_j, \nabla v_i) + \frac{1}{2} \sum_{i,j\in\mathcal{N}_E\cup\mathcal{N}_H} ((\nabla\cdot b_{ij})v_j, v_i)) + \sum_{i,j=1}^{n} (c_{ij}v_j, v_i).\end{aligned}$$

With $\gamma_{ij} := \frac{1}{2} \operatorname*{ess\,sup}_{x\in\Omega} (\nabla\cdot b_{ij} + 2c_{ij})$ we get

$$\begin{aligned}\langle\mathcal{B}\mathbf{v},\mathbf{v}\rangle &\geq \sum_{i\in\mathcal{N}_E} \underline{\kappa}_{ii}\|\nabla v_i\|^2 - \sum_{\substack{i,j\in\mathcal{N}_E\\ i\neq j}} \kappa_{ij}\|\nabla v_j\|\|\nabla v_i\| - \sum_{i,j=1}^{n} \gamma_{ij}\|v_i\|\|v_j\| \\ &\geq \sum_{i\in\mathcal{N}_E} \underline{\kappa}_{ii}\|\nabla v_i\|^2 - \sum_{\substack{i,j\in\mathcal{N}_E\\ i\neq j}} \kappa_{ij}\|\nabla v_i\|\|\nabla v_j\| - \lambda\|\mathbf{v}\|_X^2,\end{aligned}$$

where λ is the spectral norm of the matrix $\gamma := (\gamma_{ij})_{i,j=1}^n$. If c denotes the spectral norm of the matrix κ_{sym}, it follows

$$\langle\mathcal{B}\mathbf{v},\mathbf{v}\rangle \geq c \sum_{i\in\mathcal{N}_E} \|\nabla v_i\|^2 - \lambda\|\mathbf{v}\|_X^2$$

and the Lemma is proven. □

Example 2.5. We consider the PDAE (9) of Example 2.1. The inequality (21) is satisfied with $\lambda = \max\{\gamma_{11}, \gamma_{22}\}$ and arbitrary $c > 0$ (note that $\mathcal{N}_E = \emptyset$).

Example 2.6. We consider the PDAE (10) of Example 2.2. The inequality (21) is satisfied with $\lambda = \left\| \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \right\|_2$ and arbitrary $c > 0$.

Example 2.7. We consider the PDAE (11) of Example 2.3. The inequality (21) is satisfied with $c = \underline{\kappa}_{11}$ and $\lambda = \left\| \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \right\|_2$.

More examples can be found in [20] or [23].

3 The perturbation index

In this section we introduce an extension of the perturbation index, which is known from the theory of DAEs, to the case of ADAEs and PDAEs. In particular, it turns out that the introduced perturbation index coincides with the classical notion in the case of DAEs.

Let $\mathbf{u}$ be a weak solution of the ADAE (16) which is consistent with the initial value $\mathbf{u}_0 \in Y$, i.e. $\mathcal{A}(\mathbf{u} - \mathbf{u}_0) = 0$.

The notion of the perturbation index is based on the investigation of the sensitivity of this solution with respect to initial values, boundary values and right-hand sides.

Starting from the ADAE written in the form (19), we introduce perturbations $\boldsymbol{\delta}_\Omega : J \to X$ (of the structure (15) of the right-hand side), $\boldsymbol{\delta}_N : J \to V^*$ (of the structure (15) of the Neumann-type boundary conditions), and $\boldsymbol{\delta}_D : J \to V^*$ (of the structure (18) of the Dirichlet-type boundary conditions) and look for a solution $\hat{\mathbf{u}}_{hom} : J \to V$ of the equation

$$\mathcal{A}\dot{\hat{\mathbf{u}}}_{hom} + \mathcal{B}\hat{\mathbf{u}}_{hom} = \mathbf{f}_\Omega + \mathbf{f}_D + \mathbf{f}_N + \boldsymbol{\delta}_\Omega + \boldsymbol{\delta}_D + \boldsymbol{\delta}_N. \tag{23}$$

Subtracting (23) from (19) leads to the so-called *homogenized error equation* with respect to $\boldsymbol{\varepsilon}_{hom} := \hat{\mathbf{u}}_{hom} - \mathbf{u}_{hom} : J \to V$

$$\mathcal{A}\dot{\boldsymbol{\varepsilon}}_{hom} + \mathcal{B}\boldsymbol{\varepsilon}_{hom} = \boldsymbol{\delta}_\Omega + \boldsymbol{\delta}_D + \boldsymbol{\delta}_N =: \boldsymbol{\delta}\,. \tag{24}$$

Now we can define the perturbation index of an ADAE.

Definition 3.1. Let $\mathcal{F}$ be a family of right-hand sides such that, for any $\mathbf{f} \in \mathcal{F}$, the ADAE (16) has only one weak solution. Then the ADAE (16) has the perturbation index i_p along the solution $\mathbf{u}$ on J, if i_p is the smallest integer such that, for all $\hat{\mathbf{u}}$ having defects

$\boldsymbol{\delta}_\Omega : J \to X$ and $\boldsymbol{\delta}_D, \boldsymbol{\delta}_N : J \to V^*$, i.e.

$$\mathcal{A}\dot{\hat{\mathbf{u}}}_{hom} + \mathcal{B}\hat{\mathbf{u}}_{hom} = \mathbf{f}_\Omega + \mathbf{f}_D + \mathbf{f}_N + \boldsymbol{\delta}_\Omega + \boldsymbol{\delta}_D + \boldsymbol{\delta}_N,$$

there is on J an estimate of the form

$$\|\hat{\mathbf{u}}_{hom}(t) - \mathbf{u}_{hom}(t)\|_X \le C \left(\|\hat{\mathbf{u}}_{hom}(0) - \mathbf{u}_{hom}(0)\|_X + \sum_{j=0}^{\mathrm{i}_p - 1} \sup_{\tau \in J} \left\| \frac{\partial^j \boldsymbol{\delta}(\tau)}{(\partial \tau)^j} \right\|_* \right), \tag{25}$$

where

$$\|\boldsymbol{\delta}\|_* := \|\boldsymbol{\delta}_\Omega\|_X + \|\boldsymbol{\delta}_D\|_{V^*} + \|\boldsymbol{\delta}_N\|_{V^*}.$$

Here the constant C may depend only on $\mathcal{A}, \mathcal{B}, \mathbf{f}$ and the length $\bar{t}$ of J.

Remark 3.2. (i) In the definition it is implicitly assumed that equation (23) is solvable in J for the perturbations $\boldsymbol{\delta}$ under consideration.

(ii) Recall that the norms of the spaces V_i, V, V_i^* and V^* are defined as

$$\begin{aligned}
\|v_i\|_{V_i}^2 &:= \|v_i\|_{l_i,2,\Omega}^2 := \sum_{|\alpha| \le l_i} \|\partial^\alpha v_i\|_{0,2,\Omega}^2 , \qquad v_i \in V_i \\
\|\mathbf{v}\|_V^2 &:= \sum_{i=1}^n \|v_i\|_{V_i}^2 := \sum_{i=1}^n \sum_{|\alpha| \le l_i} \|\partial^\alpha v_i\|_{0,2,\Omega}^2 , \qquad \mathbf{v} \in V \\
\|\delta_i\|_{V_i^*} &:= \sup_{v_i \in V_i \setminus \{0\}} \frac{|\langle \delta_i, v_i \rangle|}{\|v_i\|_{V_i}}, \qquad \delta_i \in V_i^* \\
\|\boldsymbol{\delta}\|_{V^*} &:= \sum_{i=1}^n \|\delta_i\|_{V_i^*}, \qquad \boldsymbol{\delta} \in V^*.
\end{aligned}$$

(iii) Concerning the problem of existence and uniqueness of weak solutions, we refer to the literature, e.g. [4] and [24].

4 Hyperbolic PDAEs

As a first application of the above theory, we investigate the linear hyperbolic PDE

$$\sum_{j=1}^n (\nabla \cdot (b_{ij} u_j) + c_{ij} u_j) = f_i \quad \text{in } \Omega, \quad i \in [1, n]_{\mathbb{N}}, \tag{26}$$

with the data $b_{ij} \in W_\infty^1(\Omega)^d$, $c_{ij} \in L_\infty(\Omega)$, $i, j \in [1, n]_{\mathbb{N}}$, and $\mathbf{f} \in C(J, L_2(\Omega)^n)$. Moreover we assume that the PDE (26) is symmetric, i.e. we have $b_{ij} = b_{ji}$, $i, j \in [1, n]_{\mathbb{N}}$ (cf. (22)). The boundary conditions are

$$\sum_{j=1}^n ((\nu \cdot b_{ij}) u_j - m_{ij} u_j) = 0 \quad \text{on } \partial\Omega, \quad i \in [1, n]_{\mathbb{N}}, \tag{27}$$

where, as above, $B := \frac{1}{2}\left(\nu \cdot b_{ij}\right)_{i,j=1}^{n}$, $M := \left(m_{ij}\right)_{i,j=1}^{n}$ and (cf. [5, Sect. 5])

$$\begin{aligned} &M \text{ is continuous on } \partial\Omega, \\ &\boldsymbol{\xi} \cdot [(M + M^{\top})\boldsymbol{\xi}] \geq 0 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ on } \partial\Omega, \\ &\ker\,(2B - M) \oplus \ker\,(2B + M) = \mathbb{R}^n \quad \text{on } \partial\Omega. \end{aligned} \tag{28}$$

Theorem 4.1. *Consider the symmetric hyperbolic PDE (26) with the boundary conditions (27) under the assumptions (28). Assume that the weak problem*

$$\langle \mathcal{B}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f}_\Omega, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V := H^1(\Omega)^n \tag{29}$$

has a unique solution. If there exists a constant $C > 0$ such that

$$\sum_{i,j=1}^{n} \Big(\frac{1}{2}\nabla \cdot b_{ij} + c_{ij}\Big)\boldsymbol{\xi}_i\boldsymbol{\xi}_j \geq C\|\boldsymbol{\xi}\|_2^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^n \text{ a.e. in } \Omega, \tag{30}$$

then the problem (29) has the perturbation index $\mathfrak{i}_p = 1$.

Proof. To keep the notation simple we set $\mathbf{w} := \boldsymbol{\varepsilon}_{hom}$ and consider the weak problem

$$\langle \mathcal{B}\mathbf{w}, \mathbf{v} \rangle = \langle \boldsymbol{\delta}_\Omega, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V. \tag{31}$$

From the proof of Lemma 2.4 we know that

$$\begin{aligned} \langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle = \; & \frac{1}{2}\sum_{i,j=1}^{n}((\nabla \cdot b_{ij})w_j, w_i)) + \sum_{i,j=1}^{n}(c_{ij}w_j, w_i) \\ & + \sum_{i,j=1}^{n}(m_{ij}w_j, w_i)_{\partial\Omega} - \frac{1}{2}\sum_{i,j=1}^{n}((\nu \cdot b_{ij})w_j, w_i))_{\partial\Omega}. \end{aligned} \tag{32}$$

Using the boundary condition (27), we get

$$\langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle = \sum_{i,j=1}^{n}\Big(\Big(\frac{1}{2}\nabla \cdot b_{ij} + c_{ij}\Big)w_j, w_i\Big) + \frac{1}{2}\sum_{i,j=1}^{n}(m_{ij}w_j, w_i)_{\partial\Omega}.$$

Then it follows, by the semidefiniteness of $M + M^{\top}$ (see (28)) and the condition (30), that

$$\langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle \geq C\sum_{i=1}^{n}(w_i, w_i) = C\|\mathbf{w}\|_X^2.$$

The right-hand side of (31) with $\mathbf{v} = \mathbf{w}$ can be estimated by the Cauchy-Schwarz inequality: $\langle \boldsymbol{\delta}_\Omega, \mathbf{w} \rangle \leq \|\boldsymbol{\delta}_\Omega\|_X\|\mathbf{w}\|_X$. Thus we arrive at

$$\|\mathbf{w}\|_X \leq \frac{1}{C}\|\boldsymbol{\delta}_\Omega\|_X.$$

Hence the problem has the perturbation index $\mathfrak{i}_p = 1$. □

Remark 4.2. Instead of using the boundary condition (27) under the assumptions (28) it is sufficient to suppose that the symmetric part of $M_{HH} - B_{HH}$ is positive definite. Then we also have, by (32) and (30) that

$$\langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle \geq C\|\mathbf{w}\|_X^2.$$

Details about the solvability of the strong and the weak problems can be found in [5] and [14].

Theorem 4.3. *Consider the symmetric hyperbolic PDE*

$$\begin{cases} \dot{u}_i + \displaystyle\sum_{j=1}^{n} (\nabla \cdot (b_{ij} u_j) + c_{ij} u_j) = f_i & \text{in } J \times \Omega, \\ \qquad\qquad u_i(0, x) = u_{i0} & \text{in } \Omega, \end{cases} \qquad i \in [1, n]_{\mathbb{N}}\,, \tag{33}$$

with the boundary conditions (27) *under the assumptions (28). If the weak problem*

$$(\dot{\mathbf{u}}, \mathbf{v}) + \langle \mathcal{B}\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V := H^1(\Omega)^n \tag{34}$$

is uniquely solvable, it has the perturbation index $\mathfrak{i}_p = 1$.

Proof. In the proof of Theorem 4.1 we have seen that

$$\langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle \geq \sum_{i,j=1}^{n} \Big(\Big(\frac{1}{2} \nabla \cdot b_{ij} + c_{ij} \Big) w_j, w_i \Big).$$

Setting $\gamma := (\gamma_{ij})_{i,j=1}^n$ with $\gamma_{ij} := \frac{1}{2} \operatorname{esssup}(2c_{ij} + \nabla \cdot b_{ij})$, we get

$$(\dot{\mathbf{w}}, \mathbf{w}) - \lambda \|\mathbf{w}\|_X^2 \leq (\dot{\mathbf{w}}, \mathbf{w}) + \langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle = \langle \boldsymbol{\delta}_\Omega, \mathbf{w} \rangle,$$

where λ is the spectral-norm of the matrix γ.

Because of

$$\langle \boldsymbol{\delta}_\Omega, \mathbf{w} \rangle \leq \frac{1}{2}\|\mathbf{w}\|_X^2 + \frac{1}{2}\|\boldsymbol{\delta}_\Omega\|_X^2$$

we obtain

$$(\dot{\mathbf{w}}, \mathbf{w}) - \lambda \|\mathbf{w}\|_X^2 \leq \frac{1}{2}\|\mathbf{w}\|_X^2 + \frac{1}{2}\|\boldsymbol{\delta}_\Omega\|_X^2$$

and, with $\mu := 2\lambda + 1$,

$$2(\dot{\mathbf{w}}, \mathbf{w}) - \mu \|\mathbf{w}\|_X^2 \leq \|\boldsymbol{\delta}_\Omega\|_X^2.$$

Using the relation

$$(\dot{\mathbf{w}}, \mathbf{w}) = \frac{1}{2}\frac{d}{dt}(\mathbf{w}, \mathbf{w}) = \frac{1}{2}\frac{d}{dt}\|\mathbf{w}\|_X^2,$$

we get the following estimate:

$$\frac{d}{dt}\|\mathbf{w}\|_X^2 - \mu \|\mathbf{w}\|_X^2 \leq \|\boldsymbol{\delta}_\Omega\|_X^2.$$

From

$$\frac{d}{dt}\left(e^{-\mu t}\|\mathbf{w}\|_X^2\right) = e^{-\mu t}\frac{d}{dt}\|\mathbf{w}\|_X^2 - \mu e^{-\mu t}\|\mathbf{w}\|_X^2$$

we obtain

$$\frac{d}{dt}\left(e^{-\mu t}\|\mathbf{w}\|_X^2\right) \leq e^{-\mu t}\|\boldsymbol{\delta}_\Omega\|_X^2.$$

Integration yields

$$e^{-\mu t}\|\mathbf{w}\|_X^2 - \|\mathbf{w}_0\|_X^2 \leq \int_0^t e^{-\mu s}\|\boldsymbol{\delta}_\Omega\|_X^2\, ds$$

and

$$\begin{aligned}
\|\mathbf{w}\|_X^2 &\leq e^{\mu t}\|\mathbf{w}_0\|_X^2 + \int_0^t e^{\mu(t-s)}\|\boldsymbol{\delta}_\Omega\|_X^2\, ds \\
&\leq e^{\mu t}\|\mathbf{w}_0\|_X^2 + \sup_{t\in J}\|\boldsymbol{\delta}_\Omega(t)\|_X^2 \int_0^t e^{\mu(t-s)}\, ds \\
&\leq e^{\mu t}\|\mathbf{w}_0\|_X^2 + \frac{e^{\mu t}-1}{\mu}\sup_{t\in J}\|\boldsymbol{\delta}_\Omega(t)\|_X^2 \\
&\leq \max\left\{e^{\mu t}, \frac{e^{\mu t}-1}{\mu}\right\}\left(\|\mathbf{w}_0\|_X^2 + \sup_{t\in J}\|\boldsymbol{\delta}_\Omega(t)\|_X^2\right).
\end{aligned}$$

Finally we have

$$\|\mathbf{w}\|_X \leq \sqrt{\max\left\{e^{\mu t}, \frac{e^{\mu t}-1}{\mu}\right\}}\left(\|\mathbf{w}_0\|_X + \sup_{t\in J}\|\boldsymbol{\delta}_\Omega(t)\|_X\right)$$

and the problem has the perturbation index 1. □

Example 4.4. Let $d := 1$, $\Omega := (0, 1)$ and $b > 0$ be a given constant. The scalar hyperbolic problem

$$\begin{cases}
\dot{u} + (bu)' = f(t, x) & (t, x) \in J \times \Omega, \\
u(t, 0) = 0 & t \in J, \\
u(0, x) = u_0(x) & x \in \Omega,
\end{cases}$$

has the perturbation index $\mathfrak{i}_p = 1$, since the choice $m := b$ leads to the equivalent boundary conditions

$$(-b - m(0))u(t, 0) = 0, \quad (b - m(1))u(t, 1) = 0$$

which have the form (27) and satisfy (28).

Example 4.5. Consider the PDE (9) from Example 2.1 with the boundary conditions

$$u_1(t, 0) + u_2(t, 0) = 0, \quad u_1(t, 1) - u_2(t, 1) = 0.$$

If $b_{12}(0) > 0$, $b_{12}(1) > 0$, then the problem has the perturbation index $\mathfrak{i}_p = 1$.

First of all we observe that the result of Theorem 4.3 can be easily extended to the case where the matrix A is symmetric positive definite. Since $a_{11} > 0$ and $a_{22} > 0$ for all $x \in \Omega$, this condition is satisfied. Moreover, with the choice $m_{11} := m_{22} := b_{12}$ we get the equivalent boundary conditions

$$\begin{pmatrix} -m_{11}(0) & -b_{12}(0) \\ -b_{12}(0) & -m_{22}(0) \end{pmatrix} \begin{pmatrix} u_1(t,0) \\ u_2(t,0) \end{pmatrix} = 0, \qquad \begin{pmatrix} -m_{11}(1) & b_{12}(1) \\ b_{12}(1) & -m_{22}(1) \end{pmatrix} \begin{pmatrix} u_1(t,1) \\ u_2(t,1) \end{pmatrix} = 0$$

which have the form (27) and satisfy (28).

This result coincides with the estimate given in [8, Example 8].

Example 4.6. The PDE (10) from Example 2.2 with the boundary conditions (27) satisfying (28) has the perturbation index $\mathfrak{i}_p = 1$.

5 Mixed hyperbolic-parabolic PDAEs

In this section systems are considered which have a parabolic and a hyperbolic part. Many physical phenomena can be described by the help of such systems. One representative example is the compressible Navier-Stokes system which will be treated in Section 6.2.

We start with the situation where $\mathcal{N}_E = [1, n_1]_{\mathbb{N}}$, $\mathcal{N}_H = [n_1+1, n]_{\mathbb{N}}$ for some $n_1, n \in \mathbb{N}$, $n_1 < n$.

Theorem 5.1. *Consider the PDAE*

$$\begin{aligned} \dot{u}_i - \sum_{j=1}^{n_1} \nabla \cdot (K_{ij} \nabla u_j) + \sum_{j=1}^{n} (\nabla \cdot (b_{ij} u_j) + c_{ij} u_j) &= f_i, \quad i = 1, \ldots, n_1, \\ \dot{u}_i + \sum_{j=1}^{n} (\nabla \cdot (b_{ij} u_j) + c_{ij} u_j) &= f_i, \quad i = n_1 + 1, \ldots, n \end{aligned}$$

under the assumptions of Lemma 2.4. The initial condition reads as $\mathbf{u}(0,x) = \mathbf{u}_0(x)$. *If the corresponding weak problem is uniquely solvable and if* $\boldsymbol{\delta}$ *admits an estimate of the type*

$$|\langle \boldsymbol{\delta}, \mathbf{v} \rangle| \leq \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla v_i\|^2 + \|\mathbf{v}\|_X^2 \right\}^{1/2} \qquad \forall \mathbf{v} \in V, \tag{35}$$

then the PDAE has the perturbation index $\mathfrak{i}_p = 1$.

Proof. The error equation of the weak problem with $\mathbf{v} = \mathbf{w}$ reads as

$$(\dot{\mathbf{w}}, \mathbf{w}) + \langle \mathcal{B}\mathbf{w}, \mathbf{w} \rangle = \langle \boldsymbol{\delta}, \mathbf{w} \rangle.$$

Since the assumptions of Lemma 2.4 are fulfilled, the inequality (21) is valid. So we have

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_X^2 + c \sum_{i=1}^{n_1} \|\nabla w_i\|^2 - \lambda \|\mathbf{w}\|_X^2 \leq \langle \boldsymbol{\delta}, \mathbf{w} \rangle.$$

Because of

$$\langle \boldsymbol{\delta}, \mathbf{w} \rangle \leq \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \|\mathbf{w}\|_X^2 \right\}^{1/2} \leq \frac{c}{2} \left\{ \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \|\mathbf{w}\|_X^2 \right\} + \frac{1}{2c} \|\boldsymbol{\delta}\|_*^2,$$

where we have used the ε-inequality with $\varepsilon := c/2$, we get

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_X^2 + \frac{c}{2} \sum_{i=1}^{n_1} \|\nabla w_i\|^2 - \left(\lambda + \frac{c}{2}\right) \|\mathbf{w}\|_X^2 \leq \frac{1}{2c} \|\boldsymbol{\delta}\|_*^2.$$

With $\mu := 2\lambda + c$ it follows that

$$\frac{d}{dt} \|\mathbf{w}\|_X^2 - \mu \|\mathbf{w}\|_X^2 \leq \frac{1}{c} \|\boldsymbol{\delta}\|_*^2.$$

Using the same technique as in the proof of the previous theorem, we obtain the desired result. □

Remark 5.2. The estimate (35) is essentially a regularity estimate. This also applies to the comparable assumptions in the subsequent theorems.

Example 5.3. The PDE (11) from Example 2.3 has the perturbation index $\mathfrak{i}_p = 1$ provided the matrix

$$\begin{pmatrix} 2 - \nu \cdot b_{11} & -\nu \cdot b_{12} \\ -\nu \cdot b_{12} & -\nu \cdot b_{22} \end{pmatrix}$$

is positive definite on $\partial\Omega$.

The next theorem corresponds to a situation where $\mathcal{N}_E = \mathcal{N}_{E1} \cup \mathcal{N}_{E2}$ with $\mathcal{N}_{E1} = [1, n_1]_{\mathbb{N}}$, $\mathcal{N}_{E2} = [n_1 + n_2 + 1, n]_{\mathbb{N}}$ and $\mathcal{N}_H = [n_1 + 1, n_1 + n_2]_{\mathbb{N}}$ for some $n_1, n_2, n \in \mathbb{N}$, $n_1 < n_2 < n$ (i.e. we do not require that the indices are arranged in correspondence with (6)).

Theorem 5.4. *Consider the PDAE*

$$\begin{aligned} \dot{u}_i - \sum_{j=1}^{n_1} \nabla \cdot (K_{ij} \nabla u_j) + \sum_{j=1}^{n_1+n_2} \nabla \cdot (b_{ij} u_j) + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad i \leq n_1, \\ \dot{u}_i + \sum_{j=1}^{n_1+n_2} \nabla \cdot (b_{ij} u_j) + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad n_1 < i \leq n_1 + n_2, \\ \sum_{j=n_1+n_2+1}^{n} [-\nabla \cdot (K_{ij} \nabla u_j) + \nabla \cdot (b_{ij} u_j)] + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad n_1 + n_2 < i \leq n \end{aligned}$$

and assume that the conditions of Lemma 2.4 are satisfied.

If, in addition, the restriction of $\langle \mathcal{B}\mathbf{v}, \mathbf{v} \rangle$ *to the subspace*

$$V_{E2} := \bigotimes_{i=1}^{n_1+n_2} \{0\} \times \bigotimes_{i=n_1+n_2+1}^{n} V_i$$

is coercive, i.e. there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle \geq \alpha \sum_{i=n_1+n_2+1}^{n} \|v_i\|_{1,2,\Omega}^2 \quad \forall \mathbf{v} \in V_{E2}$$

and if $\boldsymbol{\delta}$ admits an estimate of the type

$$|\langle \boldsymbol{\delta}, \mathbf{v}\rangle| \leq \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla v_i\|^2 + \sum_{i=n_1+n_2+1}^{n} \|\nabla v_i\|^2 + \|\mathbf{v}\|_X^2 \right\}^{1/2} \qquad \forall \mathbf{v} \in V,$$

then the weak problem has the perturbation index $\mathbf{i}_p = 1$.

Proof. First we consider the restriction of the error equation

$$(A\dot{\mathbf{w}}, \mathbf{v}) + \langle \mathcal{B}\mathbf{w}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle \tag{36}$$

to the subspace V_{E2}, i.e.

$$\langle \mathcal{B}\mathbf{w}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in V_{E2}.$$

In particular, if we take

$$\mathbf{v} = (v_i)_{i=1}^n \quad \text{with} \quad v_i := \begin{cases} w_i,\ i \in [n_1+n_2+1, n]_{\mathbb{N}}, \\ 0, \quad \text{otherwise}, \end{cases}$$

then we get

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle - \langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle. \tag{37}$$

Notice that, by assumption,

$$|\langle \boldsymbol{\delta}, \mathbf{v}\rangle| \leq \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=n_1+n_2+1}^{n} \|w_i\|_{1,2,\Omega}^2 \right\}^{1/2} \leq \frac{1}{\alpha}\|\boldsymbol{\delta}\|_*^2 + \frac{\alpha}{4} \sum_{i=n_1+n_2+1}^{n} \|w_i\|_{1,2,\Omega}^2, \tag{38}$$

where we have used the ε-inequality with $\varepsilon := \alpha/4$. Since

$$|\langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle| = \left| \sum_{i=n_1+n_2+1}^{n} \sum_{j=1}^{n_1+n_2} (c_{ij} w_j, w_i) \right| \leq \sigma \sum_{i=n_1+n_2+1}^{n} \sum_{j=1}^{n_1+n_2} \|w_i\| \|w_j\|,$$

where

$$\sigma := \max_{\substack{i \in [n_1+n_2+1, n]_{\mathbb{N}} \\ j \in [1, n_1+n_2]_{\mathbb{N}}}} \operatorname*{esssup}_{x \in \Omega} |c_{ij}(x)|,$$

the ε-inequality yields

$$\begin{aligned} |\langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle| &\leq \sigma \sum_{i=n_1+n_2+1}^{n} \sum_{j=1}^{n_1+n_2} \left[\varepsilon \|w_i\|^2 + \frac{1}{4\varepsilon}\|w_j\|^2 \right] \\ &\leq \sigma \Big[\varepsilon(n_1+n_2) \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 + \frac{n-n_1-n_2}{4\varepsilon} \sum_{j=1}^{n_1+n_2} \|w_j\|^2 \Big]. \end{aligned}$$

The choice $\varepsilon := \alpha/(4\sigma(n_1+n_2))$ leads to

$$|\langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle| \le \frac{\alpha}{4} \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 + \frac{\sigma^2(n-n_1-n_2)(n_1+n_2)}{\alpha} \sum_{j=1}^{n_1+n_2} \|w_j\|^2.$$

Using this estimate, we conclude from (37), (38) that

$$\alpha \sum_{i=n_1+n_2+1}^{n} \|w_i\|_{1,2,\Omega}^2 \le \frac{1}{\alpha}\|\boldsymbol{\delta}\|_*^2 + \frac{\alpha}{2} \sum_{i=n_1+n_2+1}^{n} \|w_i\|_{1,2,\Omega}^2 + \frac{\sigma^2(n-n_1-n_2)(n_1+n_2)}{\alpha} \sum_{j=1}^{n_1+n_2} \|w_j\|^2,$$

hence

$$\frac{\alpha}{2} \sum_{i=n_1+n_2+1}^{n} \|w_i\|_{1,2,\Omega}^2 \le \frac{1}{\alpha}\|\boldsymbol{\delta}\|_*^2 + \frac{\sigma^2(n-n_1-n_2)(n_1+n_2)}{\alpha} \sum_{j=1}^{n_1+n_2} \|w_j\|^2. \tag{39}$$

Returning to the general error equation (36) and setting $\mathbf{v} := \mathbf{w}$, we have by Lemma 2.4 that

$$\frac{1}{2} \sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + c \sum_{j=1}^{n_1} \|\nabla w_i\|^2 + c \sum_{j=n_1+n_2+1}^{n} \|\nabla w_i\|^2 - \lambda \|\mathbf{w}\|_X^2 \le \langle \boldsymbol{\delta}, \mathbf{w}\rangle$$

$$\le \frac{1}{4\varepsilon}\|\boldsymbol{\delta}\|_*^2 + \varepsilon \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \varepsilon \sum_{i=n_1+n_2+1}^{n} \|\nabla w_i\|^2 + \varepsilon \|\mathbf{w}\|_X^2.$$

With $\varepsilon := c/2$ we get

$$\frac{1}{2} \sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + \frac{c}{2} \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \frac{c}{2} \sum_{i=n_1+n_2+1}^{n} \|\nabla w_i\|^2 - \left(\lambda + \frac{c}{2}\right) \|\mathbf{w}\|_X^2 \le \frac{1}{2c}\|\boldsymbol{\delta}\|_*^2.$$

Hence

$$\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 - \mu \|\mathbf{w}\|_X^2 \le \frac{1}{c}\|\boldsymbol{\delta}\|_*^2,$$

where $\mu := 2\lambda + c$. By definition of the X-norm, this estimate can be rewritten as

$$\sum_{i=1}^{n_1+n_2} \left[\frac{d}{dt}\|w_i\|^2 - \mu\|w_i\|^2\right] \le \frac{1}{c}\|\boldsymbol{\delta}\|_*^2 + \mu \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2.$$

The last term on the right-hand side can be estimated by means of (39):

$$\sum_{i=1}^{n_1+n_2} \left[\frac{d}{dt}\|w_i\|^2 - \mu\|w_i\|^2\right] \le \left(\frac{1}{c} + \frac{2}{\alpha^2}\right)\|\boldsymbol{\delta}\|_*^2 + \tilde{\mu} \sum_{i=1}^{n_1+n_2} \|w_i\|^2,$$

where $\tilde{\mu} := 2\mu\sigma^2(n-n_1-n_2)(n_1+n_2)/\alpha^2$. Thus we arrive at the relation

$$\sum_{i=1}^{n_1+n_2} \left[\frac{d}{dt}\|w_i\|^2 - (\mu+\tilde{\mu})\|w_i\|^2\right] \le \left(\frac{1}{c} + \frac{2}{\alpha^2}\right)\|\boldsymbol{\delta}\|_*^2.$$

After integration and together with (39) the desired estimate follows. □

In the rest of this section we consider the situation where $\mathcal{N}_E = [1, n_1]_{\mathbb{N}}$, $\mathcal{N}_H = \mathcal{N}_{H1} \cup \mathcal{N}_{H2}$ with $\mathcal{N}_{H1} = [n_1+1, n_1+n_2]_{\mathbb{N}}$ and $\mathcal{N}_{H2} = [n_1+n_2+1, n]_{\mathbb{N}}$ for some $n_1, n_2, n \in \mathbb{N}$, $n_1 < n_2 < n$.

Theorem 5.5. *Consider the PDAE*

$$\begin{aligned}\dot{u}_i - \sum_{j=1}^{n_1} \nabla \cdot (K_{ij} \nabla u_j) + \sum_{j=1}^{n_1+n_2} \nabla \cdot (b_{ij} u_j) + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad i \le n_1, \\ \dot{u}_i + \sum_{j=1}^{n_1+n_2} \nabla \cdot (b_{ij} u_j) + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad n_1 < i \le n_1 + n_2, \\ \sum_{j=n_1+n_2+1}^{n} \nabla \cdot (b_{ij} u_j) + \sum_{j=1}^{n} c_{ij} u_j &= f_i, \qquad n_1 + n_2 < i \le n\end{aligned}$$

and assume that the conditions of Lemma 2.4 are satisfied.

If, in addition, the restriction of $\langle \mathcal{B}\mathbf{v}, \mathbf{v} \rangle$ to the subspace

$$V_{H2} := \bigotimes_{i=1}^{n_1+n_2} \{0\} \times \bigotimes_{i=n_1+n_2+1}^{n} V_i$$

is coercive, i.e. there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v} \rangle \ge \alpha \sum_{i=n_1+n_2+1}^{n} \|v_i\|^2 \quad \forall \mathbf{v} \in V_{H2}$$

and if $\boldsymbol{\delta}$ admits an estimate of the type

$$|\langle \boldsymbol{\delta}, \mathbf{v} \rangle| \le \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla v_i\|^2 + \|\mathbf{v}\|_X^2 \right\}^{1/2} \qquad \forall \mathbf{v} \in V,$$

then the weak problem has the perturbation index $\mathbf{i}_p = 1$.

Proof. First we consider the restriction of the error equation

$$(A\dot{\mathbf{w}}, \mathbf{v}) + \langle \mathcal{B}\mathbf{w}, \mathbf{v} \rangle = \langle \boldsymbol{\delta}, \mathbf{v} \rangle \tag{40}$$

to the subspace V_{H2}, i.e.

$$\langle \mathcal{B}\mathbf{w}, \mathbf{v} \rangle = \langle \boldsymbol{\delta}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V_{H2}.$$

In particular, if we take

$$\mathbf{v} = (v_i)_{i=1}^{n} \quad \text{with} \quad v_i := \begin{cases} w_i, \ i \in [n_1 + n_2 + 1, n]_{\mathbb{N}}, \\ 0, \quad \text{otherwise}, \end{cases}$$

then we get

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle - \langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle. \tag{41}$$

By assumption,

$$|\langle \boldsymbol{\delta}, \mathbf{v}\rangle| \le \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 \right\}^{1/2} \le \frac{1}{\alpha}\|\boldsymbol{\delta}\|_*^2 + \frac{\alpha}{4} \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2, \tag{42}$$

where we have used the ε-inequality with $\varepsilon := \alpha/4$.

Furthermore, as in the proof of Theorem 5.4 we can show that

$$|\langle \mathcal{B}(\mathbf{w}-\mathbf{v}), \mathbf{v}\rangle| \le \frac{\alpha}{4} \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 + \frac{\sigma^2(n-n_1-n_2)(n_1+n_2)}{\alpha} \sum_{j=1}^{n_1+n_2} \|w_j\|^2.$$

Putting this estimate together with (41), (42), we see that

$$\frac{\alpha}{2} \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 \le \frac{1}{\alpha}\|\boldsymbol{\delta}\|_*^2 + \frac{\sigma^2(n-n_1-n_2)(n_1+n_2)}{\alpha} \sum_{j=1}^{n_1+n_2} \|w_j\|^2. \tag{43}$$

Returning to the general error equation (40) and setting $\mathbf{v} := \mathbf{w}$, we have by Lemma 2.4 that

$$\begin{aligned} \frac{1}{2} \sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + c \sum_{j=1}^{n_1} \|\nabla w_i\|^2 - \lambda \|\mathbf{w}\|_X^2 &\le \langle \boldsymbol{\delta}, \mathbf{w}\rangle \\ &\le \frac{1}{4\varepsilon}\|\boldsymbol{\delta}\|_*^2 + \varepsilon \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \varepsilon \|\mathbf{w}\|_X^2. \end{aligned}$$

With $\varepsilon := c/2$ we get

$$\frac{1}{2} \sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + \frac{c}{2} \sum_{i=1}^{n_1} \|\nabla w_i\|^2 - \left(\lambda + \frac{c}{2}\right) \|\mathbf{w}\|_X^2 \le \frac{1}{2c}\|\boldsymbol{\delta}\|_*^2.$$

Hence

$$\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 - \mu \|\mathbf{w}\|_X^2 \le \frac{1}{c}\|\boldsymbol{\delta}\|_*^2,$$

where $\mu := 2\lambda + c$. The rest of the proof runs as in the proof of Theorem 5.4.

Finally we give a result for a problem with perturbation index 2.

Theorem 5.6. *Consider the PDAE*

$$\begin{aligned} \dot{u}_i - \sum_{j=1}^{n_1} \nabla\cdot(K_{ij}\nabla u_j) + \sum_{j=1}^{n_1} \nabla\cdot(b_{ij}u_j) + \sum_{j=1}^{n} c_{ij}u_j &= f_i, \qquad i \le n_1, \\ \dot{u}_i + \sum_{j=n_1+n_2+1}^{n} a_{ij}\dot{u}_j + \sum_{j=n_1+1}^{n_1+n_2} \nabla\cdot(b_{ij}u_j) + \sum_{j=1}^{n} c_{ij}u_j &= f_i, \qquad n_1 < i \le n_1+n_2, \\ \sum_{j=n_1+n_2+1}^{n} \nabla\cdot(b_{ij}u_j) + \sum_{j=n_1+n_2+1}^{n} c_{ij}u_j &= f_i, \qquad n_1+n_2 < i \le n \end{aligned}$$

and assume that the conditions of Lemma 2.4 are satisfied.

If, in addition, the restriction of $\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle$ *to the subspace*

$$V_{H2} := \bigotimes_{i=1}^{n_1+n_2} \{0\} \times \bigotimes_{i=n_1+n_2+1}^{n} V_i$$

is coercive, i.e. there exists a constant $\alpha > 0$ *such that*

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle \geq \alpha \sum_{i=n_1+n_2+1}^{n} \|v_i\|^2 \quad \forall \mathbf{v} \in V_{H2}$$

and if $\boldsymbol{\delta}$ *admits the estimates*

$$|\langle \boldsymbol{\delta}, \mathbf{v}\rangle| \leq \|\boldsymbol{\delta}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla v_i\|^2 + \|\mathbf{v}\|_X^2 \right\}^{1/2} \qquad \forall \mathbf{v} \in V$$

and

$$|\langle \dot{\boldsymbol{\delta}}, \mathbf{v}\rangle| \leq \|\dot{\boldsymbol{\delta}}\|_* \left\{ \sum_{i=1}^{n_1} \|\nabla v_i\|^2 + \|\mathbf{v}\|_X^2 \right\}^{1/2} \qquad \forall \mathbf{v} \in V,$$

then the weak problem has the perturbation index $\mathbf{i}_p = 2$.

Proof. As in the proof of Theorem 5.5 we restrict the error equation

$$(A\dot{\mathbf{w}}, \mathbf{v}) + \langle \mathcal{B}\mathbf{w}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle \tag{44}$$

to the subspace V_{H2} and take the same particular test function

$$\mathbf{v} = (v_i)_{i=1}^n \quad \text{with} \quad v_i := \begin{cases} w_i,\ i \in [n_1 + n_2 + 1, n]_{\mathbb{N}}, \\ 0, \quad \text{otherwise.} \end{cases}$$

Since $\langle \mathcal{B}(\mathbf{w} - \mathbf{v}), \mathbf{v}\rangle = 0$, we get

$$\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle = \langle \boldsymbol{\delta}, \mathbf{v}\rangle.$$

By the assumption w.r.t. $\boldsymbol{\delta}$,

$$|\langle \boldsymbol{\delta}, \mathbf{v}\rangle| \leq \frac{1}{2\alpha} \|\boldsymbol{\delta}\|_*^2 + \frac{\alpha}{2} \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2,$$

where we have used the ε-inequality with $\varepsilon := \alpha/2$.

Since $\langle \mathcal{B}\mathbf{v}, \mathbf{v}\rangle$ is coercive on V_{H2}, it follows that

$$\sum_{i=n_1+n_2+1}^{n} \|w_i\|^2 \leq \frac{1}{\alpha^2} \|\boldsymbol{\delta}\|_*^2. \tag{45}$$

In a next step we differentiate the restriction of the error equation to V_{H2} w.r.t. t and get

$$\langle \mathcal{B}\dot{\mathbf{w}}, \mathbf{v}\rangle = \langle \dot{\boldsymbol{\delta}}, \mathbf{v}\rangle \quad \forall \mathbf{v} \in V_{H2}.$$

Using the particular test function

$$\mathbf{v} = (v_i)_{i=1}^n \quad \text{with} \quad v_i := \begin{cases} \dot{w}_i, \ i \in [n_1+n_2+1, n]_{\mathbb{N}}, \\ 0, \quad \text{otherwise}, \end{cases}$$

we obtain, as in the first part of this proof, the estimate

$$\sum_{i=n_1+n_2+1}^{n} \|\dot{w}_i\|^2 \le \frac{1}{\alpha^2}\|\dot{\boldsymbol{\delta}}\|_*^2. \tag{46}$$

Returning to the general error equation (44) and setting $\mathbf{v} := \mathbf{w}$, we have by Lemma 2.4 that

$$\frac{1}{2}\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + \sum_{i=n_1+1}^{n_1+n_2} \sum_{j=n_1+n_2+1}^{n} (a_{ij}\dot{w}_j, w_i) + c\sum_{j=1}^{n_1} \|\nabla w_i\|^2 - \lambda\|\mathbf{w}\|_X^2 \le \langle \boldsymbol{\delta}, \mathbf{w}\rangle.$$

Putting the second term of the left-hand side to the right-hand side and using the estimate

$$\begin{aligned} \left| \sum_{i=n_1+1}^{n_1+n_2} \sum_{j=n_1+n_2+1}^{n} (a_{ij}\dot{w}_j, w_i) \right| &\le \sum_{i=n_1+1}^{n_1+n_2} \sum_{j=n_1+n_2+1}^{n} \|a_{ij}\|_{0,\infty,\Omega} \|\dot{w}_j\| \|w_i\| \\ &\le \eta \left\{ \sum_{i=n_1+n_2+1}^{n} \|\dot{w}_i\|^2 \right\}^{1/2} \left\{ \sum_{i=n_1+1}^{n_1+n_2} \|w_i\|^2 \right\}^{1/2}, \end{aligned}$$

where

$$\eta^2 := \sup_{\xi_j \in \mathbb{R}:\, n_1+n_2<j\le n} \frac{\sum_{i=n_1+1}^{n_1+n_2} \left\{ \sum_{j=n_1+n_2+1}^{n} \|a_{ij}\|_{0,\infty,\Omega}\xi_j \right\}^2}{\sum_{j=n_1+n_2+1}^{n} \xi_j^2},$$

we get by the help of the ε-inequality

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + c\sum_{j=1}^{n_1} \|\nabla w_i\|^2 - \lambda\|\mathbf{w}\|_X^2 &\le \frac{1}{4\varepsilon_1}\|\boldsymbol{\delta}\|_*^2 + \varepsilon_1 \sum_{i=1}^{n_1} \|\nabla w_i\|^2 + \varepsilon_1\|\mathbf{w}\|_X^2 \\ &\quad + \frac{\eta^2}{4\varepsilon_2} \sum_{i=n_1+n_2+1}^{n} \|\dot{w}_i\|^2 + \varepsilon_2 \sum_{i=n_1+1}^{n_1+n_2} \|w_i\|^2. \end{aligned}$$

With $\varepsilon_1 := c/2$ and $\varepsilon_2 := 1/2$ it follows that

$$\begin{aligned} \frac{1}{2}\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 + \frac{c}{2}\sum_{i=1}^{n_1} \|\nabla w_i\|^2 - \Big(\lambda + \frac{c}{2} + \frac{1}{2}\Big)\|\mathbf{w}\|_X^2 &\le \frac{1}{2c}\|\boldsymbol{\delta}\|_*^2 + \frac{\eta^2}{2} \sum_{i=n_1+n_2+1}^{n} \|\dot{w}_i\|^2 \\ &\le \frac{1}{2c}\|\boldsymbol{\delta}\|_*^2 + \frac{\eta^2}{2\alpha^2}\|\dot{\boldsymbol{\delta}}\|_*^2, \end{aligned}$$

where we have used (46). So we arrive at

$$\sum_{i=1}^{n_1+n_2} \frac{d}{dt}\|w_i\|^2 - \mu\|\mathbf{w}\|_X^2 \le \frac{1}{c}\|\boldsymbol{\delta}\|_*^2 + \frac{\eta^2}{\alpha^2}\|\dot{\boldsymbol{\delta}}\|_*^2,$$

where $\mu := 2\lambda + c + 1$. By definition of the X-norm, this estimate can be rewritten as

$$\sum_{i=1}^{n_1+n_2} \left[\frac{d}{dt}\|w_i\|^2 - \mu\|w_i\|^2\right] \le \frac{1}{c}\|\boldsymbol{\delta}\|_*^2 + \frac{\eta^2}{\alpha^2}\|\dot{\boldsymbol{\delta}}\|_*^2 + \mu \sum_{i=n_1+n_2+1}^{n} \|w_i\|^2.$$

Using (45), we get

$$\sum_{i=1}^{n_1+n_2} \left[\frac{d}{dt}\|w_i\|^2 - \mu\|w_i\|^2\right] \le \left(\frac{1}{c} + \frac{\mu}{\alpha^2}\right)\|\boldsymbol{\delta}\|_*^2 + \frac{\eta^2}{\alpha^2}\|\dot{\boldsymbol{\delta}}\|_*^2.$$

After integration and with (45) again we conclude that the weak problem has the perturbation index $\mathfrak{i}_p = 2$.

6 Applications

In this section we apply the above results to two nonlinear problems of great practical interest. Like other authors too (e.g. [25]), we use a linearization approach. The reason is that the generalization of Definition 3.1 to the case of nonlinear PDAEs causes several difficulties. For instance, it could happen that the family $\mathcal{F}$ is too small.

6.1 The compressible Euler equations

These equations read as

$$\begin{aligned} \rho\dot{u} + \rho(u\cdot\nabla)u + \nabla p &= f, \\ \dot{\rho} + \nabla\cdot(\rho u) &= f_{d+1}, \\ p &= r(\rho), \end{aligned}$$

where p is the pressure and $r(\rho)$ is a given function defining the equation of state. The fluid-density $\rho = \rho(t,x)$ and the velocity field $u = (u_1(t,x),\ldots,u_d(t,x))^\top$ are the unknown functions. The boundary and the initial conditions are

$$\begin{aligned} \nu\cdot u &= 0 \quad \text{on } J\times\partial\Omega, \\ u(0,x) &= u_0(x), \quad x\in\Omega, \\ \rho(0,x) &= \rho_0(x), \quad x\in\Omega. \end{aligned}$$

More information about the mathematical theory and the physical background of the Euler equations can be found in [18], [2], [15], and [16].

First we write the equations component-wise and use the fact that

$$\partial_x p = \partial_x r(\rho) = \partial_\rho r\,\partial_x\rho.$$

In general, the following inequalities are satisfied:

$$r > 0 \text{ and } c^2 := \partial_\rho r > 0.$$

In the three-dimensional case (i.e. $d = 3$) we get

$$\begin{aligned}
\rho\dot{u}_1 + \rho(u_1\partial_x u_1 + u_2\partial_y u_1 + u_3\partial_z u_1) + c^2\partial_x\rho &= f_1, \\
\rho\dot{u}_2 + \rho(u_1\partial_x u_2 + u_2\partial_y u_2 + u_3\partial_z u_2) + c^2\partial_y\rho &= f_2, \\
\rho\dot{u}_3 + \rho(u_1\partial_x u_3 + u_2\partial_y u_3 + u_3\partial_z u_3) + c^2\partial_z\rho &= f_3, \\
\dot{\rho} + \rho(\partial_x u_1 + \partial_y u_2 + \partial_z u_3) + u_1\partial_x\rho + u_2\partial_y\rho + u_3\partial_z\rho &= f_4.
\end{aligned}$$

Next we linearize the system at the point $(u_1, u_2, u_3, \rho)^\top$ and obtain a system in the new unknown variables $\mathbf{w} = (w_1, w_2, w_3, w_4)^\top$

$$A\dot{\mathbf{w}} + \sum_{j=1}^{n} \mathcal{L}_{ij} w_j = \tilde{\mathbf{f}} \tag{47}$$

with

$$\mathcal{L}_{ij} w := \tilde{b}_{ij} \cdot \nabla w + \tilde{c}_{ij} w, \qquad i, j \in [1, n]_{\mathbb{N}},$$

and

$$A := \begin{pmatrix} \rho\ 0\ 0\ 0 \\ 0\ \rho\ 0\ 0 \\ 0\ 0\ \rho\ 0 \\ 0\ 0\ 0\ 1 \end{pmatrix}, \quad \tilde{\mathbf{f}} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix},$$

$$\tilde{b}_{11} := \tilde{b}_{22} := \tilde{b}_{33} := \begin{pmatrix} \rho u_1 \\ \rho u_2 \\ \rho u_3 \end{pmatrix}, \quad \tilde{b}_{12} := \tilde{b}_{13} := \tilde{b}_{21} := \tilde{b}_{23} := \tilde{b}_{31} := \tilde{b}_{32} := \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{b}_{14} := \begin{pmatrix} c^2 \\ 0 \\ 0 \end{pmatrix}, \tilde{b}_{24} := \begin{pmatrix} 0 \\ c^2 \\ 0 \end{pmatrix}, \tilde{b}_{34} := \begin{pmatrix} 0 \\ 0 \\ c^2 \end{pmatrix},$$

$$\tilde{b}_{41} := \begin{pmatrix} \rho \\ 0 \\ 0 \end{pmatrix}, \tilde{b}_{42} := \begin{pmatrix} 0 \\ \rho \\ 0 \end{pmatrix}, \tilde{b}_{43} := \begin{pmatrix} 0 \\ 0 \\ \rho \end{pmatrix}, \tilde{b}_{44} := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

$$\tilde{C} := \begin{pmatrix} \rho\partial_x u_1 & \rho\partial_y u_1 & \rho\partial_z u_1 & \dot{u}_1 + u_1\partial_x u_1 + u_2\partial_y u_1 + u_3\partial_z u_1 \\ \rho\partial_x u_2 & \rho\partial_y u_2 & \rho\partial_z u_2 & \dot{u}_2 + u_1\partial_x u_2 + u_2\partial_y u_2 + u_3\partial_z u_2 \\ \rho\partial_x u_3 & \rho\partial_y u_3 & \rho\partial_z u_3 & \dot{u}_3 + u_1\partial_x u_3 + u_2\partial_y u_3 + u_3\partial_z u_3 \\ \partial_x \rho & \partial_y \rho & \partial_z \rho & \partial_x u_1 + \partial_y u_2 + \partial_z u_3 \end{pmatrix}.$$

This system can be symmetrized by the following scaling. Using the new variable $\tilde{w}_4 := c^2 w_4/\rho$ and multiplying the fourth equation by c^2, we obtain a symmetric hyperbolic system in the new variables $(w_1, w_2, w_3, \tilde{w}_4)^\top$. For simplification of the presentation we write again $(w_1, w_2, w_3, w_4)^\top$ instead of $(w_1, w_2, w_3, \tilde{w}_4)^\top$. Then the system (47) can be written in the form (1), (4), (5) with the settings $b_{ij} := \tilde{b}_{ij}$ and $c_{ij} := \tilde{c}_{ij} - \nabla \cdot \tilde{b}_{ij}$.

Theorem 6.1. *Let the linearized compressible Euler equations be uniquely solvable. Then the linearized system has the perturbation index 1.*

Proof. Since both $u = (u_1, u_2, u_3)^\top$ and $(w_1, w_2, w_3)^\top$ satisfy the boundary condition $\nu \cdot u = 0$ resp. $\sum_{i=1}^3 \nu_i w_i = 0$ on $J \times \partial\Omega$, we have

$$2B_{HH} = \varrho \begin{pmatrix} \nu\cdot u & 0 & 0 & \nu_1 \\ 0 & \nu\cdot u & 0 & \nu_2 \\ 0 & 0 & \nu\cdot u & \nu_3 \\ \nu_1 & \nu_2 & \nu_3 & \nu\cdot u \end{pmatrix} = \varrho \begin{pmatrix} 0 & 0 & 0 & \nu_1 \\ 0 & 0 & 0 & \nu_2 \\ 0 & 0 & 0 & \nu_3 \\ \nu_1 & \nu_2 & \nu_3 & 0 \end{pmatrix} \text{ and } M_{HH} = 0 \text{ on } J \times \partial\Omega.$$

Therefore, $\mathbf{w}^\top B_{HH} \mathbf{w} = w_4 \sum_{i=1}^3 \nu_i w_i = 0$, and we can use Remark 4.2 to obtain the assertion. □

6.2 The compressible Navier-Stokes equations

The compressible Navier-Stokes equations read as

$$\begin{aligned} \rho\dot{u} - \mu\Delta u - (\mu + \mu')\nabla(\nabla\cdot u) + \rho(u\cdot\nabla)u + \nabla p &= f, \\ \dot{\rho} + \nabla\cdot(\rho u) &= f_{d+1}, \\ p &= r(\rho), \end{aligned}$$

where $\mu > 0$, $\mu' \geq 0$, p is the pressure, and $r(\rho)$ is a given function. The fluid-density $\rho = \rho(t, x)$ and the velocity field $u = (u_1(t, x), \ldots, u_d(t, x))^\top$ are the unknown functions. Furthermore, the boundary and initial conditions are given by

$$\begin{aligned} u &= g \quad \text{on } J \times \partial\Omega, \\ u(0, x) &= u_0(x), \quad x \in \Omega, \\ \rho(0, x) &= \rho_0(x), \quad x \in \Omega. \end{aligned}$$

The reader is referred to the classical references [1] and [13] for the physical aspects. Here we only consider the two-dimensional case (i.e. $d = 2$). A generalization for the three-dimensional case can be done in a straightforward manner.

First we write the equations component-wise and use the information

$$\partial_x p = \partial_x r(\rho) = \partial_\rho r \partial_x \rho.$$

As in the previous section, we have

$$r > 0 \text{ and } c^2 := \partial_\rho r > 0.$$

We get

$$\begin{aligned}
\rho\dot{u}_1 - (2\mu+\mu')\partial_{xx}u_1 - (\mu+\mu')\partial_{xy}u_2 - \mu\partial_{yy}u_1 & \\
+\rho(u_1\partial_x u_1 + u_2\partial_y u_1) + c^2\partial_x\rho &= f_1, \\
\rho\dot{u}_2 - \mu\partial_{xx}u_2 - (\mu+\mu')\partial_{xy}u_2 - (2\mu+\mu')\partial_{yy}u_2 & \\
+\rho(u_1\partial_x u_2 + u_2\partial_y u_2) + c^2\partial_y\rho &= f_2, \\
\dot{\rho} + \rho(\partial_x u_1 + \partial_y u_2) + u_1\partial_x\rho + u_2\partial_y\rho &= f_3.
\end{aligned}$$

Next we linearize the system at the point $(u_1, u_2, \rho)^\top$ and obtain a system in the unknown variables $\mathbf{w} = (w_1, w_2, w_3)^\top$

$$A\dot{\mathbf{w}} + \mathcal{L}\mathbf{w} = \tilde{\mathbf{f}}$$

with

$$A := \begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\mathbf{f}} := \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad K_{11} := \begin{pmatrix} 2\mu+\mu' & 0 \\ 0 & \mu+\mu' \end{pmatrix},$$

$$K_{12} := K_{21} := \frac{\mu+\mu'}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad K_{22} := \begin{pmatrix} \mu+\mu' & 0 \\ 0 & 2\mu+\mu' \end{pmatrix},$$

$$b_{11} := \begin{pmatrix} \rho u_1 \\ \rho u_2 \end{pmatrix}, \quad b_{12} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b_{13} := \begin{pmatrix} \partial_\rho r \\ 0 \end{pmatrix},$$

$$b_{21} := \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad b_{22} := \begin{pmatrix} \rho u_1 \\ \rho u_2 \end{pmatrix}, \quad b_{23} := \begin{pmatrix} 0 \\ \partial_\rho r \end{pmatrix},$$

$$C := \begin{pmatrix} \rho\partial_x u_1 & \rho\partial_y u_1 & \dot{u}_1 + u_1\partial_x u_1 + u_2\partial_y u_1 \\ \rho\partial_x u_2 & \rho\partial_y u_2 & \dot{u}_2 + u_1\partial_x u_2 + u_2\partial_y u_2 \\ \partial_x\rho & \partial_y\rho & \partial_x u_1 + \partial_y u_2 \end{pmatrix}.$$

Again, the linearized system can be symmetrized by the following scaling. Using the new variable $\tilde{w}_3 := c^2 w_3/\rho$ and multiplying the third equation by c^2, we obtain a symmetric problem. As in the previous section the system can be written in the form (1), (4), (5).

Theorem 6.2. *Let the linearized compressible Navier-Stokes equations be uniquely solvable. Then the linearized system has the perturbation index 1.*

Proof. In the following we show that the assumptions of Theorem 5.1 are fulfilled. First we have $\underline{\kappa}_{11} = \underline{\kappa}_{22} = \mu + \mu'$ and $\kappa_{12} = \kappa_{21} = \frac{1}{2}(\mu + \mu')$. Hence the matrix

$$\kappa = \frac{1}{2}(\mu + \mu') \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

is positive definite. Since we have prescribed Dirichlet boundary conditions on the whole boundary, the assumptions of Lemma 2.4 w.r.t. the boundary matrices are satisfied trivially. So we may apply Theorem 5.1, and the problem has the perturbation index $\mathfrak{i}_p = 1$. □

7 Conclusion

We dicussed systems of partial differential equations of the form

$$A\dot{\mathbf{u}} + \mathcal{L}\mathbf{u} = \mathbf{f}$$

with a possibly singular coefficient matrix A and a linear differential operator $\mathcal{L}$ with respect to the space variables subject to general boundary conditions.

Stability results with respect to perturbations of the problem data (of the right-hand side as well as of the boundary data) were derived. The main tool for these estimates was the availability of a Gårding-type inequality.

The similarity of the perturbation results with those of the perturbation index for ordinary differential equations motivated a definition of a perturbation index for the class of problems considered here. In particular, the knowledge of the index is important for the selection of appropriate numerical methods.

As an application, linearizations of the compressible Euler and Navier-Stokes equations are considered.

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